ERGODIC H^1 SPACES

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1. Introduction

Let X be a probability space on which $\mathbb{R}^n$ acts as an ergodic flow. Cotlar [4] proved that one can define Hilbert transforms ($n = 1$) or Riesz transforms ($n > 1$). These allow to define the space $H^1(X)$ in a fashion analogous to the classical case. In this paper we prove that the main properties of $H^1(\mathbb{R}^n)$ extended to $H^1(X)$.

For $n = 1$, the characterization of $H^1(X)$ in terms of maximal functions as well as the identification of $H^1(X)^*$, were obtained by Coifman and Weiss [2].

In section 3 we extend their results to the general case, $n > 1$. In section 2 we extend to the ergodic case, for $n = 1$, a decomposition theorem for functions in $H^1(X)$, that in the classical case is due to Coifman [1].

2. The one dimensional theory

Let (X, μ) be a probability space. Let (T_s), $s \in \mathbb{R}$ be an ergodic flow on X; i.e., (T_s) satisfies: a) $T_{s+v} = T_sT_v$; b) $T_0 = I$; c) $f(T_sx)$ is measurable on $X \times \mathbb{R}$ whenever $f(x)$ is measurable on X; d) T_s is measure preserving for all s; e) the only invariant sets are those of measure zero or one. (From now on we will identify sets, and functions, that differ only on sets of measure zero).

LEMMA. *Let $0 \subset X$ be a measurable set such that for each x in 0, the set $0^x = \{t \in \mathbb{R};\ T_tx \in 0\}$ is open in $\mathbb{R}$. Then 0 can be decomposed in a disjoint union of sets, $0 = \cup I_i$, where the I_i's are measurable sets such that, for each x in I_i, the orbit through x is a disjoint union of intervals of length between 2^k and 2^{k+1}, and k is an integer depending only on the set I_i. (Such as I_i will be called an "ergodic interval" and the number 2^{k+1} will be called the "length" of the interval).*

Proof. 0^x being open, decomposes canonically as disjoint union of intervals. This decomposition commutes with the action of the flow. This means that if $y = T_sx$ then the intervals in 0^y are translations by s of the intervals in 0^x. Now for any integer k, we define the set I_k to be the set of points x in X such that the interval in 0^x containing the origin has length between 2^k and 2^{k+1}. It is then obvious that $0 = \cup$ I$_i$ and the I_i's are ergodic intervals if they are measurable. It is also clear that the I_i's are pairwise disjoint.

To prove that the I_i's are measurable we observe first that, if A is any measurable set in X and J is any interval in $\mathbb{R}$, then the set:

$$A + J = \{y \in X;\ y = T_sx,\ s \in J,\ x \in A\}$$

† Some of the results in this paper are contained in the author's Ph.D. thesis, written under the direction of R. Coifman at Washington University.

is measurable. This is an easy consequence of the properties of the flow. In particular if $J_n = (-2^{-n}, 0)$ then the set

$$0_n = (X - 0) \cap (0 + J_n)$$

is measurable and so is the "left boundary" of 0 *i.e.*, the set $\partial 0 = \cap_1^\infty 0_n$. If we define now a function $d{:}\partial 0 \to (0, \infty)$ by $d(x) = \sup \{t;\ T_t x \in 0\}$ we have that d is measurable. (Just observe that $d^{-1}(b, \infty) = \partial 0 \cap (0 + (-\infty, b))$. Therefore for any integer k, the set $A_k = d^{-1}[2^k, 2^{k+1})$ is measurable. But the ergodic interval I_k, of length 2^{k+1}, is nothing but the set $A_k + (0, d(x))$, which is measurable.

Definition. Let ψ be in $L^1(\mathbb{R})$ and f in $L^1(X)$, the convolution of f and ψ is the function.

$$f*\psi(x) = \int_{\mathbb{R}} f(T_{-s}x)\psi(s)\, ds.$$

It is clear that $f*\psi$ is in $L^1(X)$ and $\| f*\psi \|_1 \leq \| f \|_1 \| \psi \|_1$. (The context makes clear in which space we are taking norms).

It was shown by Coifman [1] that any function in $H^1(\mathbb{R})$ can be decomposed in sum of functions a_i, where each a_i is supported on an interval I_i, has average zero and $|a_i(s)| < |I_i|^{-1}$ for any s. From such a decomposition one can get the main results about $H^1(\mathbb{R})$. Our program is to show that this can also be done in the ergodic case.

