FUNCTIONAL DIFFERENTIAL EQUATIONS OF ADVANCED TYPE

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0. Introduction

The aim of the present paper is to study difference differential equations of advanced type

$$\dot{x}(t) = f(t, x(t), x(t + T)), \tag{1}$$

where T is a fixed positive number, $0 \leq t < \infty$, x is a column vector in $\mathbb{R}^n$, and $f: \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ ($\mathbb{R}^+ = \{s \in \mathbb{R} \mid s \geq 0\}$), we focus our attention at the initial value problem and related existence theorems, considering these problems in a new setting.

For the meaning of (1) we call attention of the reader to G. Plackzek [1], L. Fox, D. F. Mayers, J. R. Ockendon and H. B. Tayler [2], J. Tinberger [3] and N. Georgescu-Roegen [4]; where such equations arise from actual models.

A very interesting model is that of S. Sargan [5]: "In this model the $x_i(t)$ are taken to be the outputs of a set of commodities, or it may be said that the actual supply of commodity i at time t is $x_i(t)$. This is taken to generate a demand for commodity j as a current input which is equal to $a_{ji}x_i(t)$. Moreover, it is assumed that the rate of increase in the stock of commodity j, which is associated with the increase in the level of the activity producing commodity i, would be $b_{ji}\dfrac{dx_i}{dt}$. Thus, the total demand for the commodity j can be written:

$$\textstyle\sum_{i=1}^{m} a_{ji}x_i(t) + \sum_{i=1}^{m} b_{ji}\dfrac{dx_i}{dt} + y_j(t); \qquad a_{ji},\ b_{ji} \geq 0,$$

where $y_j(t)$ is the exogenous demand for the j^{th} commodity. If it is assumed that supply follows demand with a constant time lag, then the equations take the form

$$x_j(t + \epsilon\,\mu_j) = \textstyle\sum_{i=1}^{m} a_{ji} \times x_i(t) + \sum_{i=1}^{m} b_{ji}\dfrac{dx_i}{dt} + y_j(t)\text{''}.$$

I. The initial value problem

The authors of recent papers, like S. Doss and S. Nasr [6], S. Sugiyama [7], T. Dlotko and M. Kuzma [8], C. Anderson [9], A. Sobolewska [10], T. Kato and J. B. McLeod [11], and P. O. Frederickson [12] think on initial

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conditions in the same way we do in ordinary differential equations: as a fixed vector in $\mathbb{R}^n$. So they make the following definition.

Definition 1. A solution of (1) is a continuous function on $[0, \infty)$ into $\mathbb{R}^n$, with a piece-wise continuous derivative that satisfies (1) and such that

$$x(0) = x_0. \tag{2}$$

This definition has the inconvenience that even with conditions like lipschitz and continuity on t, the problem (1) – (2) is not well posed. So we must examine with care the implications of initial conditions like (2).

If we have

$$\dot{x}(t) = x(t+1); \qquad x(0) = x_0 \tag{3}$$

and

$$\dot{y}(t) = y(t); \qquad y(0) = x_0, \tag{4}$$

we know that for the ordinary equation (4), it is sufficient to know the value of the solution at $t = 0$, to assure that this solution can be defined on the interval $[0, \epsilon)$, for some $\epsilon > 0$. It is to be noted that when we specify the value of $y(t)$ at $t = 0$, we also know the value of the vector field $\dot{y}(t)$ at zero, that is, with the above information we know how to start the solution. But if we look at the advanced equation (3), we observe that knowing the value of x at zero, it is not enough to know the value of the vector field at that point, *i.e.*, we don't have the starting direction. If we wish to know that direction, a possibility, maybe the natural one, is to know the value of x at $t = 1$. Let us suppose that with this additional information we are able to define x at $[0, \epsilon]$, for some $\epsilon > 0$, then, in contrast to the ordinary case, in which we can continue the solution after ϵ, in this case we would be as at the beginning, *i.e.*, we know the value of x at ϵ but not that of the vector field, thus we need to know x at $1 + \epsilon$. We have two possibilities to consider, the first is to specify the value of x at the interval $[1, 2)$, so that we can determine its value at $[0, 1)$; the other one is to try to determine x on $[0, \infty)$ with the aid of the following fact: to define x in a neighborhood of zero we need to know x in a neighborhood of 1, and for this we need to know the value of x in a neighborhood of 2, and so on. From this, we must know at least the values of x at $0, 1, 2, 3, \cdots$. This latter consideration is our choice.

