PROJECTIVE STABLE STEMS OF SPHERES

BY R. JAMES MILGRAM, J. STRUTT AND P. ZVENGROWSKI

1. Introduction

Let P^n denote n-dimensional real projective space, and $c: S^n \to P^n$ the double covering map. We shall say that a class $[f]$ in $\pi_n(X)$ is projective if and only if there is a homotopy commutative factorization

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ & c\searrow \quad \nearrow & \\ & P^n & \end{array}$$

We then denote the subset of projective homotopy classes of $\pi_n(X)$ by $\pi_n^{\mathrm{Proj}}(X)$. This clearly contains the zero class, and if X is an H-space or if n is in the stable range ($n < 2m - 1$ where x is $(m-1)$-connected), $\pi_n^{\mathrm{Proj}}(X)$ is a subgroup of $\pi_n(X)$.

The notion of projective homotopy class was first considered by J. H. C. Whitehead ([18]) (he called f a "symmetric map"). Whitehead proved that if $\pi_{n+1}(S^n)$ contained a non-zero projective class, then $n + 1 \equiv 3 \pmod 4$. The same result was also proved by Conner and Floyd ([7]). Since then, $\pi_{n+k}^{\mathrm{Proj}}(S^n)$ has been computed through the 3-stem by Bredon ([5], [6]), Rees ([12]), and the second author ([15]), all using different methods. (Bredon considered a more general problem about Z_2-equivariant maps; his results extend through the 6-stem. Rees obtained results for the non-stable case as well.) Projective homotopy classes were later used by Zvengrowski ([19]) to study skew linear vector fields on spheres. In this connection, the second author was led to compute the projective homotopy of certain Stiefel manifolds ([16]).

The aim of the present note is to extend the computations of $\pi_{n+k}^{\mathrm{Proj}}(S^n)$ (in the stable range) to much higher stems. The method applies S-duality to reduce the problem to one concerning the stable homotopy of truncated projective spaces and then makes use of Mahowald's calculations for these spaces ([10]). Results are obtained for the 2-primary component of $\pi_{n+k}^{\mathrm{Proj}}(S^n)$ through the 20-stem, and are listed in Tables I and II. Recently, the projective homotopy of spheres has also been studied in [21], [22], [23].

2. S-duality

We ask which classes $[f]$ in $\pi_n(X)$ admit a factorization

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ & c\searrow \quad \nearrow & \\ & P^n. & \end{array}$$

For X $(r-1)$-connected, this is equivalent to admitting a factorization

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "c"'] & X \\ & P_r^n \arrow[u] \end{tikzcd}$$

where P_r^n denotes the truncated projective space P^n/P^{r-1} (this follows from the fact that $[P^{r-1}, X]$ is trivial and from the homotopy extension property).Thus in the stable range, e.g., $n < 2r-1$, $\pi_n^{\mathrm{Proj}}X$ is isomorphic to

$$\mathrm{Im}(c^{\#}: \{P_r^n, X\} \to \{S^n, X\}).$$

LEMMA (2.1). *Let $p: P_r^{n+1} \to S^{n+1}$ be the map given by pinching $P_r^n \subset P_r^{n+1}$ to a point. Then, assuming $n < 2r-1$,*

$$\pi_n^{\mathrm{Proj}}(X) \approx \mathrm{Ker}[p^{\#}: \{S^{n+1}, SX\} \to \{P_r^{n+1}, SX\}].$$

Proof. Consider the Puppe sequence

$$S^n \xrightarrow{c} P_r^n \to P_r^{n+1} \xrightarrow{p} S^{n+1} \xrightarrow{Sc} SP_r^n \to \cdots$$

and apply the functor $\{-, SX\}$, thus obtaining an exact sequence

$$\{P_r^{n+1}, SX\} \xleftarrow{p^{\#}} \{S^{n+1}, SX\} \xleftarrow{(Sc)^{\#}} \{SP_r^n, X\}.$$

Then $\pi_n^{\mathrm{Proj}}(X) \approx \mathrm{Im}\, c^{\#} = \mathrm{Im}(Sc)^{\#} = \mathrm{Ker}\, p^{\#}$.