Definition. An ergodic atom A is a function A, living on an ergodic interval I and such that:

a) $\| A \|_\infty \leq (\mu(I))^{-1}$

b) $\ell(I)$ is finite

c) For all x in I, $\int_{I_0^x} A(T_s x)\, ds = 0$, where I_0^x is the interval on I^x that contains the origin. Let ψ be a C^∞ function with support in $(-1, 1)$ and H a positive number. If f is in $L^1(X)$ we define

$$M(H, \psi)f(x) = \sup_{|t|<\epsilon<H} |(f*\psi_\epsilon)(T_t x)|$$

and

$$m(\psi)f(x) = \lim_{h\to\infty} M(H, \psi)f(x) \qquad (\psi_\epsilon(s) = \psi(s/\epsilon)^{-1}).$$

We also define a maximal operator independent of the particular ψ by

$$M(H)f(x) = \sup M(H, \psi)f(x) \cdot A(\psi)^{-1}$$

where the sup is over all C^∞ functions, with support in $(-1, 1)$, and $A(\psi)$ is a normalizing factor defined as:

$$A(\psi) = \| \psi \|_\infty + \| \psi' \|_\infty + \| \psi'' \|_\infty. \tag{2.1}$$

Finally, $Mf(x) = \lim_{H\to\infty} M(H)f(x)$.

Remark. The operator M allows us to estimate integrals of the type $\int f(T_s x)\psi(s)\,ds$, where ψ is a C^∞ function with support in $(a-h, a+h)$ and such that $\int\psi \sim h$, $|\psi'(s)| < h^{-1}$ and $|\psi''(s)| < h^{-2}$ for all s. In fact if we define $\phi(u) = h\psi(a-uh)$, then ϕ has support in $(-1, 1)$ and

$$(2.2)\qquad \left|\int f(T_s x)\psi(s)\,ds\right| = |f*\phi_h(T_a x)| \le Mf(T_{a+s}x)\cdot A(\phi)$$

for any s, $|s| < h$. A simple computation gives $A(\phi) \sim \int \psi$.

It is shown in [5] that in the case of $\mathbb{R}^n$, the maximal function M characterizes H^1. Therefore we define $H^1(X)$ as follows:

Definition. $H^1(X) = \{f \in L^1(X) \text{ s.t. } Mf \in L^1(X)\}$. We norm this set by $\|f\|_{H^1} = \|Mf\|_1$.

More notation. For f in $L^1(X)$ and a fixed x in X, the function of s, $f_x(s) = f(T_s x)$ is a locally integrable function in $\mathbb{R}$. In the case $X = \mathbb{R}$ and $T_s x = x + s$ we can consider all the operators defined in the general case, which we will represent by the same symbols. The context will make clear which operator we are dealing with. We will make frequent use of the following identity.

$$(2.3)\qquad (0f)(T_s x) = (0f_x)(s)$$

where 0 is any operator commuting with the action of the flow.

PROPOSITION 2.4. *Let A be an ergodic atom, then A is in H^1 and $\|A\|_{H^1} \le C$, where C is an absolute constant.*

Proof. We will use the fact that if a is a real atom (*i.e.*, supp $a \subset J$, $\int a = 0$ and $\|a\|_\infty < |J|^{-1}$, where J is an interval) then $\|Ma\|_1 \le C$. See [1].

Let now A be an ergodic atom with support in the ergodic interval I. For x fixed we can write $A_x(s) = \Sigma_i \ell(I)\mu(I)^{-1} a_{i,x}(s)$ where the $a_{i,x}$ are real atoms supported in intervals of length $\ell(I)$. We pick $L > \ell(I)$ and $n > L$ and we have:

$$\begin{aligned}\int_X M(L)A(x)\,dx &= \int_X (2N)^{-1} \int_{(-N,N)} M(L)A_x(s)\,ds\,dx\\ &\le \int_X (2N)^{-1} \int_{(-N,N)} \Sigma_i M(L)\ell(I)\mu(I)^{-1} a_{i,x}(s)\,ds\,dx.\end{aligned}$$

But the atoms $a_{i,x}$, whose support is outside $(-4N, 4N)$ do not contribute to the value of the integral and we can then assume that our sum is, in the rest of the proof, restricted to the atoms with support in $(-4N, 4N)$. Therefore our integral is, in absolute value, less than:

$$\begin{aligned}&\ell(I)\mu(I)^{-1}(2N)^{-1} \int_X \Sigma \int_{\mathbb{R}} M(L)a_{i,x}(s)\,ds\,dx\\ &\qquad \le C \int_X (2N)^{-1}\mu(I)^{-1}\Sigma\ell(I)\,dx\\ &\qquad\qquad \le C \int_X \mu(I)^{-1}(2N) \int_{(-4N,4N)} \chi_I(T_s x)\,ds\,dx \le C.\end{aligned}$$

As a corollary we get that any function of type $\Sigma c_i A_i(x)$, where the A_i's are ergodic atoms and $\Sigma |c_i|$ is finite, is in H^1 with a norm bounded by $\Sigma |c_i|$. But the important fact is that the converse is also true.

