ON THE STABLE HOMOTOPY TYPE OF STUNTED PROJECTIVE SPACES

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1. Introduction

Let RP^n denote the real projective space. If $m < n$, we have a natural inclusion $RP^m \subset RP^n$, and we denote by $RP_n{}^{n+k}$ the quotient space of RP^{n+k} by RP^{n-1}.

The object of this paper is to determine necessary and sufficient conditions for the stunted real projective spaces $RP_n{}^{n+k}$ and $RP_m{}^{m+k}$ to be of the same stable homotopy type. This is achieved for a great number of values of n, m and k, (see Theorem 1.3).

Note that if ξ_k denotes the Hopf bundle over RP^k, then $RP_n{}^{n+k}$ can be identified with the Thom space $T(n\xi_k)$. We are then concerned here with the classification, up to stable homotopy type, of Thom spaces of bundles over real projective spaces.

The complete classification for complex and quaternionic projective spaces was obtained in [3].

For the real case, T. Kobayashi and M. Sugarawa in [5], have obtained some necessary conditions.

Let $\phi(k)$ be the number of integers in the closed interval $[1, k]$ congruent to 0, 1, 2 or 4 modulo 8, and let $A_k = 2^{\phi(k)}$. By the work of J. F. Adams [1] and I. M. James [6], one has

THEOREM 1.1. *If n and k are integers for which either $n \equiv 0 \pmod{\frac{1}{2} A_k}$ or $n + k + 1 \equiv 0 \pmod{\frac{1}{2} A_k}$, then the spaces $RP_n{}^{n+k}$ and $RP_m{}^{m+k}$ are not of the same stable homotopy type, if $n - m \equiv \frac{1}{2} A_k \pmod{A_k}$.*

We now extend this theorem in the following way. Let $\epsilon(n, k)$ be the function which depends on the mod 4 values of n and the mod 8 values of k, given by the following table, where a blank in the table, means that the function $\epsilon(n, k)$ is not defined:

Table 1.2

n \ k	0	1	2	3	4	5	6	7
0	1/2	1/2	1/2		1/2			
1	1	1	1	1	1/2			1
2	1	1/2	1/2	1/2	1/2		1	1
3	1/2	1/2	1	1	1/2	1	1	1

Then for $k \geq 12$, we have

THEOREM 1.3. *If n and k do not satisfy the conditions of* (1.1) *and $\epsilon(n, k)$ is* *defined, then the spaces* RP_n^{n+k} *and* RP_m^{m+k} *are of the same stable homotopy type if and only if* $m \equiv n \bmod (\epsilon(n, k)A_k)$.

Notice that for the pairs (n, k) which satisfy the conditions of (1.1), we have $\epsilon(n, k) = \frac{1}{2}$, and so they are indeed exceptional.

We conjecture that if the function $\epsilon(n, k)$ is extended by defining $\epsilon(n, k) = \frac{1}{2}$ for all the remaining cases, then (1.1) and (1.3) will give the complete classification of the stable homotopy types of the stunted real projective spaces.

2. The necessary conditions

We begin by establishing, with $k \geq 9$, the following

PROPOSITION 2.1. *Let N be the order of the torsion subgroup of* $\widetilde{KO}(RP_n^{n+k})$. *Then for* RP_n^{n+k} *and* RP_m^{m+k} *to be stably homotopically equivalent, it is necessary that* $m \equiv n \left(\bmod \frac{N}{2}\right)$.

Proof. Suppose $f{:}\Sigma^{n-m}RP_m^{m+k} \to RP_n^{n+k}$ is a homotopy equivalence. Applying Steenrod squares, we find $m \equiv n \pmod 8$.

Let $\beta_i \in \widetilde{KO}(RP_i^{k+i})$ be a generator of the torsion subgroup. We have from [1], that $\psi^3\beta_i = \beta_i$. Now $f^*\beta_n = t\sigma^r\beta_m$, where t is an odd integer and σ is the Bott periodicity, while $n - m = 8r$. Thus

$$f^*(\psi^3 - 1)\beta_n = (\psi^3 - 1)t\sigma^r\beta_m = 0,$$

but $\psi^3\sigma^r\beta_m = 3^{4r}\sigma^r\beta_m$, and so we obtain the condition $3^{4r} - 1 \equiv 0 \pmod N$, which implies $4r \equiv 0 \left(\bmod \frac{N}{4}\right)$ and so (2.1) follows.