Definition 2. An initial condition for (1) is a sequence of vectors $\{x_k\}_{k=0}^{\infty}$. A solution for (1) with this initial condition is a function defined in the interval $[0, \infty)$ with the values in $\mathbb{R}^n$, piece-wise continuous with a piece-wise continuous derivative, that satisfies (1) and

$$x(kT) = x_k \quad \text{for} \quad k = 0, 1, 2, \cdots \tag{5}$$

(See figure 1).

The problem (1) – (5) is well posed, and a way to see this is to show that (1) – (5) is equivalent to an ordinary differential equation in a certain Banach space.

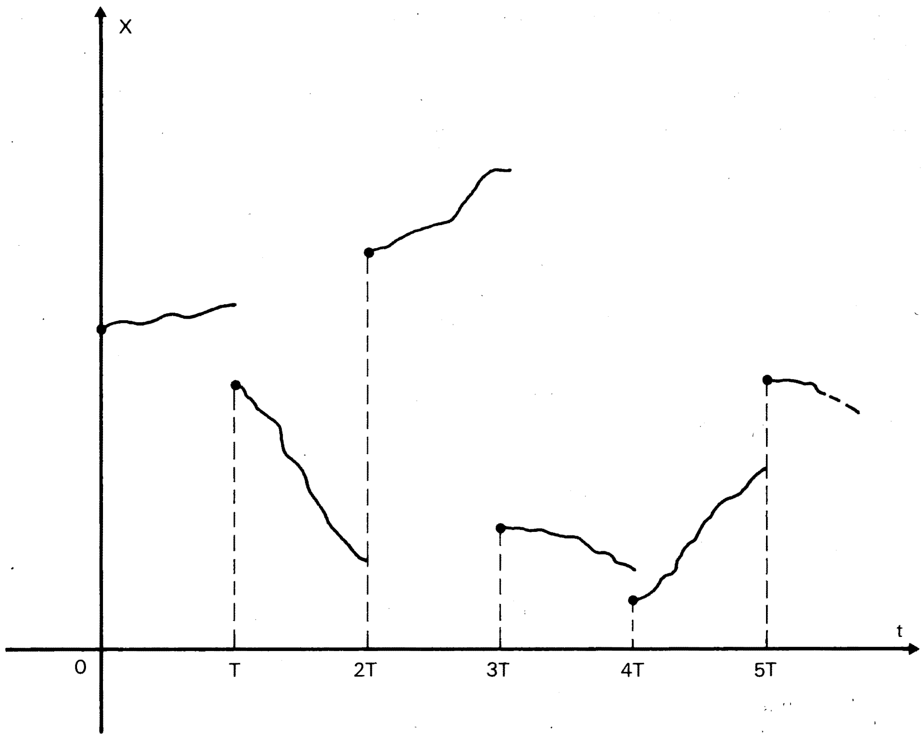

Fig. 1

2. The equivalence

Let S be a complete metric linear space formed with sequences of $\mathbb{R}^n$, and G a function from $[0, T) \times S$ to S defined by

$$G(t, \{y_k\}_{k=0}^{\infty}) = \{f(t + kt, y_k, y_{k+1})\}_{k=0}^{\infty}. \tag{6}$$

then the problem (1) – (5) and the problem

$$\dot{y}(t) = G(t, y(t)), \qquad t \in [0, T) \tag{7}$$

with

$$y(0) = \{x_k\}_{k=0}^{\infty} \tag{8}$$

are equivalent in the sense that if $x(t)$ is a solution of (1) – (5) then

$$y(t) = \{x(t + kT)\}_{k=0}^{\infty}, \qquad t \in [0, T) \tag{9}$$

is a solution of (7) – (8), and if $y(t)$ is a solution of (7) – (8) then

$$x(t) = y_k(t - kT) \quad \text{if} \quad t \in [kT, (k+1)T), \qquad k = 0, 1, 2, \cdots. \tag{10}$$

where $y(s) = \{y_k(s)\}_{k=0}^{\infty}$, is a solution of (1) – (5).