THEOREM 2.5. *If f is in $H^1(X)$ and $\int f = 0$, then f can be written as $f(x) = \Sigma c_i A_i(x)$, where the A_i's are atoms and $\Sigma c_i \le C \|f\|_{H^1}$.*

Proof. For $\lambda > 0$ we can consider the set $0_\lambda = \{x \in X; (Mf)(x) > \lambda\}$. Let λ_0 be the infimum of the λ's for which $\mu(0_\lambda) < 1$. Let (λ_k) be a doubly infinite, monotone sequence of numbers that converges to ∞ as k goes to ∞ and to λ_0 as k goes to $-\infty$. We also assume that $\lambda_{k+1} < 4\lambda_k$, all k. Let $0(k) = 0(\lambda_k)$. For x fixed we consider the set:

$$0(x, k) = \{s \in \mathbb{R}; Mf_x(s) > \lambda_k\}.$$

$0(x, k)$ is open, therefore is union of intervals, $J(i, k, x)$, of finite length (since $\mu(0(k)) < 1$ and the flow is ergodic). This means that we can proceed as in [1] to get a smooth partition of the characteristic function, $\chi_{k,x}$, of $0(x, k)$. $\chi_{k,x} = \Sigma_j \psi_{j,k,x}$ where each ψ lives on an interval, the distance from the support of ψ to the complement of $0(x, k)$ is comparable to the size of the support and

$$\|\psi\|_\infty \le c, \|\psi'\|_\infty \le C|\operatorname{supp}\psi|^{-1}, \|\psi''\|_\infty \le C|\operatorname{supp}\psi|^{-2}.$$

We write then

$$\begin{aligned} f_x(s) &= \Sigma_j(f_x - m_{j,k,x})\psi_{j,k,x}(s) \\ &\quad + f(1-\chi_{k,x})(s) + \Sigma_j m_{j,k,x}\psi_{j,k,x}(s) \\ &\equiv b_{k,x}(s) + g_{k,x}(s). \end{aligned}$$

Where

$$b_{k,x}(s) = \Sigma_j(f_\lambda - m_{j,k,x})\psi_{j,k,x}(s)$$

and

$$m_{j,k,x} = \left(\int \psi_{j,k,x}\right)^{-1} \int f_x(s)\psi_{j,k,x}(s)\,ds.$$

Observe that

$$b_{k,x}(s) = \Sigma_j, a_{j,k,x}(s)$$

with

$$a_{j,k,x}(s) = (f_x - m_{j,k,x})\psi_{j,k,x}(s)$$

and each $a_{j,k,x}$ lives in an interval and has average zero. Also, because of (2.2), we have

$$|m_{j,k,x}| \le (Mf)(x')$$

with

$$x' \text{ in } X - 0(k). \text{ Therefore: } |m_{j,k,x}| \leq \lambda_k.$$

This means that $|g_{k,x}(s)| < \lambda_k$ for all s. Now if $k \to \infty$ then $\mu(0(k)) \to 0$, therefore $b_{k,x} \to 0$ and we have

$$f_x = \lim_{k\to\infty} g_{k,x}$$

while if $k \to -\infty$ then $f_x(1 - \chi_{k,x})$ goes to zero, and since f has average zero and the flow is ergodic, all the $m_{j,k,x}$ also go to zero. We can then write:

$$f_x = \Sigma_{-\infty}^{\infty} (g_{k+1,x} - g_{k,x}) = \Sigma_{-\infty}^{\infty} (b_{k,x} - b_{k+1,x}).$$

But since the intervals corresponding to the decomposition of $0(k + 1, x)$ are subintervals of those of $0(k, x)$ we can write

$$b_{k,x} - b_{k+1,x} = \Sigma_i (a_{i,k,x} - \Sigma_j a_{j,k+1,x}),$$

where the sum in j is taken over all the $a_{j,k+1,x}$, whose support is contained in the support of $a_{i,k,x}$. Now for fixed s we have $b_{k,x}(s) - b_{k+1,x}(s) = (a_{i,k,x} - \Sigma a_{j,k+1,x})(s)$ for some particular i. This means that, since $b_{k,x}(s) - b_{m+1,x}(s) = g_{k+1,x}(s) - g_k(s)$, each of the functions $A_{i,k,x} = (a_{i,k,x} - \Sigma a_{j,k+1,x})$ has support in a real interval and $|A_{i,k,x}| \leq \lambda_{k+1} + \lambda_k \leq 5\lambda_k$. Now, according to our first lemma we know that $0(k) = \cup_i I_{i,k}$ with the $I_{i,k}$ being ergodic intervals. Observe that the orbits of each $I_{i,k}$ are made up of intervals supporting the $A_{i,k,x}$. Therefore if for each $I_{i,k}$ we define a function $A'_{i,k}(x)$ to be zero outside $I_{i,k}$, and to agree with the $A_{i,k,x}$ in the orbits, i.e., $A'_{i,k}(T_s x) = A_{i,k,x}(s)$, we have, first of all the $A_{i,k}$ are well defined since everything we have done commutes with translations, and second $|A_{i,k}| < 5\lambda_k$. Since it is clear that each $A_{i,k}$ has average zero on each of the intervals forming the orbits, and it is also obvious that $f(x) = \Sigma_k \Sigma_i A'_{i,k}$ we have:

$$f(x) = \Sigma_k \Sigma_i 5\lambda_k \mu(I_{i,k}) A_{i,k},$$

where

$$A_{i,k} = (5\lambda_k \mu(I_{i,k}))^{-1} A'_{i,k}.$$

To finish the proof we just observe that $\Sigma_k \Sigma_i 5\lambda_k \mu(I_{i,k}) = \Sigma_k 5\lambda_k \mu(0(k)) \leq C \int_X Mf(x)\, dx$.