THEOREM (2.2). *If $n < 2r-1$ and X is $(r-1)$-connected, then*

$$\pi_n^{\mathrm{Proj}}(X) \approx \mathrm{Ker}[(J \wedge 1)_*: \pi_{M-2}(S^{M-n-2} \wedge X) \to \pi_{M-2}(P_{M-n-2} \wedge X)],$$

where M is any sufficiently large power of 2 and $J: S^{M-n-2} \hookrightarrow P_{M-n-2}$ is the inclusion map into the bottom cell.

Proof. Recall that the $(M-1)$-dual of P_r^{n+1} is P_{M-n-2}^{M-r-1}, where M is divisible by the order of the canonical line bundle ξ_{n+1-r} in $J(P^{n+1-r})$ and $M-n-2 > 0$ (c.f. [4]). Since this order is a power of 2 ([1]), one can simply take M to be a sufficiently large power of 2. Also, the $(M-1)$-dual of the projection map $P_r^{n+1} \xrightarrow{p} P_{n+1}^{n+1} = S^{n+1}$ is the inclusion $S^{M-n-2} \xrightarrow{J} P_{M-n-2}^{M-r-1}$ into the bottom cell.

We now have the following communtative diagram, where the vertical isomorphism are given respectively by $(M-1)$-duality, stability, and the cellular approximation theorem:

$$\begin{array}{ccc}
\{S^{n+1}, SX\} & \xrightarrow{p^{\#}} & \{P_r^{n+1}, SX\} \\
\downarrow\approx & & \downarrow\approx \\
\{S^{M-1}, S^{M-n-2} \wedge SX\} & \xrightarrow{(J \wedge 1)_{\#}} & \{S^{M-1}, P_{M-n-2}{}^{M-r-1} \wedge SX\} \\
\| & & \| \\
\{S^{M-2}, S^{M-n-2} \wedge SX\} & \xrightarrow{(J \wedge 1)_{\#}} & \{S^{M-2}, P_{M-n-2}{}^{M-r-1} \wedge X\} \\
\uparrow\approx & & \uparrow\approx \\
\pi_{M-2}(S^{M-n-2} \wedge X\} & \xrightarrow{(J \wedge 1)_{*}} & \pi_{M-2}(P_{M-n-2}{}^{M-r-1} \wedge X)
\end{array}$$

By 2.1, $\pi_n{}^{\mathrm{Proj}}(X) \approx \operatorname{Ker} p^{\#} \approx \operatorname{Ker}(J \wedge 1)_*$.

COROLLARY 1. *Let $\alpha \in \pi_n(X)$ be represented by $f: S^n \to X$. Then $\alpha \in \pi_n{}^{\mathrm{Proj}}(X)$ if and only if $(J \wedge f): S^{M-1} \to P_{M-n-1} \wedge X$ is null homotopic.*

COROLLARY 2. *For $k < n-1$,*

$$\pi_{n+k}{}^{\mathrm{Proj}}(S^n) \approx \operatorname{Ker}[J_*: \pi_{M-n-2}(S^{M-n-k-2}) \to \pi_{M-n-2}(P_{m-n-k-2})].$$

The Adams spectral sequence can now be used to compute Ker J_*. For this, we make extensive use of Mahowald's computations of $\pi_{N+k}(P_N)$, particularly his tables of $\operatorname{Ext}_A(H^*(P_N), Z_2)$ in order to see which elements of $\operatorname{Ext}_A(H^*(S^N), Z_2)$ map to 0 under J (see §4). First, however, we deduce several properties about projective stable stems which will shorten the computations described in §4.