The utility of this fact lies in the possibility of applying the well known facts for ordinary differential equations in a Banach space to differential equations of advanced type.

THEOREM 1: *Let* $f: \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ *be uniformly continuous, and suppose there exist positive numbers* $\lambda_1, \lambda_2, \lambda_3$, *and an* $n-$ *vector* z, *such that*

i) $|f(t, x, y) - f(t, \tilde{x}, \tilde{y})| \leq \lambda_1 |x - \tilde{x}| + \lambda_2 |y - \tilde{y}|$

for $t \in \mathbb{R}^+; x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^n$

ii) $|f(t, z, z)| \leq k$ for $t \in \mathbb{R}^+$.

Then, given any bounded sequence, there exists a unique solution of (1) – (5).

Proof. Let $S = \ell_\infty$, the *B-Space* of bounded sequences with sup norm and G defined by (6). Then, for any

$$\{x_k\}_{k=0}^{\infty}, \{y_k\}_{k=0}^{\infty} \in \ell_\infty.$$

$$\begin{aligned}
&\| G(t, \{x_k\}_{k=0}^{\infty}) - G(t, \{y_k\}_{k=0}^{\infty})\| \\
&\qquad = \|\{f(t + kT, x_k, x_{k+1})\}_{k=0}^{\infty} - \{f(t + kT, y_k, y_{k+1})\}_{k+1}^{\infty}\| \\
&\qquad = \sup_{k=0,1,2,\ldots} \| f(t + kT, x_k, x_{k+1}) - f(t + kT, y_k, y_{k+1})\| \\
&\qquad \leq (\lambda_1 + \lambda_2)\|\{x_k\}_{k=0}^{\infty} - \{y_k\}_{k=0}^{\infty}\|.
\end{aligned}$$

So G is lipschitz. Moreover $G([0, T) \times \ell_\infty) \subset \ell_\infty$, since

$$\begin{aligned}
\| G(t, \{x_k\}_{k=0}^{\infty})\| &\leq \| G(t, \{x_k\}_{k=0}^{\infty}) - G(t, \{z\}_{k=0}^{\infty}\| + \| G(t, \{z\}_{k=0}^{\infty}\| \\
&\leq (\lambda_1 + \lambda_2) \text{ dist } (z, \{x_k\}_{k=0}^{\infty}) + k = \tilde{k}.
\end{aligned}$$

Analogously, it can be seen that G is a continuous function and satisfies, for a fixed sequence, $\{x_k\}_{k=0}^{\infty} \in \ell_\infty$ the relation:

$$\| G(t, \{y_k\}_{k=0}^{\infty})\| \leq (\lambda_1 + \lambda_2)\|\{y_k\}_{k=0}^{\infty} - \{x_k\}_{k=0}^{\infty}\| + \tilde{k}.$$

All conditions for existence and uniqueness in the interval $[0, T)$ are satisfied (see J. Dieudonné [13]). This completes the proof of the theorem.

Observation: If f satisfies the conditions of Theorem 1, then, given

$$\{x_k\}_{k=0}^{\infty}, \{y_k\}_{k=0}^{\infty} \in \ell_\infty,$$

if $x(t)$, $y(t)$, are the solutions of (1) with these initial conditions, we have:

$$|x(t) - y(t)| \leq \|\{x_k\}_{k=0}^{\infty} - \{y_k\}_{k=0}^{\infty}\| e^{(\lambda_1+\lambda_2)T}. \tag{11}$$

3. Continuous solutions

An interesting case, specially in actual applications, is that of continuous solutions of (1); that is, to find, among the solutions according to definition 2 those that satisfy definition 1. If we put this condition into problem (7) we have:

LEMMA 1: *For a solution $x(t)$ of* (1) – (5) *to be a continuous solution it is necessary and sufficient that the solution $X(t)$ of* (7) – (8) *associated to $x(t)$, satisfies* $\lim_{t\to T} X(t)$ *exists, and the boundary condition*

$$\Pi X(0) = X(T) \tag{12}$$

where Π is the shift operator, Π: $S \to S$ *such that*

$$\Pi \{y_k\}_{k=0}^{\infty} = \{y_{k+1}\}_{k=0}^{\infty}. \tag{13}$$

LEMMA 2: *The function* X: $[0, T] \to S$ *satisfies* (7)-(12) *if and only if*

$$X(t) = \{x_0\}_{k=0}^{\infty} + \sum_{i=1}^{\infty} \Delta^i \int_0^T G(s, X(s))\, ds + \int_0^t G(s, X(s))\, ds, \qquad t \in [0, T] \tag{14}$$

for some $x_0 \in \mathbb{R}^n$, where Δ^k, is the k iteration of Δ, Δ given by

$$\Delta\{y_k\}_{k=0}^{\infty} = \int \{0, y_0, y_1, y_2, \cdots\} \tag{15}$$

Proof: It is simple to see that (14) is equivalent to

$$X(t) = \{x_0 + \int_0^{kt+t} f(s, x(s), x(s + T))\, ds\}_{k=0}^{\infty} \tag{16}$$

where $x(t)$ is the associated function to $X(t)$ by (10), and from this the lemma follows.

With lemmas 1, 2, we can prove:

THEOREM 2: *Given* f: $\mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ *and* w: $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, *continuous functions such that*
i) *w is non decreasing in the second and third variables, that is, if $s_1 \le s_2$, $u_1 \le u_2$ then*

$$w(t, s_1, u_1) \le w(t, s_2, u_2), \text{ for fixed } t. \tag{17}$$

ii) *Given $t \in R^+$ and n-vectors, x_1, x_2, y_1, y_2, f satisfies*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le w(t, |x_1 - y_1|, |x_2 - y_2|) \tag{18}$$

iii) *Given a non negative number r, the integral*

$$\Omega(r) = \int_0^{\infty} w(t, r, r)\, dt$$

is convergent, $\Omega(0) = 0$, and the function Ω so defined is continuous.
iv) *if we define*

$$l = \sup_{r \in \mathbb{R}^+} (r - \Omega(r)),$$

l is either a positive number or infinite.

Then there exists a continuous and bounded solution $x(t)$ with $x(0) = x_0$, if

x_0 satisfies

$$\beta = \int_0^\infty |f(s, x_0, x_0)|\, ds < l. \tag{19}$$

Proof: *If x_0 satisfies* (19) *there exists a positive real number ξ such that $\beta \leq \xi - \Omega(\xi)$*. Define the subset A of $C([0,T]; l_\infty)$, the continuous functions $[0,T] \to l_\infty$ as those functions that satisfy $\| x(t) - \{x_0\}_{k=0}^\infty \| \leq \xi$ for all t in $[0,T]$. Then A is convex, closed and bounded.

For any $X \in A$ define the function $H[X]$ by the relation

$$H[X](t) = \{x_0\}_{k=0}^\infty + \sum_{i=1}^\infty \Delta^i \int_0^T G(s, X(s))\, ds + \int_0^t G(s, X(s))\, ds. \tag{20}$$

From lemma 2, to find fixed points of H is equivalent to solve (7)–(12). We now apply the Schauder fixed point theorem to assert the existence of a fixed point of H in A. Using (16) it follows:

$$\begin{aligned} \| H[X](t) - \{x_0\}_{k=0}^\infty \| &\leq \int_0^\infty |f(s, x(s), x(s+T))|\, ds \\ &\leq \int_0^\infty w(s, |x(s) - x_0|, |x(s+T) - x_0|)\, ds \\ &+ \int_0^\infty |f(s, x_0, x_0)|\, ds \leq \Omega(\xi) + \beta \leq \xi, \end{aligned}$$

where $x(t)$ is the associated function to $X(t)$. Obviously $H[X]$ is a continuous function for $t \in [0,T]$, thus, H: $A \to A$.