We are now in position to characterize the dual of H^1.

Let b be an L^1 function and consider the function

$$b^{\#}(x) = \sup \mu(I)^{-1} \int_I |b(y) - m_{I(y)} b|\, dy$$

where the sup is taken over all ergodic intervals containing x, and where $m_{I(y)}$ is defined as follows: For y fixed we look at the orbit through y, $I^y = \{s \in \mathbb{R}; T_s y \in I\}$. Let $I(y)$ be the interval in I^y containing the origin. Then $m_{I(y)} b =$

$|I(y)|^{-1} \int_{I(y)} b(T_s y)\, ds$ observe that if $z = T_s y$, s in $I(y)$, then $m_{I(y)} = m_{I(z)}$, *i.e.* $m_{I(y)}$ is constant on each of the intervals forming the orbits.

Definition. b in $L^1(X)$ is said to be in B.M.O. (or to have bounded mean oscillation) if $b^{\#} \in L^{\infty}$. We norm B.M.O. by setting $\| b \|_{\text{B.M.O.}} = \| b \|_1 + \| b^{\#} \|_{\infty}$.

LEMMA 2.6. *If $b \in$ B.M.O., then $|b| \in$ B.M.O.*

Proof. Since $\| b(x) | - | m_{I(x)} b \| \leq | b(x) - m_{i(x)} b |$ we have

$$\mu(I)^{-1} \int_I \| b(y) | - | m_{I(y)} b \| \leq C. \tag{2.7}$$

Therefore it is enough to show that

$$\mu(I)^{-1} \int_I \| m_{I(y)} b | - m_{I(y)} b | \, dy \leq C$$

but this is an easy consequence of (2.7) and the fact that over the intervals forming the orbits $m_{I(y)} b$ is constant.

As a consequence of (2.6) we have that for any $N > 0$, the function $b_N(x) =$ Max $(-N, \min(b(x), N))$ is in B.M.O. if b is, and $\| b_N \|_{\text{B.M.O.}} \leq 5 \| b \|_{\text{B.M.O.}}$ Since b_N is bounded we can consider

$$\int b_N(x) A(x)\, dx$$

for any atom x. But using the cancellation properties of atoms we can write

$$\int_I b_N(x) A(x)\, dx = \int_I (b_N(x) - m_{I(x)} b_N) A(x)\, dx$$

with I being the support of A, and finally:

$$\begin{aligned} \left| \int_I b_N(x) A(x) A(x)\, dx \right| &\leq \int_I | b_N(x) - m_{I(x)} b_N | \, | A(x) | \, dx \\ &\leq \| b_N \|_{\text{B.M.O.}} \leq 5 \| b \|_{\text{B.M.O.}}. \end{aligned}$$

Using theorem (2.5) we have $| \int b_N(x) f(x)\, dx | \leq 5 \| b \|_{\text{B.M.O.}} \| f \|_{H^1}$ which means that b defines a linear functional by $< b, f > = \lim_{N \to \infty} \int b_N f$ and the functional norm is less than $5 \| b \|_{\text{B.M.O.}}$.

Conversely, it is not difficult to see [3; p. 119] that any linear functional L, on H^1 can be represented by a function b in such a way that for any bounded function with support on an ergodic interval I, we have

$$Lf = \int_I f(x) b(x)\, dx. \tag{2.8}$$

Now if A is an atom supported on I we have

$$| LA | = \left| \int_I A(x) b(x)\, dx \right| = \left| \int_I A(x) (b(x) - m_{I(x)} b)\, dx \right|.$$

If f is any bounded function supported in the ergodic interval I, then

$$A(x) = (f(x) - m_{I(x)} f) \mu(I)^{-1} \| f \|_{\infty}^{-1}$$

is an ergodic atom so we have

$$|\int_I (f(x) - m_{I(x)}f) b(x)\, dx \,|\, \mu(I)^{-1} \| f\|_\infty{}^{-1}$$
$$= |\int_I f(x) (b(x) - m_{I(x)}b \,|\, dx \,|\, \mu(I)^{-1} \| f\|_\infty{}^{-1} = |\, LA| \le \| L\|.$$

Taking the sup over all f in $L^\infty(I)$ we get

$$\int_I |\, b(x) - m_{I(x)}b \,|\, dx\, \mu(I)^{-1} \le \| L \|.$$

Since it is clear that $\int |\, b \,| \le 2 \, \| L \|$ we have that b is in B.M.O. and $\| b \|_{\text{B.M.O.}} \le 3 \, \| L \|$.

We can summarize the above discussion in the following theorem.