3. Some preliminary results

THEOREM (3.1). (i) *Suppose $n + K$ is odd. Then $2 \cdot \pi_{n+k}(S^n) \subset \pi_{n+k}{}^{\mathrm{Proj}}(S^n)$.*

(*ii*) *$\pi_{n+k}{}^{\mathrm{Proj}}(S^n) = 0$ for $k + 2 \leq \rho(n+k+2)$, where ρ is the Radon-Hurwitz function ($\rho(m) = 8\mathrm{a} + 2^b$ where $m = c2^{4a+b}$ with c odd and $0 \leq \mathrm{b} \leq 3$).*

(iii) *On the 2-primary parts, $\pi_{n+k}{}^{\mathrm{Proj}}(S^n) = \pi_{n+k}(S^n)$ for $2 < k + 2 \leq \rho(n+k+1)$.*

Proof. For (i), it suffices to observe that when $n + k$ is odd, the composition

$$S^{n+k} \xrightarrow{c} P^{n+k} \longrightarrow P^{n+k}/P^{n+k-1} = S^{n+k}$$

has degree 2 and is clearly projective. So if $[f]$ is any class in $\pi_{n+k}(S^n)$, the above composition followed by f represents $2 \cdot [f]$ and is projective.

For (ii), we consider the vector field problem. When $k + 2 \leq \rho(n+k+2)$, the fibration $V_{n+k+2,k+2} \to S^{n+k+1}$ has a cross section (i.e., S^{n+k+1} has $k+1$ orthonormal vector fields.). If n is large, this is equivalent to $P_n{}^{n+k+1}$ being reducible ([9], Chapter 15, or [11]). Taking S-duals, this is equivalent to $P_{M-n-k-2}{}^{M-n-1}$ being coreducible. Clearly, this implies that Ker $J_* = 0$.

(iii) is proved in [22], using the Kahn-Priddy theorem.

Remark. In Table 1, statement (ii) above amounts to saying that zeroes occur along the diagonals $n + k = s \cdot 2^m - 2$, s odd, for $0 \le k \le \rho(2^m) - 2$. Similarly, the adjacent diagonal $n + k = s \cdot 2^m - 1$ will contain the entire 2-primary part of $\pi_{n+k}(S^n)$, $1 \le k \le \rho(2^m) - 2$. This implies the corollary.

COROLLARY 1. *Any class in $\pi_k(S^0)$ having even order, $k > 0$, is realized in $\pi_{n+k}{}^{\text{Proj}}(S^n)$ for suitable n.*

Indeed, we need only choose m so that $k < \rho(2^m) - 2$ and then take $n = 2^m - 1 - k$.

COROLLARY 2. *Any class in $\pi_k(S^0)$, $k > 0$, is realized in $\pi_{n+k}{}^{\text{Proj}}(S^n)$ for suitable n* (see also [22]).

Proof. Choosing n as above, $\pi_{n+k}{}^{\text{Proj}}(S^n)$ will also contain odd order elements since $n + k$ is odd here and (i) above applies. Since $\pi_{n+k}{}^{\text{Proj}}(S^n)$ is a group in the stable range, Cor. 2 is proved.

THEOREM (3.2). *Let $i(k)$ denote the order of $J(P^{k+1})$. Then if $[f]$ is an element of the projective k-stem, so is $[S^{i(k)}f]$, where S denotes suspension.*

COROLLARY 3. $\pi_{n+k}{}^{\text{Proj}}(S_n) = \pi_{n+k+i(k)}{}^{\text{Proj}}(S^{n+i(k)})$.

Proof of 3.2. Suspend the map p in the Puppe sequence (2):

$$S^rP_n{}^{n+k+1} \to S^{r+n+k+1},$$

and also consider the same map p in higher dimension:

$$P_{r+n}{}^{r+n+k+1} \to S^{r+n+k+1}.$$

Notice that both maps are quotient maps collapsing the next to top skeleton. The spaces $S^rP_n{}^{n+k+1}$ and $P_{r+n}{}^{r+n+k+1}$ are equal to the Thom spaces $T(n\xi_{k+1}+r)$ and $T(r+n)\xi_{k+1})$, respectively, where ξ_{k+1} is the canonical line bundle over P^{k+1} and r is the trivial bundle of dimension r ([8], Chapter 15, Theorem 1.8). If r is a multiple of $i(k)$, then $J((r+n)\xi_{k+1}) = J(n\xi_{k+1})$ $(= J(n\xi_{k+1}+r))$, so that by Proposition 2.6 of [4], the two Thom spaces have the same stable homotopy type. The dimensions are the same, however, and we are in the stable range, so the Thom spaces in fact have the same homotopy type; i.e., $S^rP_n{}^{n+k+1} \simeq P_{r+n}{}^{r+n+k+1}$. Furthermore, the following diagram commutes up to homotopy:

$$\begin{array}{ccc} S^rP_n{}^{n+k+1} & \xrightarrow{S^rp} & S^{r+n+k+1} \\ \wr & & \| \\ P_n{}^{r+n+k+1} & \xrightarrow{p} & S^{r+n+k+1}. \end{array}$$

This can be seen by cellular approximation since the horizontal maps collapse the $(r+n+k)$-skeleton.

The result now follows, since $\pi_{n+k}{}^{\mathrm{Proj}}(S^n) = \mathrm{Ker}\, p^{\#}$ (see 2.1).

Since $J(P^{k+1}) = \tilde{K}O(P^{k+1}) = Z_{2^{f(k+1)}}$ ([1], Example 6.3, and [2], Theorem 7.4), we conclude that the projective k-stem has periodicity $2^{f(k+1)}$.

4. Computation of $\pi_{n+k}{}^{\mathrm{Proj}}(S^n)$ (2-primary part)

In the stable range, the exact homotopy sequence of

$$S^N \xrightarrow{J} P_N \longrightarrow P_{N+1},$$

together with Theorem 2.1, implies that

$$\pi_{n+k}{}^{\mathrm{Proj}}(S^n) = \mathrm{Im}[\delta : \pi_{N+k+1}(P_{N+1}) \longrightarrow \pi_{N+k}(S^N)].$$

Mahowald's tables [10] also include this exact homotopy sequence together with Im δ, at least for stems 15 to 28. The exact homotopy sequences in stems below 15, together with Im δ, are a routine calculation based on the knowledge of the groups $\pi_*(S^0)$, $\pi_*(P_N)$ (c.f. also [8]). Thus the groups $\pi_{n+k}{}^{\mathrm{Proj}}(S^n)$ are known through $k \leq 28$.

Example: For $n \equiv 5(16)$ and $k = 15 < n-1$,

$$\pi_{n+k}{}^{\mathrm{Proj}}(S^n) = \mathrm{Im}[\delta : \pi_{n+15}(P_{N+1}) \longrightarrow \pi_{n+15}(S^N)],$$

where $N \equiv M - 5 - 15 - 2 \equiv 10(16)$, so $\pi_{n+15}{}^{\mathrm{Proj}}(S^n) = Z_2$ is found in the table for P_{10} [10].

We are not only interested in the subgroup $\pi_{n+k}{}^{\mathrm{Proj}}(S^n)$ but in the generators as well. Thus, in the above case, $\pi_{n+15}(S^n) = Z_{32} \oplus Z_2$, so $\pi_{n+15}{}^{\mathrm{Proj}} = Z_2 \subset Z_{32} + Z_2$ is not in itself sufficient to determine this subgroup uniquely. Such further information can often be obtained by an examination of the spectral sequences in [10].

Elements in Ker J_* can be found by first computing

$$\mathrm{Ker}[J_{\#} : \mathrm{Ext}_A(H^*(S^N), Z_2) \longrightarrow \mathrm{Ext}_A(H^*(P_N), Z_2),$$

i.e. finding the kernel of the mapping induced by J on the E_2 terms of the Adams spectral sequences for $\pi_*(S^N)$ and $\pi_*(P_N)$. Ker J_* can then be found, taking care to note that if $J_{\#}(\alpha) = 0$ then the homotopy class corresponding to α might map under J_* to zero or to a non-zero class of higher filtration.