Also, if $t_1 \geq t_2$ we have

$$\begin{aligned} \|H[X](t_1) - H[X](t_2)\| &\leq \sup_{k=0,1,2,\ldots} \int_{kT+t_1}^{kT+t_2} |f(s, x(s), x(s+T))|\, ds \\ &\leq \sup_{k=0,1,2,\ldots} \int_{kT+t_1}^{kT+t_2} w(s, \xi, \xi)\, ds \\ &+ \sup_{k=0,1,2,\ldots} \int_{kT+t_1}^{kT+t_2} |f(s, x_0, x_0)|\, ds. \end{aligned}$$

This does not depend on X and it follows from the continuity and integrability of $w(s, \xi, \xi)$, $f(s, x_0, x_0)$, that the set $H[A]$ is a family of equicontinuous functions, and since the sequence $H[X](t)$ (for fixed t) is uniformly convergent to $x_0 + \int_0^\infty f(s, x(s), x(s+T))\, ds$ (where $x(t)$ is the one associated to X) for $X \in A$; that is, $\{H[X](t) | X \in A\}$ for each t, is a conditionally compact set. Ascoli's theorem (J. Dieudonné [13]) asserts that H is a compact operator.

It follows from the continuity of Ω and from $\Omega(0) = 0$, that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\Omega(\delta) < \epsilon$. For any $X_1, X_2 \in A$ such that $\| X_1 - X_2 \|$

$< \delta$ we have:

$$\begin{aligned}\|H[X_1](t) - H[X_2](t)\| \\ &\leq \int_0^\infty |f(s, x_1(s), x_1(s+T)) - f(s, x_2(s), x_2(s+T)|\, ds \\ &\leq \int_0^\infty w(s, |x_1(s) - x_2(s)|, |x_1(s+T) - x_2(s+T)|)\, ds \\ &\leq \int_0^\infty w(s, \delta, \delta)\, ds = \Omega(\delta).\end{aligned}$$

This is precisely the statement that H is a continuous mapping. The conditions of the Schauder theorem are satisfied and the proof of the theorem is complete.

THEOREM 3. *Let the functions f, w, satisfy the conditions of the above theorem, moreover, suppose that l is infinite and*
v) *There is a n-vector z such that*

$$\int_0^\infty |f(s, z, z)|\, ds < \infty,$$

vi) $$r - \Omega(r) > 0 \quad \text{for all} \quad r > 0.$$

Then, given any n-vector x_0, there exists a unique continuous and bounded solution, for which $\lim_{t\to\infty} x(t)$ exists, and it depends together with the whole solution continuously on x_0.

Proof: Suppose that $x_1(t)$, $x_2(t)$ are two continuous and bounded solutions of (1), with the same initial condition $x_1(0) = x_2(0) = x_0$, and define θ by

$$\theta = \sup_{0\leq t<\infty} |x_1(t) - x_2(t)|$$

then

$$\begin{aligned}|x_1(t) - x_2(t)| &= \left|\int_0^t [f(s, x_1(s), x_1(s+T)) - f(s, x_2(s), x_2(s+T))]\, ds\right| \\ &\leq \int_0^t w(s, |x_1(s) - x_2(s)|, |x_1(s+T) - x_2(s+T)|)\, ds \\ &\leq \int_0^t w(s, \theta, \theta)\, ds = \Omega(\theta).\end{aligned}$$

From the above inequality we get $\theta \leq \Omega(\theta)$, so we must have $\theta = 0$.

We want to show continuous dependence on initial conditions. For this, let us suppose that $x_1(t)$, $x_2(t)$ are bounded and continuous with $x_1(0) = x_1$, $x_2(0) = x_2$. If we define

$$\mu = \sup_{0\leq t<\infty} |x_1(t) - x_2(t)| \quad \text{we get:}$$

$$\begin{aligned}|x_1(t) - x_2(t)| &\leq |x_1 - x_2| + \int_0^\infty |f(s, x_1(s), x_1(s+T)) \\ &\qquad - f(s, x_2(s), x_2(s+T)|\, ds \\ &\leq |x_1 - x_2| + \int_0^\infty w(s, |x_1(s) - x_2(s)|, \\ &\qquad |x_1(s+T) - x_2(s+T)|)\, ds \\ &\leq |x_1 - x_2| + \Omega(\mu),\end{aligned}$$

so we must have $\mu - \Omega(\mu) \leq |x_1 - x_2|$, consequently, if $|x_1 - x_2| \to 0$, $\mu - \Omega(\mu) \to 0$, but this only happens if $\mu \to 0$, and this proves the theorem.