THEOREM 2.9. *To any continuous linear function L in* H^1 *corresponds a* B.M.O. *function b s.t.* $Lf = \lim_{N\to\infty} \int b_N f$ *for any f in* H^1 *and conversely, any* B.M.O. *function gives rise to a continuous linear functional. Furthermore the functional norm and the* B.M.O. *norm are equivalent.*

The usual definition of H^1 is in terms of the Hilbert transform. We are going to show that it is equivalent to our definition. For f in $L^1(X)$ we define the ergodic Hilbert transform of f as

$$\tilde{f}(x) = \lim_{N\to\infty} \tilde{f}_N(x),$$

where $N > 0$ and

$$\tilde{f}_N(x) = \text{P.V.} \int s^{-1}\eta(sN^{-1}) f) T_{-s}x)\, ds$$

(η is a fixed C^∞ function, bounded by 1, identically 1 in $(-3, 3)$ and zero outside $(-4, 4)$). In the case $X = \text{R}$ we would have $\tilde{g}_N(t) = \text{P.V.} \int s^{-1}(sN^{-1}) f) t - s)\, ds$. As usual $\tilde{f}_N(T_s x) = (f_x)_N(s)$. It can be seen in [4] that the ergodic Hilbert transform is well defined for all f in $L^1(X)$. If we define $\tilde{H}_1(X)$ as the space of functions in $L^1(X)$ for which $\tilde{f}$ is also in L^1, and we norm it by setting $\| f \|_* = \| f \|_1 + \| \tilde{f} \|_1$, then we have the following theorem.

THEOREM 2.10. $H_1 = \tilde{H}_1$ *as sets and* $\exists C_1, C_2$ *s.t.* $\| f \|_* \le C_1 \| f \|_{H^1} \le C_2 \| f \|_*$.

Proof. If A is an atom, then a repetition of the argument is (2.4) gives $\| A \|_* \le C$ where C is an absolute constant. By theorem (2.5) we have $\| f \|_* \le C_1 \| f \|_{H_1}$.

We claim that to proof the other part it is enough to show that for any locally integrable function g in $\mathbb{R}$, or any $N > 0$, we have

$$\int_{(-N,N)} M_N g(s)\, ds \le C \int_{(-20N,20N)} (| g(s) | + | \tilde{g}_{4N}(s) |)\, ds. \tag{2.11}$$

Indeed if (2.11) holds we can write

$$\int_X (M_N f)(x)\,dx = \int_X (2N)^{-1} \int_{(-N,N)} (M_N f_x)(s)\,ds\,dx$$
$$\leq \int_X (2N)^{-1} \int_{(-20N,20N)} (|f_x(s)|$$
$$+ |\tilde{f}_{x,4N}(s)|)\,ds \leq C \int_X (|f(x)| + |\tilde{f}_{4N}(x)|)\,dx$$

and a limiting argument finishes the proof.

Proof of (2.11). It was proved in [5] that if g is in $H^1(\mathbb{R})$ then

$$\int_{\mathbb{R}} Mg \leq \int_{\mathbb{R}} (|g| + |\tilde{g}|).$$

Assume now that g has support in $(-1, 1)$ and average zero, then

$$\int_{(-1,1)} M_1 g \leq \int_{(-1,1)} Mg \leq \int (|g| + |\tilde{g}|)$$

but

$$\int |\tilde{g}| = \int_{(-2,2)} |\tilde{g}| + \int_{|x|>2} |\tilde{g}(x)|$$

and it is an easy exercise to check that, since g has average zero, the second integral is dominated by the integral of $|g|$. For the first integral we observe that since g has support in $(-1, 1)$ then

$$\tilde{g}_1(s) = \int \frac{1}{t} \eta(t) g(s-t)\,dt = \int \frac{1}{t} g(s-t)\,dt$$

for $|s| < 2$, since then $|t| < 3$ and $\eta(t) = 1$. Therefore we have

$$\int_{(-1,1)} M_1 g \leq 2 \int_{(-2,2)} (|g| + |\tilde{g}_1|).$$

If g does not have average zero, we write $g = (g - \int g)\chi_{(-1,1)} + \int g \cdot \chi_{(-1,1)}$ and use the result above. If g is supported in $(-N, N)$, then we define $f(x) = g(xN)$, supported in $(-1, 1)$. Observing that $(M_1 f)(x) = (M_N g)(xN)$ and $(\tilde{f}_1)(x) = (\tilde{g}_N)(xN)$ and changing variables we get

$$\int_{(-N,N)} (M_N g)(x) \leq C \int_{(-2N,2N)} (|g| + |\tilde{g}_N|)(x)$$

with C independent of N. Finally if g is locally integrable we choose a C^∞ function δ, bounded by 1 and *s.t.* $\delta(s) = 0$ for $|s| > 4N$, $\delta(s) = 1$ if $|s| < 3N$ and $|\delta(x) - \delta(y)| \leq CN^{-1}|x - y|$ for x, y in $(-8N, 8N)$. We write $g = g\delta + g(1 - \delta) \equiv g^1 + g^2$ and we have that for $|s| < N$, $M_N g(s) = M_N g^1(s)$ and g^1 has support in $(-4N, 4N)$. Therefore

$$\int_{(-N,N)} M_N g = \int_{(-N,N)} M_N g^1 \leq \int_{(-4N,4N)} M_{4N} g^1 \leq \int_{(-8N,8N)} (|(g^1)\tilde{}_{4N}| + |g^1|).$$

Finally we compare $(g^1)_{\widetilde{4N}}(s)$ with $g_{4N}(s)$ for $|s| < 8N$.

