ON OPTIMAL STOPPING OF INDEPENDENT RANDOM VARIABLES APPEARING ACCORDING TO A RENEWAL PROCESS, WITH RANDOM TIME HORIZON

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0. Suppose that some random, identically distributed quantities appear at the renewal epochs of a renewal process independently of each other and of that process. We observe the consecutive quantities and want to stop the process when a possibly large value occurs. Our decision about stopping must be made up to a time T, which is assumed to be random, not known to us in advance. More precisely, we look for a stopping rule to maximize the probability that at the moment of stopping we get the quantity which is largest among all those which have appeared and will appear up to the time T. This problem (see below for the rigorous formulation) is a natural generalization of Problem III in [1], where the random quantities appear according to a Poisson process and T is fixed non-random. A similar problem has been formulated independently in [6]. The approach used in this note is an elaboration of the one in [1]. It is proved that a solution of our problem exists and its form (unfortunately not very explicit) is found. We outline also a general method which sometimes permits to obtain the solution in a simpler form. As an example, we consider the exponential distributions case, where everything can be calculated explicitly and the obtained results seem quite interesting.

1. Assume that we are given:

a) $\xi_1, \xi_2, \cdots$, a sequence of independent, identically distributed random variables with a continuous distribution function F,

b) $\rho_1, \rho_2, \cdots$, a sequence of i.i.d. positive random variables with a continuous distribution function G,

c) T, a positive random variable with a distribution function H.

Moreover, we assume that

d) $\xi_1, \xi_2, \cdots, \rho_1, \rho_2, \cdots, T$ are independent.

The ρ's are interpreted as time periods between consecutive appearances of the ξ's.

Denote $R_n = \rho_1 + \cdots + \rho_n$ for $n \in N$, $R_0 = 0$, and let $N(t)$ be the corresponding counting process, *i.e.* $N(t) = \max\{n \geq 0\colon R_n \leq t\}$ for $t \geq 0$. Furthermore, we denote by $\mathfrak{F}_n$ the σ-field generated by the random variables $\xi_1, \cdots, \xi_n, \rho_1, \cdots, \rho_n, T\chi_{[0,R_n]}(T)$, $(n \in N)$, where $\chi_{\bullet}$ is the characteristic function of the event $\bullet$, and by $\mathfrak{I}$ we denote the class of all stopping times with respect to the (increasing) family $(\mathfrak{F}_n)_{n \in N}$. $\mathfrak{F}_n$ is interpreted as the class of all events which are known at the moment when ξ_n occurs.

We will consider the following problem:

(P) Find a stopping time $\tau^* \in \mathfrak{I}$, such that

$$P(\tau^* \leq N(T), \xi_{\tau^*} = \xi_1 \vee \cdots \vee \xi_{N(T)}) = \sup_{\tau \in \mathfrak{I}} P(\tau \leq N(T), \xi_\tau = \xi_1 \vee \cdots \vee \xi_{N(T)}).$$

THEOREM. *Assume a), b), c), d). Problem (P) can be reduced to an optimal stopping problem for some Markov chain with some reward function. A solution of (P) exists and has the following form:*

$$\tau^* = \inf \{n: \xi_n = \xi_1 \vee \cdots \vee \xi_n, (R_n, F(\xi_n)) \in \Gamma\}, \tag{1.1}$$

where Γ is a subset of $\mathbb{R}_+ \times [0, 1]$.

Proof. First, observe that $F(\xi_n)$ is uniformly distributed on [0, 1] and $\{\xi_n \geq \xi_m\} = \{F(\xi_n) \geq F(\xi_m)\}$ a.s., therefore we may additionally assume that $\xi_1, \xi_2, \cdots$ are uniformly distributed themselves.

Define $Y_n = P(n \leq N(T), \xi_n = \xi_1 \vee \cdots \vee \xi_{N(T)} \mid \mathfrak{F}_n)$. We have

$$Y_n = \textstyle\sum_{k=n}^{\infty} P(R_k \leq T, R_{k+1} > T, \xi_n \geq \xi_{n+1}, \cdots, \xi_n \geq \xi_k \mid \mathfrak{F}_n)\chi_{\{\xi_n = \xi_1 \vee \cdots \vee \xi_n\}}$$

$$= \chi_{\{\xi_n=\xi_1\vee\cdots\vee\xi_n\}} E(\textstyle\sum_{k=n}^{\infty} x^{k-n} P(R_{k-n} \leq t - (r_1 + \cdots + r_n),$$

$$R_{k-n+1} > t - (r_1 + \cdots + r_n))]_{t=T} \mid T\chi_{[0, r_1+\cdots+r_n]}(T))]_{r_1+\cdots+r_n=R_n, x=\xi_n}$$

where the symbol $]_{t=T}$ (and, analogously, $]_{r_1+\cdots+r_n=R_n, x=\xi_n}$) means that having calculated the probability we substitute T in the place of t.