Example: Consider the equation

$$\dot{x}(t) = f(t, x(t+1)) = z\, e^{-t-|x(t+1)|},\ z \in \mathbb{R}^n$$

which satisfies the conditions of theorem 2, with $w(t, a) = |z| e^{-t}(1 - e^{-a})$. Since

$$|f(t, y) - f(t, \bar{y})| \leq |z| e^{-t}(1 - e^{-|y-\bar{y}|}),$$

we also note that if $|z| \leq 1$ the solution is unique.

We can also show a theorem with Carathéodory type conditions.

THEOREM 4. *Let* $f: \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ *be a continuous function, and* $m: \mathbb{R}^+ \to \mathbb{R}^+$ *an integrable one, such that*

$$|f(s, x, y)| \leq m(s) \quad \text{for} \quad x, y \in \mathbb{R}^n,\ s \in \mathbb{R}^+$$

then, given any $x_0 \in \mathbb{R}^n$, *there exists a continuous and bounded solution of* (1)–(2).

COROLLARY: *Let f be continuous in* $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^{2n}$ *with values in* $\mathbb{R}^n$, *and bounded by a constant K. Then given a positive number N and any* $x_0 \in \mathbb{R}^n$, *there exists a solution of* (1) *which is continuous in the interval* $[0, N]$

Proof: Define the function

$g(t, x, y) = \alpha(t) f(t, x, y)$ where

$$\alpha(t) = \begin{cases} 1 & \text{if} \quad t \in [0, N+T) \\ e^{N+T-t} & \text{if} \quad t \in [N+T, \infty) \end{cases}$$

and consider the equation

$$\dot{x} = g(t, x(t), x(t+T)).$$

4. The linear case

If we have

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, x(t+T) \tag{21}$$

where the functions $A(t)$, $B(t)$ are uniformly continuous and uniformly bounded, the equivalent ordinary equation in l_∞ is

$$\dot{y}(t) = \theta(t) y(t) \tag{22}$$

where the linear operator $\theta(t)$ is given by

$$\theta(t)\{y_k\}_{k=0}^{\infty} = \{A(t) y_k + B(t) y_{k+1}\}_{k=0}^{\infty}.$$

Then, $\theta(t)$ is continuous and uniformly bounded, so the problem (22)–(8) satisfies the existence and uniqueness condition in the interval $[0, T)$.

THEOREM 5: *Let $\Phi(t)$ be the fundamental operator of* (22). *A necessary and sufficient condition for the equation* (21) *to have a continuous and bounded solution, different from the trivial one, is that* $\Pi\Phi(T)$ *be singular. In fact, the number of linear independent continuous solutions of* (21) *is equal to the number of linear independent sequences who satisfy* $\{\pi - \Phi(T)\}X = 0$.

Example: Consider the equation

$$\dot{x}(t) = bx(t+1), \qquad \ell n\, 2 \leq b < \ell n\, 3. \tag{23}$$

In this case the operator $\theta(t)$ is equal to the infinite matrix

$$\theta = \begin{pmatrix} 0 & b & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & b & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & b & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & b & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}, \tag{24}$$

so $\Phi(t)$ is defined by

$$\Phi(t) = \begin{pmatrix} 1 & bt & \frac{b^2t^2}{2!} & \frac{b^3t^3}{3!} & \cdots \\ 0 & 1 & bt & \frac{b^2t^2}{2!} & \cdots \\ 0 & 0 & 1 & bt & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

and if $Q = \Phi(1) - \Pi$, we have

$$\| \dot{E} - Q \| \leq e^b - 2 < 1$$

consequently Q is an invertible operator. Finally from theorem 5 it follows that the only continuous and bounded solution of (23) is the trivial one.

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