$|(g^1)_{\widetilde{4N}}(s) - \tilde{g}_{4N}(s) \cdot \delta(s)| = |\int (\delta(t) - \delta(s)) \cdot (t-s)^{-1}\eta((t-s)4N^{-1}g(t)\,dt| \leq C \int_{|t|<20N} N^{-1}|g(t)|\,dt.$

Hence

$$\int_{|s|<8N} |(g^1)_{\widetilde{4N}}(s)|\,ds \leq \int ds_{|s|<8N} \int_{|t|<20N} CN^{-1}|g(t)|\,dt + \int_{|s|<8N} |\tilde{g}_{4N}(s)|\,ds \leq C \int_{|s|<20N} (|g(s)| + |\tilde{g}_{4N}(s)|)\,ds.$$

Coifman and Weiss in [2] identified the dual of H^1 as a different type of B.M.O. space. Theorem (2.10) can be used to see that it agrees with our definition of B.M.O.

Definition. A locally integrable function g, defined on $\mathbb{R}$, is said to be in B.M.O. if

$$g^{\#}(s) = \sup |I|^{-1} \int_I |g(t) - m_I g|\,dt$$

is in L^∞ (The sup is over all intervals containing s, and $m_I g = |I|^{-1} \int_I g(t)\,dt$). If f *is in* $L^1(X)$ we will say that f is in B.M.O.O. (B.M.O. in orbits) iff $f_x(s)$ is in real B.M.O. and $\| f_x^{\#} \|_\infty$ is bounded by a constant independent of x. If we norm B.M.O.O. by using this constant it is clear that B.M.O.O. is contained in B.M.O. and the inclusion is continuous. To prove the converse we make the following remarks. First, because of theorem (2.10) one can identify the dual of H^1 with the space of functions of the type $u + \tilde{v}$ with u, v in $L^\infty(X)$. Second, it is proved in [5] that in the real case such functions are always in B.M.O., so by looking at the orbits we have the same result in the ergodic case (f in $L^\infty(X)$ iff $f_x(s)$ is in $L^\infty(\mathbb{R})$ uniformly!). Third, we know that $(H^1)^* =$ B.M.O.

3. The n-dimensional case

If we have $\mathbb{R}^n$ acting as an ergodic flow in X, we can define a maximal operator as in §2, but now we take convolutions with functions with support in the unit ball $B(0, 1)$. (Notation: $B(0, N) = = \{t \in R^n;\ |t| < N\}$. The normalization factor $A(\phi)$ is now $\Sigma_\alpha \| D^\alpha \phi \|_\infty$ with $\alpha = (\alpha_1, \cdots, \alpha_n)$ being multi indexes and $|\alpha| < n + 1$. The role of the Hilbert transform is played by the Riesz transforms. For technical reasons we will work with maximal Riesz transforms. For $H > 0$, $i = 1, 2, \cdots n$ we define

$$R^*_i f(x) = \lim_{H\to\infty} R^*_{i,H} f(x)$$

where

$$R^*_{i,H} f(x) \equiv \sup_{0<\delta<\epsilon<H} |(f * K_{j,\epsilon,\delta})(x)|$$

$$K_{j,\epsilon,\delta} \equiv \delta_j |s|^{-n-1}(\eta(s\epsilon^{-1}) - \eta(y\delta^{-1}))$$

and η is a fixed C^∞

function identically 1 in $B(0, 3)$, supported in $B(0, 4)$ and bounded by 1.

We can define $H^1(X) = \{f \in L^1(X); Mf \in L^1(X)\}$ with norm $\|f\|_{H^1} = \|Mf\|_1$, and we can also define $\bar{H}^1(X) = \{f \in L^1(X); R^*{}_if \equiv L^1(X)\}$ with norm $\|f\|_* = \|f\|_1 + \Sigma_1{}^n \|R^*{}_if\|_1$. One can get $\|f\|_{H^1} < C\|f\|^*$ in exactly the same way as the one dimensional case. Unfortunately, theorem (2.5) uses our lemma on decomposition of a set into ergodic intervals, and the proof of that lemma seems to be one dimensional. Still the main results of §2 remain valid.

THEOREM 3.1. *The spaces $H^1(X)$ and $\bar{H}^1(X)$ are equal and have equivalent norms.*

THEOREM 3.2. *The dual of $H^1(X)$ can be identified with the space* B.M.O.O. *i.e., (functions that are in* B.M.O. ($\mathbb{R}^n$ on each orbit with uniform norm). Remember that g $\in$ B.M.O. (R^n) *means* $g^\# \in L^\infty(\mathbb{R}^n)$ *where*

$$g^\#(s) = \sup |Q|^{-1} \textstyle\int_Q |g - m_Q g|$$

with the sup *taken over cubes containing s and* $m_Q g = |Q|^{-1} \int_Q g$.