The Kernel of $J_{\#}$ is computed from Mahowald's spectral sequences, which actually give the E_{16} term of a spectral sequence which converges to $\mathrm{Ext}_A(H^*(P_N), Z_2)$. The E_1-term of this spectral sequence is

$$\begin{aligned} E_1{}^{s,t} &= \textstyle\sum_{k\geq 0} \mathrm{Ext}_A{}^{s,t}(H^*(S^{N+k}), Z_2) \\ &= \textstyle\sum_{k\geq 0} \mathrm{Ext}_A{}^{s,t-k}(H^*(S^N), Z_2) \\ &= \textstyle\sum_{k\geq 0} H^{s,t-k-N}(A). \end{aligned}$$

TABLE I

$\pi_{n+k}{}^{\mathrm{proj}}(S^n)$ (2-primary part)

Note: C, D_8, D_9,… explained in Table II′

	$n \equiv 0$ (16)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$k = 0$	0	$2Z$	0	$2Z$	0	$2Z$	0	$2Z$	0	$2Z$	0	$2Z$	0	$2Z$	0	$2Z$
1	0	0	Z_2	0	0	0	Z_2	0	0	0	Z_2	0	0	0	Z_2	0
2	0	Z_2	Z_2	0	0	Z_2	Z_2	0	0	Z_2	Z_2	0	0	Z_2	Z_2	0
3	Z_4	Z_2	Z_4	0	Z_8	Z_2	Z_4	0	Z_4	Z_2	Z_4	0	Z_8	Z_2	Z_4	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	Z_2	0	Z_2	Z_2	0	0	0	0	Z_2	0	Z_2	Z_2	0	0	0
7	Z_8	0	Z_8	0	Z_8	0	Z_8	0	Z_{16}	0	Z_8	0	Z_8	0	Z_8	0
8	D_8	0	0	Z_2^2	Z_2	0	0	Z_2^2	Z_2	0	0	Z_2^2	Z_2	0	0	Z_2^2
9	D_9	Z_2	Z_2^3	Z_2^2	Z_2	0	Z_2^3	Z_2^2	Z_2^2	Z_2	Z_2^3	Z_2^2	Z_2	0	Z_2^3	Z_2^2
10	0	Z_2	Z_2	0	0	Z_2	Z_2	0	0	Z_2	Z_2	0	0	Z_2	Z_2	0
11	D_{11}	Z_2	Z_8	0	Z_8	Z_2	Z_8	0	Z_8	Z_2	Z_8	0	Z_8	Z_2	Z_8	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	Z_2^2	0	0	0	Z_2	0	Z_2	0	Z_2	0	0	0	0	0	0
15	D_{15}	Z_2	Z_{16}	0	$Z_{32}Z_2$	Z_2	$Z_{16}Z_2$	0	$Z_{32}Z_2$	Z_2	Z_{16}	0	$Z_{32}Z_2$	Z_2	$Z_{16}Z_2$	0
16	D_{16}	0	0	Z_2^2	Z_2	0	0	Z_2^2	Z_2	0	0	Z_2^2	Z_2	0	0	Z_2^2
17	D_{17}	0	Z_2^3	Z_2^2	Z_2^2	0	Z_2^4	Z_2^3	Z_2^2	0	Z_2^3	Z_2^2	Z_2^2	0	Z_2^4	Z_2^3
18	0	Z_4Z_2	Z_2^2	Z_4	0	Z_8Z_2	Z_2^2	Z_4	0	Z_4Z_2	Z_2^2	Z_4	0	Z_8Z_2	Z_2^2	Z_4C
19	D_{19}	Z_2	Z_8	0	Z_8Z_2	Z_2^2	Z_8Z_2	0	Z_8	Z_2	Z_8	0	Z_8Z_2	Z_2	Z_8	0
20	0	Z_4	0	Z_8	Z_2	Z_4	Z_2	Z_4	0	Z_4	0	Z_8	Z_2	Z_4	Z_2	Z_4

Given an element α in $H^{s,t-k-N}(A)$, the corresponding element in $E_1^{s,t}$ is denoted ${}_K\alpha$ or α_K. Clearly, the map

$$J_{\#}: \mathrm{Ext}_A(H^*(S^N), Z_2) \to \mathrm{Ext}_A(H^*(P_N), Z_2)$$

sends α to ${}_0\alpha$. Thus, Ker $J_{\#}$ can be found from the spectral sequence for P_N by seeing for which elements α, ${}_0\alpha$ does not survive in the spectral sequence. Note that this can only happen when ${}_0\alpha$ is hit by a differential in the spectral sequence (c.f. [10], p. 27). It is also important to note that occasionally there may be a relation and ${}_0\alpha$ can be listed as an element under a different name with the same s, t coordinates.