COROLLARY 2.2. *Assume* $k \geq 9$. *If the spaces* RP_n^{n+k} *and* RP_m^{m+k} *are of the same stable homotopy type, then always* $m \equiv n \pmod{\frac{1}{2}A_k}$. *Moreover, for those values of n and k for which* $\epsilon(n, k) = 1$, *we must, in fact, have* $n \equiv m \pmod{A_k}$.

3. The sufficient conditions

Our function $\epsilon(n, k)$ is determined precisely by the necessary condition of (2.2), and our goal is to construct a homotopy equivalence between $\Sigma^{m-n}RP_n^{n+k}$ and RP_m^{m+k} whenever such an equivalence cannot be outruled by these necessary conditions.

If $\epsilon(n, k) = 1$, evidently there is nothing to prove, since it is well known (cf. [2]) that Thom spaces of stably homotopically equivalent bundles are of the same stable homotopy type. We restrict ourselves to those cases for which $\epsilon(n, k) = \frac{1}{2}$.

Let us denote by $f_n{:}RP^k \to BO_n$ the map that classifies $n\xi_k$, i.e. n times the Hopf bundle over RP^k.

If $t = \frac{1}{2}A_k$, then $f_t{:}RP^k \to BO_n$ is trivial over the skeleton RP^{K-1}, where K is the largest integer less than or equal to k with the property that $K \equiv 0$, 1, 2 or

4 mod 8. Now the space $RP_K{}^k$ admits a retraction to S^K and so we have a factorization of f_t up to homotopy as:

$$RP^k \xrightarrow{\pi} RP_K{}^k \xrightarrow{r} S^K \xrightarrow{\theta_K} BO_t$$

where $\{\theta\}$ can be taken as the generator of $\pi_K(BO_t)$.

The map $f_{n+t}:RP^k \to BO_{n+t}$, can then be described as:

$$RP^k \xrightarrow{\Delta} RP^k \times RP^k \xrightarrow{1 \times \rho} RP^k \times S^K \xrightarrow{f_n \times \theta_K} BO_n \times BO_t \xrightarrow{\mu} BO_{n+t},$$

where ρ is the composite $r \circ \pi$, the map μ is the Whitney sum operation and Δ is the diagonal map.

Let $g = (1 \times \rho)\Delta$. Then we have a map of Thom spaces

$$(RP^k)^{(n+t)\xi} \xrightarrow{T(g)} (RP^k \times S^K)^\alpha$$

where we denote by α the product bundle $n\xi_k \times \theta_K$. Hence $(RP^k \times S^K)^\alpha \cong RP_m{}^{n+k} \wedge (S^K)^{\theta_K}$. If β_K is the element in the Image of J obtained from θ_K, then

$$(S^K)^{\theta_K} \cong S^t \cup_{\beta_K} e^{t+K}$$

so that we have a cofibration:

$$RP_n{}^{n+k} \wedge S^{t+K-1} \xrightarrow{1 \wedge \beta_K} RP_n{}^{n+k} \wedge S^t \to (RP^k \times S^K)^\alpha$$

Let us denote by $Y_{m,t}$ the space obtained from the cofibration

$$RP_n{}^{n+k-K} \wedge S^{t+K-1} \xrightarrow{i \wedge \beta_K} RP_n{}^m \wedge S^t \to Y_{m,t}.$$

The $(n + k + t)-$ skeleton of $(RP^n \times S^K)^\alpha$ is thus $Y_{n+k,t}$.

The following proposition is the key to the proof of sufficiency in (1.3).

PROPOSITION 3.1. *If the attaching map for $Y_{n+k-1,t}$ is nul-homotopic, then the spaces $\Sigma^t RP_n{}^{n+k}$ and $RP_n{}^{n+k+t}$ are of the same homotopy type.*

Proof. Consider the diagram:

$$\begin{array}{ccccccc}
 & \nearrow^{i \wedge \beta_K} & RP_n{}^{n+k-1} \wedge S^t & \longrightarrow & Y_{n+k-1,t} & \xrightarrow{r} & RP_n{}^{n+k-1} \wedge S^t \\
RP_n{}^{n+k-K} \wedge S^{t+K-1} & & \downarrow j & & \downarrow j & & \downarrow j \\
 & \searrow_{i \wedge \beta_K} & RP_n{}^{n+k} \wedge S^t & \longrightarrow & Y_{n+k,t} & \xrightarrow{\hat{r}} & RP_n{}^{n+k} \wedge S^t
\end{array}$$

where the vertical maps are inclusions and r, $\hat{r}$ are retractions given by the nul-homotopy of $i \wedge \beta_K$. Thus the diagram commutes up to homotopy.