For the proof of theorem (3.1) we just point out that the same localization techniques used in (2.11) work also the other way around to give that, for any locally integrable function f in $\mathbb{R}^n$ we have

$$\textstyle\int_{B(0,N)} R^*{}_{i,N} f(x)\, dx \le C \int_{B(0,16N)} (M_{8N} f)(x)\, dx$$

which implies $\|f\|_* \le C\|f\|_{H^1}$. Full details can be seen in [8].

Theorem 3.1 also tells us that $H^1(X)$ can be thought as $\{f \in L^1(X); R_if \in L^1(X), i = 1, \cdots, n\}$ where $R_if(x) = \lim\limits_{H\to\infty} R_{i,H}f(x)$ *and* $R_{i,H}f(x) = \lim\limits_{\delta\to 0} f * k_{i,H,\delta}f(x)$. It is then easy to see [5; p 145] that any linear functional in H^1 can be represented by a function of the form $g_0 + \Sigma_1{}^n R_i g_i$ with $g_0, g_1, \cdots g_n$ in $L^\infty(X)$, and such a function is in B.M.O.O. So to prove theorem (3.2) we need to show that any function in B.M.O.O. gives rise to a continuous linear functional in $H^1(X)$.

Let ϕ be a fixed $C^\infty(\mathbb{R}^n)$ radial function supported in $B(0, 1)$. For f, locally integrable, we define the *area function* as

$$Sf(s) = \lim_{H\to\infty} S_H f(s)$$

where

$$(S_H f(s))^2 = \textstyle\iint_{|v-s|<t<H} |\nabla (f*\phi_t)(v)|^2 t^{1-n}\, dv\, dt.$$

$$\left(|\nabla (f*\phi_t)(v)|^2 = \left|\frac{\partial f*\phi_t}{\partial t}\right|^2 + \Sigma_1{}^n \left|\frac{\partial f*\phi_t}{\partial v_i}\right|^2 \right).$$

For f in $L^1(X)$ we define $(S_H f)(x)$ as $(S_H f_x)(0)$, and $(Sf)(x) = \lim\limits_{H\to\infty} (S_H f)(x)$.

It is known [1] that every function in $H^1(\mathbb{R}^n)$ can be written in the form $\Sigma\lambda_i a_i(x)$, where $\Sigma\lambda_i$ can be taken as the H^1 norm of the function and the a_i's are atoms; *i.e.* a_i lives on a cube Q_i, has average zero and $|a_i| < |Q_i|^{-1}$.

LEMMA 3.3. *$f \in H^1(\mathbb{R}^n)$ implies* $\| Sf \| \leq C \| f\|_{H^1}$.

Proof. It is enough to show that if a is an atom supported in Q, centered at the origin we have $\| Sa \|_1 \leq C$.

If $|v| > Q$, the fact that a has average 0 gives easily that $((Sa)(v))^2 \leq |Q|^2 |v|^{-2n-2}$, therefore

$$\int_{|v|>4|Q|} (Sa)(v)\, dv \leq \int_{|v|>4|Q|} (|Q|^2 |v|^{-2n-2})^{1/2}\, dv \leq C.$$

On the other hand S is easily seen to be bounded in L^2, which means that

$$\int_{|v|<|Q|4} (Sa)(v) \leq |Q|^{1/2} \int |Sa|^2 \leq |Q|^{1/2} (\int |a|^2)^{1/2} \leq C.$$

LEMMA 3.4. *If f is in $H^1(X)$ then* $\| Sf \|_1 \leq C \| f \|_{H^1(X)}$.

Proof. Repeat the argument in (2.11).

The area function is also related to B.M.O.

LEMMA 3.5. *Let f be in* B.M.O. $(\mathbb{R}^n)$, *then* $\int_{|v-s|<H} (S_H f)(v)\, dv \leq C_0 H^n$ *where C_0 is independent of s and H.*

Proof. Assume $s = 0$. Since the gradient of a constant is 0 we can substitute f by $f - m_H f$, with $m_H f = |B(0, 4H)|^{-1} \int_{B(0,4H)} f$. Also since $|v| < H$ we can substitute $f - m_H f$ by $(f - m_H f)\chi_{B(0,4H)}$. Now using that S is bounded in L^2 we have

$$\int_{|v|<H} (S_H f)(v)\, dv \leq CH^n |B(0, 4H)|^{-1/2} (\int_{B(0,4H)} (|f - m_H f|^2)^{1/2} < CH^n.$$

This last inequality is true because f is in B.M.O. and the theorem of John and Niremberg [6].

Let $\mathscr{S}$ be the class of testing functions and $\mathscr{S}_0$ those functions in $\mathscr{S}$ whose Fourier transform vanishes in an interval around the origin. Let $\mathscr{S}_0$ be the class of functions of the form

$$F(x) = \int_{\mathbb{R}^n} f(T_{-s}x)\phi(s)\, ds$$

with $f \in H^1(X)$, $\phi \in \mathscr{S}_0$.

LEMMA 3.6. *The class S_0 is dense in* $H_0^1(X) = \{f \in H^1(X); \int f = 0\}$.