Having found Ker $J_{\#}$ in this way, one then takes into account the Adams differentials, if any are present, to see if ${}_0\alpha$ survives to the E_∞ term of the

TABLE II

Projective Generators in $\pi_{n+k}(S^n)$

$n = 0$	(16)	1	2	3	4	5	6	7
$k = 0$	0	$2h_0$	0	$2h_0$	0	$2h_0$	0	$2h_0$
1	0	0	h_1	0	0	0	h_1	0
2	0	h_1^2	h_1^2	0	0	h_1^2	h_1^2	0
3	$2h_2$	$4h_2$	$2h_2$	0	h_2	$4h_2$	$2h_2$	0
6	0	h_2^2	0	h_2^2	h_2^2	0	0	0
7	$2h_3$	0	$2h_3$	0	$2h_3$	0	$2h_3$	0
8	D_8	0	0	h_1h_3 c_0	 c_0	0	0	h_1h_3 c_0
9	D_9	$h_1^2h_3 + h_1c_0$	$h_1^2h_3$ h_1c_0 P_1h_1	$h_1^2h_3$ h_1c_0	 h_1c_0	0	$h_1^2h_3$ h_1c_0 P_1h_1	$h_1^2h_3$ h_1c_0
10	0	$h_1(P_1h_1)$	$h_1(P_1h_1)$	0	0	$h_1(P_1h_1)$	$h_1(P_1h_1)$	0
11	D_{11}	$4P_1h_2$	P_1h_2	0	P_1h_2	$4P_1h_2$	P_1h_2	0
14	0	h_3^2 d_0	0	0	0	$h_3^2 +$	0	$h_3^2 +$
15	D_{15}	 h_1d_0	$2\bar{h}_4$	0	$\bar{h}_4$ h_1d_0	 h_1d_0	$2\bar{h}_4$ h_1d_0	0
16	D_{16}	0	0	P_1c_0 h_1h_4	P_1c_0	0	0	P_1c_0 h_1h_4
17	D_{17}	0	$h_1^2h_4$ $h_1(P_1c_0)$ P_2h_1	$h_1^2h_4$ $h_1(P_1c_0)$	 $\bar{e}_0$ $h_1(P_1c_0) +$	0	$h_1^2h_4$ $\bar{e}_0$ $h_1(P_1c_0)$ P_2h_1	$h_1^2h_4$ $\bar{e}_0 +$ $h_1(P_1c_0)$
18	0	$h_1(P_2h_1)$ $2h_2h_4$	$h_1(P_2h_1)$ $4h_2h_4$	 $2h_2h_4$	0	$h_1(P_2h_1)$ h_2h_4	$h_1(P_2h_1)$ $4h_2h_4$	 $2h_2h_4$
19	D_{19}	$4(P_2h_2)$	P_2h_2	0	P_2h_2 c_1	$4P_2h_2$ c_1	P_2h_2 c_1	0
20	0	$2g$	0	g	$4g$	$2g$	$4g$	$2g$

TABLE II (cont.)

