A NOTE ON THE PRODUCT OF OPERATOR-VALUED MEASURES

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Abstract

An example is given to show that the product of two, non-commuting, regular, countably additive, nonnegative operator-valued measures may not be countably additive.

Given a Hilbert space $\mathscr{H}$ and two nonnegative operator-valued measure spaces $(S, \mathscr{S}, \mu)$ and $(T, \mathscr{T}, \nu)$ where $(S, \mathscr{S})$ and $(T, \mathscr{T})$ are measure spaces and μ and ν are strongly countably additive measures whose values are nonnegative operators from $\mathscr{H}$ into $\mathscr{H}$. For any rectangle $A \times B$, $A \in \mathscr{S}$, $B \in \mathscr{T}$, we can form $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$. S. K. Berberian [2] proves von Neumann's theorem that if μ and ν commute and if μ and ν satisfies some topological regularity conditions, then $\mu \times \nu$ is strongly countably additive on the rectangles. R. Dudley [3] showed that $\mu \times \nu$ need not be countably additive if one drops the regularity conditions. Berberian [2, p. 114] raises the question of amalgamating two non-commuting nonnegative operator valued measures. After a brief discussion he concludes that a nonnegative amalgamation for non-commuting spectral measures may not exist. The purpose of this note is to show, by means of an example, that in the absence of commutativity one cannot expect to obtain even an amalgam, not necessarily nonnegative, for two nonnegative operator valued measures. This question often arises in other fields of analysis such as dilation theory and the study of stochastic processes [1] and [5].

Proposition. *There exists a strongly countably additive nonnegative operator-valued measure μ on the Borel subsets $\mathscr{B}$ of* $[0, 2\pi]$ *which is regular (for the definition of regularity of such measures see* [2]*), but $\mu \times \mu$ is not strongly countably additive on the algebra of all rectangles.*

Proof: Let $\mathscr{H}$ be a separable Hilbert space with an orthogonal basis e_k, $-\infty < k < \infty$. Let U denote the shift operator on $\mathscr{H}$ taking e_k to e_{k+1}, for every k, and E be its spectral measure (E is obviously regular). Let P denote the orthogonal projection onto the subspace spanned by e_k, $k \geq 0$. Now let $\mu(A) = PE(A)P$. Then we assert $(\mu \times \mu)(A \times B) = \mu(A)\mu(B)$ is not strongly countably additive on the algebra of rectangles. Because otherwise the set function $F(A, B) = (PE(A)Pe_0, PE(B)Pe_0)$ must be countably additive and hence extends to a measure $F(\cdot, \cdot)$ on the Borel subsets of $[0, 2\pi] \times [0, 2\pi]$.

Letting $x_n = Pe_n$ we obtain

$$(x_n, x_m) = \int_0^{2\pi} \int_0^{2\pi} e^{i(mx-ny)} dF(x, y).$$

But this is impossible as one can see in Abreu's paper [1] where he discusses some interesting problems on harmonizable sequences. Q.E.D.

Remark. In dealing with prediction theory of stationary processes one sometimes faces the question of whether the inner product of two Hilbert space-valued measures is a measure [5]. A negative answer to this question was provided in [4] by R. Dudley and L. Pakula in response to a question raised by P. Masani. The inner product of the Hilbert space valued measure $\nu(A) = PE(A)Pe_0$ with itself is another example showing that the inner product of two Hilbert space-valued measures need not extend to a measure on the Borel sets of the plane.

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