Let $g(t, x)$ denote the moment generating function of $N(t)$, *i.e.*

$$\begin{aligned} g(t, x) &= E(x^{N(t)}) \quad \text{for} \quad x \in [0, 1], t \geq 0, \\ g(t, x) &= 0 \quad \text{for} \quad t < 0. \end{aligned} \tag{1.2}$$

Then, the conditional expectation in the expression above is equal to

$$E(\textstyle\sum_{j=0}^{\infty} x^j \chi_{\{T \geq r_1+\cdots+r_n\}} P(N(t - (r_1 + \cdots + r_n)) = j)]_{t=T} \mid T\chi_{[0, r_1+\cdots+r_n]}(T))$$

$$= E(g(T - (r_1 + \cdots + r_n), x) \mid T\chi_{[0, r_1+\cdots+r_n]}(T))$$

$$= \chi_{\{T > r_1+\cdots+r_n\}} \frac{1}{P(T > r_1 + \cdots + r_n)} E(g(T - (r_1 + \cdots r_n), x))$$

$$+ \chi_{\{T = r_1+\cdots+r_n\}},$$

hence we get

$$\begin{aligned} Y_n &= \chi_{\{\xi_n=\xi_1\vee\cdots\vee\xi_n, T>R_n\}} \frac{1}{1 - H(R_n)} \\ &\quad \cdot E(g(T - (r_1 + \cdots + r_n), x))]_{r_1+\cdots+r_n=R_n, x=\xi_n} \\ &= \chi_{\{\xi_n=\xi_1\vee\cdots\vee\xi_n, T>R_n\}} \frac{1}{1 - H(R_n)} \int_{R_n}^{+\infty} g(t - R_n, \xi_n)\, dH(t). \end{aligned} \tag{1.3}$$

It is clear that if we define $Y_\infty = 0$, then for each $\tau \in \mathfrak{I}$

$$E(Y_\tau) = P(\tau \leq N(T), \xi_\tau = \xi_1 \vee \cdots \vee \xi_{N(T)}), \tag{1.4}$$

therefore our problem (P) can be formulated now as follows: find a $\tau^* \in \mathfrak{I}$ to maximize $E(Y_\tau)$. Next, we observe that if for any $\tau \in \mathfrak{I}$ we define $\tau' = \inf\{n \geq \tau: n \leq N(T), \xi_n = \xi_1 \vee \cdots \vee \xi_n\}$ (we adopt the usual convention: $\inf\phi = +\infty$), then τ' is, clearly, a stopping time w.r.t. $(\mathfrak{F}_n)_{n\in N}$ and $E(Y_\tau) \leq E(Y_{\tau'})$. This means that we may confine the stopping times to those belonging to $\mathfrak{I}_0$, where

$$\mathfrak{I}_0 = \{\tau \in \mathfrak{I}: \tau = n \Rightarrow n \leq N(T), \xi_n = \xi_1 \vee \cdots \vee \xi_n, n \in N\}.$$

Now, as in [1], we define the consecutive moments when "leaders" appear, *i.e.*

$$\tau_1 = \begin{cases} 1 & \text{if } \rho_1 \leq T \\ +\infty & \text{if } \rho_1 > T \end{cases}, \quad \tau_{k+1} = \inf\{n: n \leq N(T), n > \tau_k, \xi_n \geq \xi_{\tau_k}\}, \quad k \in N$$

It is evident that $\tau_k \in \mathfrak{I}_0$ for $k \in N$.

Calculations quite similar to those we have done above (and therefore omitted) show that

$$P(\tau_{k+1} = m, R_{\tau_{k+1}} \leq s, \xi_{\tau_{k+1}} \geq y \mid \mathfrak{F}_{\tau_k})$$
$$= \xi_{\tau_k}^{m-\tau_k-1}(y - \xi_{\tau_k}) \frac{1}{1 - H(R_{\tau_k})} \int_{R_{\tau_k}}^{+\infty} G^{*m-\tau_k}(t \wedge s - R_{\tau_k})\, dH(t)$$

if $m > \tau_k$, $y \geq \xi_{\tau_k}$, and $= 0$ otherwise, where G^{*n} denotes the distribution function of R_n. Thus, if we define $\mathfrak{F}_{\tau_0}$ as the trivial σ-field, and