Let $\Sigma^t x^{n+k}$ be a generator of $H^{n+t+k}(\Sigma^t RP_n{}^{n+k}; Z)$ it follows from the above diagram that

$$j^* \hat{r}^* (\Sigma^t x^{n+k}) = r^* j^* (\Sigma^t x^{n+k}) = 0. \tag{3.2}$$

Now if we look at the composition $\hat{r} \circ g'$

$$\begin{array}{ccccc} RP_{n+t}{}^{n+t+k} & \xrightarrow{T(g)} & (RP^k \times S^K)^\alpha. & & \\ & \searrow^{g'} & \uparrow & & \\ & & Y_{n+k,t} & \xrightarrow{\hat{r}} & RP_n{}^{n+k} \wedge S^t \end{array}$$

where g' is a cellular approximation of $T(g)$, it is easy to see that $\hat{r} \circ g'$ induces isomorphisms on integral cohomology and so is a homotopy equivalence, as follows. By definition of $T(g)$, it follows that g'^* is an isomorphism in dimensions $\leq n + k + t - 1$. Now (3.2) guarantees that $(\hat{r} \circ g')^*$ is an isomorphism in dimensions $\leq n + k + t$, since $\hat{r}^*$ is also an isomorphism in dimensions $\leq n + k + t - 1$.

Note that if $k = K$, i.e. the cases when $k \equiv 0, 1, 2$ or $4 \bmod 8$, then $RP_n{}^{n+k-K} = S^n$ and $i \wedge \beta_K$ can be identified with the composition

$$S^{n+K-1} \xrightarrow{\beta_K} S^n \xrightarrow{i} RP_n{}^{n+K-1} \tag{3.3}$$

suspended t-times. This suspension is unnecessary when n is large enough and so the hypothesis of (3.1) becomes simply the vanishing of Im J generators in appropriate stunted projective spaces.

If we take the dual of diagram (3.3), we obtain from (3.1)

PROPOSITION 3.4. *If* $K \equiv 0, 1, 2$ *or* $4 \bmod 8$ *and there exists factorization of* β_K *of the form*

$$\begin{array}{ccc} S^{m+k-1} & \xrightarrow{\beta_K} & S^m \\ & {}_{\pi}\searrow \quad \nearrow & \\ & RP_{m+1}{}^{m+k-1} & \end{array}$$

then the spaces $\Sigma^t RP_m{}^{m+k}$ *and* $RP_{m+t}{}^{m+k+t}$ *are of the same homotopy type.*

Note that if we had replaced RP_{m+1}^{m+k-1} in (3.4) by $RP_m{}^{m+k-1}$, we would be considering the problem of projectivity of β_K on S^m as described in [4].

Since a factorization such as that of (3.4) implies projectivity, we call the condition of (3.4) "strongly projective".

The most difficult vanishing theorems are marked by an "x" in Mahowald's table of [7] and was obtained by different methods in [4]. Modifying the beginning of the induction in [4] we have: For $s < m - 2$:

PROPOSITION 3.5. *The generator β of* Im J *in the stable s-stem is strongly projective on S^m provided* $3 \leq \nu_2(m) < \phi(s)$.

The inequality $\nu_2(m) < \phi(s)$ explains why we have to exclude the pairs (n, k) of (1.1).

Much easier to establish, as we need only to consider at most 8 cells, and using the techniques of [7], we have:

LEMMA 3.6. *The generators β of* Im J *in the stable $k - 1$ stem are strongly projective on S^n where $n \equiv 4(8)$.*

The following lemma is even easier:

LEMMA 3.7. *In $RP_m^{\ m+3}$ we have*
a) η^2 vanishes if $m \equiv 3 \pmod 4$
b) ξ_n vanishes if $m \equiv 1 \pmod 4$

Proposition (3.5) together with (3.6) and (3.7) establish all of table (1.2) if we use duality, except for the case $n \equiv 2 \pmod 4$ and $k \equiv 3 \pmod 8$.

In this last case we must establish that the map

$$RP_n^{\ n+1} \wedge S^{t+k-2} \xrightarrow{1 \wedge \beta_{k-1}} RP_n^{\ n+1} \wedge S^t \to RP_n^{\ n+k-1} \wedge S^t$$

is nul-homotopic. But since $n \equiv 0 \pmod 2$ this map is a sum of two maps, and since $k - 2 \equiv 1 \pmod 8$ we have $\beta_{k-1} = \rho\eta^2$ and both maps in the sum factor through maps whose homotopy classes lie in $\pi_{n+2}(P_n^{\ n+3})$, but one easily sees that $\pi_{4k+4}(P_{4k+2}^{4k+5}) = 0$ (see e.g. Paetcher [8]) and so our results follows.

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