	8	9	10	11	12	13	14	15
$k=0$	0	$2h_0$	0	$2h_0$	0	$2h_0$	0	$2h_0$
1	0	0	h_1	0	0	0	h_1	0
2	0	h_1^2	h_1^2	0	0	h_1^2	h_1^2	0
3	$2h_2$	$4h_2$	$2h_2$	0	h_2	$4h_2$	$2h_2$	0
6	0	h_2^2	0	h_2^2	h_2^2	0	0	0
7	h_3	0	$2h_3$	0	$2h_3$	0	$2h_3$	0
8	h_1h_3	0	0	h_1h_3 c_0	c_0	0	0	h_1h_3 c_0
9	$h_1^2h_3$ h_1c_0	$h_1^2h_3 + h_1c_0$	$h_1^2h_3$ h_1c_0 P_1h_1	$h_1^2h_3$ h_1c_0	h_1c_0	0	$h_1^2h_3$ h_1c_0 P_1h_1	$h_1^2h_3$ h_1c_0
10	0	$h_1(P_1h_1)$	$h_1(P_1h_1)$	0	0	$h_1(P_1h_1)$	$h_1(P_1h_1)$	0
11	P_1h_2	$4P_1h_2$	P_1h_2	0	P_1h_2	$4P_1h_2$	P_1h_2	0
14	0	d_0	0	0	0	0	0	0
15	$\bar{h}_4$ h_1d_0	h_1d_0	$2\bar{h}_4$	0	$\bar{h}_4$ h_1d_0	h_1d_0	$2\bar{h}_4$ h_1d_0	0
16	P_1c_0	0	0	P_1c_0 h_1h_4	P_1c_0	0	0	P_1c_0 h_1h_4
17	$\bar{e}_0$ $h_1(P_1c_0)$	0	$h_1^2h_4$ $h_1(P_1c_0)$ P_2h_1	$h_1^2h_4$ $h_1(P_1c_0)$	$\bar{e}_0$ $h_1(P_1c_0)$ +	0	$h_1^2h_4$ $\bar{e}_0$ $h_1(P_1c_0)$ P_2h_1	$h_1^2h_4$ $\bar{e}_0$ + $h_1(P_1c_0)$
18	0	$h_1(P_2h_1)$ $2h_2h_4$	$h_1(P_2h_1)$ $4h_2h_4$	$2h_2h_4$	0	$h_1(P_2h_1)$ h_2h_4	$h_1(P_2h_1)$ $4h_2h_4$	c $2h_2h_4$
19	P_2h_2	$4P_2h_2$	P_2h_2	0	P_2h_2 c_1	$4P_2h_2$	P_2h_2	0
20	0	$2g$	0	g	$4g$	$2g$	$4g$	$2g$

Adams spectral sequence. Here ${}_0\alpha$ might vanish by virtue of being a boundary or by not being a cycle. If ${}_0\alpha$ survives to E_∞, then the homotopy class corresponding to α cannot be in Ker J_*.

Finally, extra care must be taken with the case $n \equiv 0(16)$, where Adams differentials on the class I_{k+1} give rise to groups A and B in [10], which depend on m, where $n \equiv 2^m(2^{m+1})$. These groups in turn effect Im δ and this effect is not shown in Mahowald's tables of Im δ. It can usually be computed from Theorem 7.4 of [10] (or Prop. 7.5, but the definition of A_k there is incorrect). For $k \leqslant 13$ one can also compare with [8], Table 2.

Table I lists the groups $\pi_{n+k}{}^{\mathrm{Proj}}(S^n)$ for $k \leqslant 20$, and Table II lists the generators of these groups as far as possible. In some cases $J_\#(\alpha) = 0$ for an element $\alpha \in \mathrm{Ext}_A(H^*(S^N), Z_2)$ but $J_*(\bar{\alpha}) = J_*(\bar{\beta})$, where β is a class in $\mathrm{Ext}_A(H^*(S^n), Z_2)$ of filtration higher than that of α, and $\bar{\alpha}$, $\bar{\beta}$ are the corresponding homotopy classes. In this case the class $\bar{\alpha} + \bar{\beta}$ is projective rather than $\bar{\alpha}$. This information is not carried in the Adams spectral sequences, although sometimes it is derivable in other ways. For cases where we cannot decide we simply list the projective generator as $\bar{\alpha}+$ (usually omitting the bar). In §5 sample computations of the generators are given for the 16-stem.