$$X_k = \begin{cases} \delta & \text{if } k = 0 \\ (\tau_k, R_{\tau_k}, \xi_{\tau_k}) & \text{if } k > 0, \tau_k < +\infty \\ \partial & \text{if } k > 0, \tau_k = +\infty \end{cases} \tag{1.5}$$

where δ and ∂ are labels for the initial and the final states, respectively, we see that $(X_k)_{k\in N\cup\{0\}}$ is a Markov chain with respect to the σ-fields $(\mathfrak{F}_{\tau_k})_{k\in N\cup\{0\}}$. The state space of this chain is $(N \times R_+ \times [0, 1]) \cup \{\delta\} \cup \{\partial\}$, and the transition function is

$$p(n, r, x; m \times [0, s] \times [0, y])$$
$$= P(\tau_{k+1} = m, R_m \leq s, \xi_m \leq y \mid \tau_k = n, R_n = r, \xi_n = x)$$
$$= \begin{cases} x^{m-n-1}(y - x) \dfrac{1}{1 - H(r)} \displaystyle\int_r^{+\infty} G^{*m-n}(t \wedge s - r)\, dH(t) & \text{if } n < m, x \leq y, H(r) < 1 \\ 0 & \text{otherwise.} \end{cases}$$

∂ is an absorbing state, and it is clear what the transition probabilities involving δ and ∂ should be.

Now, it is easy to see that our problem can be reduced to the problem of optimal stopping of the chain $(X_k)_{k\in N\cup\{0\}}$, with the reward function defined as

$$f(n, r, x) = f(r, x) = \frac{1}{1 - H(r)} \int_r^{+\infty} g(t - r, x)\, dH(t) \tag{1.6}$$

if $H(r) < 1$, and $= 0$ if $H(r) = 1$; $f(\delta) = f(\partial) = 0$.

Indeed, to any $\tau \in \mathfrak{I}_0$ we can associate a stopping time σ with respect to $(\mathfrak{F}_{\tau_k})_{k\in N\cup\{0\}}$, defining $\sigma = k$ on the set $\{\tau = \tau_k < +\infty\}$, $k = 1, 2, \cdots$, and $\sigma = +\infty$ on $\{\tau = +\infty\}$. By (1.3), (1.5), (1.6) we have $E(Y_\tau) = E(f(X_\sigma))$.

Recall that the optimal stopping problem for Markov chains is solved by the following procedure (cf. e.g. [4]): If $\mathbb{P}$ is the operator connected with the transition function then we define $f_1 = f$ (f being the reward function), $f_{n+1} = \max(f, \mathbb{P}f_n)$, $f^* = \lim_{n\to\infty} f_n$, and $\Gamma^* = \{f = f^*\}$. An optimal (finite a.s.) stopping rule exists iff the first hitting time of the set Γ^* by the chain is a.s. finite. That first hitting time is optimal.

In our case,

$$\mathbb{P}f(n, r, x) = \int f(s, y) p(n, r, x; dm, ds, dy) \tag{1.7}$$

$$= \frac{1}{1 - H(r)} \int_x^1 \int_r^{+\infty} \int_0^{t-r} f(s + r, y) \sum_{k=1}^{\infty} x^{k-1}\, dG^{*k}(s)\, dH(t)\, dy,$$

$\mathbb{P}f(\partial) = 0 = f(\partial)$, consequently the set Γ^* must have the form $\Gamma^* = (N \times \Gamma) \cup \{\partial\}$, where $\Gamma \subset \mathbb{R}_+ \times [0, 1]$. The state ∂ is attained by the chain (X_k) with probability one, so an optimal stopping rule (for the problem of stopping the chain) exists. It is evident now that the solution of (P) exists and is given by (1.1). The theorem is proved.

Note that by a trivial modification of (1.1) we obtain an optimal stopping rule which is a.s. finite.

2. In the general case there is no effective way of finding the set Γ. However, sometimes (X_k) and f satisfy the conditions of the so called "monotone case" ([3]) and then much more can be said about the optimal stopping rule. One should consider the set $\Delta^* = \{\mathbb{P}f \leq f\}$. If this set is such that

1° for each state z, $P_z(\exists_k X_k \in \Delta^*) = 1$,

2° for each $z \in \Delta^*$, $P_z(\exists_k X_k \notin \Delta^*) = 0$,

where P_z is the probability connected with the chain starting from z, then the first hitting time of Δ^* is optimal for stopping the chain (X_k) with the reward function f (cf. [1], [2], [5]).