Proof. Pick ψ in $\mathscr{S}_0$ s.t. $\hat{\psi}$ has support in $B(0, 2)$ and is 1 in $B(0, 1)$. For $k = 1, 2, \cdots$ we consider the function $\phi_k = \psi_{1/k}(s) - \psi_k(s)$. If $h \in H_0^1(X)$ we consider the sequence $h * \phi_k = F_k$. We claim F_k converges to h in H^1. First of all it is clear that for $i = 1, \cdots, n$, $R_i F_k = R_i h * \phi_k$, which means that it is enough to show F_k converges to h in $L^1(X)$. Since the operators $h \to h * \phi_k$ are uniformly bounded in $L^1(X)$ it is enough to consider a dense class. Now the class S of functions of the type $\int_{\mathbb{R}^n} g(T_{-s}X)\eta(s)\, ds$ with $g \in L^1(X)$, $\eta \in L_0$ is dense in $L_0^1(X)$. This is a consequence of the ergodicity of the flow plus the fact that $\mathscr{S}_0$ is dense in $L_0^1(\mathbb{R}^n)$ [7; p 230]. Let then h be in S, *i.e.*

$$h(x) = f * \eta(S) \quad \text{with} \quad \eta \in L_0,$$

then $F_k(x) = h*\phi_k(x) = f*(\phi_k*\eta)(x)$ but since $\hat{\eta}$ has support away from origin and $\hat{\phi}_k$ is 1 on $1/k < |s| < k$, we have $\phi_k*\eta = \mu$for k big enough.

LEMMA 3.7. *If ϕ is as in our definition of the area function and F_1, F_2 are in S_0 then:*

$$\int_X F_1(x)F_2(x)\,dx = C\int_X \int_0^\infty t(\nabla F_1(x,t))\cdot(\nabla F_2(x,t))\,dx\,dt$$

where

$$\nabla F_i(x,t) = F_i*\nabla(\phi_t)(x) \qquad i = 1, 2.$$

Proof. Let $F_i = f_i*\eta_i$, $\eta_i \in L_0$. Now

$$\int_{R^n} \eta_1(s)\eta_2(s)\,ds = C\int_{R^n}\int_0^\infty t(\nabla\eta_1(s,t))\cdot(\nabla\eta_2(s,t))\,ds\,dt \quad \text{with} \quad \nabla\eta_i = \eta_i*\nabla(\phi_t).$$

To see this we just use Plancherel's theorem plus the fact that $\hat{\phi}$ is radial, and the right hand side becomes

$$\int_{R^n} \hat{\eta}_1(s)\hat{\eta}_2(s)\int_0^\infty |ts|^2(\hat{\phi}(t|s|))^2\frac{dt}{t},$$

and the integral in t is independent of 2. The lemma follows from Fubini's theorem.

LEMMA 3.8. *If b is in* B.M.O.O. *and for $N > 0$, $b_N = \max(-N, \min(b(x), N))$ then for any h in S_0 we have*

$$\int_X h(x)b_N(x)\,dx \le C\|h\|_{H^1}$$

and C is independent of N and h.

Proof. Lemmas (3.4) to (3.7) insure us that the proof in [5] works in this case. We include it for completeness. To simplify the notation we will drop our N.

We observe first of all that a density argument allows us to extend the identity in (3.7) to our case. Also from lemma (3.5) we get:

(3.8) If $H_x(w) = \sup\{H > 0;\ (S_H f_x(s))^2 \le 100\,C_0\}$

then $|\{s;\ |s| < H;\ H_x(s) > H\}| \ge CH^n$.

If (3.8) were not true then, for some H, $|\{s,\ |s| < H;\ H_x(s) \le H\}|$ would be almost equal to $|B(0,H)|$ which contradicts (3.5).

If H is a positive number we have

$$\left|\int_X h(x)b(x)\,dx\right| \le C\int_X\int_0^\infty t|\nabla h(x,t)||\nabla b(x,t)|\,dx\,dt$$

$$= C\int_X H^{-n}\int_{|v|<H}\int_0^\infty t|\nabla(h_x*\phi_t)(v)||\nabla(b_x*\phi_t)(v)|\,dt\,dv\,dx.$$

Using (3.8) and Fubini's theorem we have:

$$\leq C \int_X H^{-n} \int_{|w|<H} \int_0^{H_x(w)} \int_{|v-w|<t} t^{1-n} |\nabla(f_x * \phi_t)(v)| \nabla(b_x * \phi_t)(v)| \, dv\, dt\, dw\, dx$$

$$\leq C \int_X H^{-N} \int_{|w|<H} (Sf)(T_w x) \cdot (S_{Hx}(w) b)(T_w x) \, dw\, dx$$

$$< C.10 C_0 \int_X H^{-n} \int_{|w|<H} (Sf)(T_w x) \, dw\, dx$$

$$= 10\, C\, C_0 \| Sf \|_1 < C \| f \|_{H^1(X)}$$

Since the class S_0 is dense in H_0^1, we can extend the linear functional given by b, continuously to H_0^1 and we get theorem (3.2).

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