In our case 1° is always satisfied, since $\partial \in \Delta^*$, and it is clear that we should investigate the inequality (cf. (1.7))

$$\frac{1}{1 - H(r)} \int_x^1 \int_r^{+\infty} \int_0^{t-r} f(s + r, y) \sum_{k=1}^{\infty} x^{k-1} dG^{*k}(s)\, dH(t)\, dy \leq f(r, x). \tag{2.1}$$

Let Δ denote the set of pairs (r, x) for which this inequality is satisfied. Taking into account that our chain "goes to the right and upwards" we obtain the following

Criterion: if the set Δ has the form

$$\Delta = \{(r, x) \in \mathbb{R}_+ \times [0, 1]: 1 \geq x \geq x(r), r \in \mathbb{R}_+\}$$

where $x(\cdot)$ is a *non-increasing* function, then

$$\tau^* = \inf [n: \xi_n = \xi_1 \vee \cdots \vee \xi_n, (R_n, F(\xi_n)) \in \Delta\} \tag{2.2}$$

is the solution of (P).

As an example we will consider the case of exponentially distributed random variables. Assume that T has the exponential distribution with parameter μ ($\mu > 0$). Straightforward calculations give (cf. (1.6), (1.7) and (1.2))

$$f(r, x) = \mu \int_0^{+\infty} g(t, x) e^{-\mu t} \, dt = E(x^{N(T)}) \overset{def}{=} f(x),$$

$$\mathbb{P}f(n, r, x) = \frac{1}{1 - x} \int_x^1 f(y) \, dy (1 - f(x)),$$

where we have used the fact that

$$\sum_{k=1}^{\infty} x^{k-1} G^{*k}(t - r) = \frac{1 - g(t - r, x)}{1 - x}$$

If, in addition, we assume that $\rho_1, \rho_2, \cdots$ are exponentially distributed with parameter λ ($\lambda > 0$) then we have

$$f(x) = \frac{\mu}{\lambda(1 - x) + \mu}, \quad \mathbb{P}f(n, r, x) = \frac{\mu}{\lambda(1 - x) + \mu} \log \frac{\lambda(1 - x) + \mu}{\mu}$$

Thus, inequality (2.1) has the form $\log \frac{\lambda(1 - x) + \mu}{\mu} \leq 1$ and, remembering that $x \in [0, 1]$, we get $\Delta = \mathbb{R}_+ \times [x_0, 1]$, where

$$x_0 = \left(1 - (e - 1) \frac{\mu}{\lambda}\right)^+. \tag{2.3}$$

In this case even more can be obtained, namely, it is easy to calculate the "the maximal reward", *i.e.* the probability that using the optimal stopping rule we get the maximal ξ. To this end, let τ^* be given by (2.2). Now

$$\begin{aligned} P_{\max} &= P(\tau^* \leq N(T), \xi_{\tau^*} = \xi_1 \vee \cdots \vee \xi_{N(T)}) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{m} P(N(T) = m) P(F(\xi_1) < x_0, \cdots, F(\xi_{n-1}) < x_0, \\ &\qquad F(\xi_n) \geq x_0, F(\xi_n) = F(\xi_n) \vee \cdots \vee F(\xi_m)) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{m} \left(\frac{\lambda}{\lambda + \mu}\right)^m \frac{\mu}{\lambda + \mu} {x_0}^{n-1} \frac{1}{m - n + 1} (1 - x_0)^{m-n+1}. \end{aligned}$$

Hence, if $x_0 = 0$ then

$$P_{\max} = \frac{\mu}{\lambda + \mu} \log \frac{\lambda + \mu}{\mu},$$

and if $x_0 > 0$ then

$$P_{\max} = \sum_{n=1}^{\infty} \frac{\mu}{\lambda + \mu} x_0^{n-1} \left(\frac{\lambda}{\lambda + \mu}\right)^{n-1} \log \frac{\lambda(1 - x_0) + \mu}{\mu}$$

$$= \frac{\mu}{\lambda(1 - x_0) + \mu} \log \frac{\lambda(1 - x_0) + \mu}{\mu} = \frac{1}{e}.$$

Denote $\alpha = \frac{\lambda}{\mu}$. In the proposition below we summarize our results obtained for the exponential case.

PROPOSITION. *Suppose that T and ρ_n are exponentially distributed with parameters μ, λ respectively.*

1° *If $\alpha \leq e - 1$ then $\tau^* \equiv 1$ is the solution of problem (P), and the probability that using this policy we obtain the maximal ξ is* $\frac{1}{1 + \alpha} \log (1 + \alpha)$.

2° *If $\alpha > e - 1$ then $\tau^* = \inf \{n : F(\xi_n) \geq x_0\}$ is the solution of problem (P), where x_0 is given by* (2.3), *and the corresponding probability is* $\frac{1}{e}$.

It is rather striking that in all reasonable cases ($\alpha > e - 1$) the maximal probability does not depend on λ, μ!

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