5. Sample Computation: $k = 16$

For $k = 16$ one has $N \equiv (14 - n) \pmod{16}$. Thus,

$$\pi_{n+16}{}^{\mathrm{Proj}}(S^n) \approx \mathrm{Ker}\,[J_* : \pi_{N+16}(S^N) \to \pi_{N+16}(P_N)]$$

$$\mathrm{Im}\,[\delta : \tau_{N+17}(P_{N+1}) \to \pi_{N+16}(S^N)].$$

The stable 16-stem is $Z_2 + Z_2$, with generators $h_1 h_4$ and $P_1 c_0$ in Mahowald's notation (η^* and $\eta\rho$ respectively).

For $n \equiv 1, 2, \pmod 4$ we find $\mathrm{Im}\,\delta = 0$, while for $n \equiv 3 \pmod 4$ $\mathrm{Im}\,\delta = Z_2 + Z_2$, so in these cases the choice of generators is trivial. When $n \equiv 4, 8, 12$ (16), we have $N \equiv 10, 6, 2,$ (16) and in each case $\mathrm{Im}\,\delta = Z_2$. Now $P_1 c_0$ has filtration 7, and in each of these cases the spectral sequence reveals no term in degree 16 of filtration 7. Thus $J_{\#}(P_1 c_0) = 0$ here. Finally, since there are also no terms in filtration greater than 7, $J_*(P_1 c_0) = 0$ *so* $P_1 c_0$ is the projective generator here.

Now consider the case $n \equiv 0$ (16), i.e. $N \equiv 14$ (16). In this case the class ${}_0(P_1 c_0)$ is a non-zero class in the E_2 term, as can be verified by checking all the possible differentials $\delta_r{}'$ up to this stage. Indeed,

$${}_0(P_1 c_0) = h_0{}^2({}_5 P_1 h_2).$$

However, ${}_0(P_1 c_0)$ is generally the image of an Adams differential as seen from

TABLE II′

$C =$	$h_1(P_2 h_1)$	$n \equiv 15(32)$	
	0	$n \equiv 31(32)$	
$D_8 =$	$h_1 h_3$	$m = 4$	
	0	$m > 4$	*where* $n \equiv 2^m(2^{m+1})$
$D_9 =$	$h_1{}^2 h_3,\ h_1 c_0$	$m = 4, 5$	
	$h_1{}^2 h_3 + h_1 c_0$	$m > 5$	
$D_{11} =$	$P_1 h_2$	$m = 4, 5, 6$	
	$2P_1 h_2$	$m > 6$	
$D_{15} =$	$h_1 d_0,\ \bar{h}_4$	$4 \le m \le 7$	$(\bar{h}_4 = h_0{}^3 h_4)$
	$h_1 d_0,\ 2\bar{h}_4$	$m > 7$	
$D_{16} =$	$P_1 c_0$	$4 \le m \le 8$	
	0	$m > 8$	
$D_{17} =$	$\bar{e}_0,\ h_1(P_1 c_0) +$	$4 \le m \le 9$	
	$\bar{e}_0$	$m > 9$	
$D_{19} =$	$P_2 h_2$	$4 \le m \le 10$	
	$2P_2 h_2$	$m > 10$	

[10] Theorem 7.4 and Prop. 7.1 In the notation of [10], $k = 17$, $q = 8$, and $n \equiv 2^m(2^{m+1})$ here. We then see that

$$\delta_{m-1}(h_0{}^{8-m}I_{17}) = {}_0(P_1c_0),$$

as long as $8 - m \geq 0$. Thus $\pi_{n+16}{}^{\mathrm{Proj}}(S^n) = Z_2$, generated by P_1c_0, in these cases. However, if $m > 8$, then no Adams differential hits ${}_0(P_1c_0)$ and it follows that $\pi_{n+16}{}^{\mathrm{Proj}}(S^n) = 0$ here.

STANFORD UNIVERSITY, STANFORD, CA. U.S.A.
STATE UNIVERSITY OF NEW YORK, POTSDAM, N.Y., U.S.A.
UNIVERSITY OF CALGARY, ALBERTA, CANADA.

REFERENCES

[1] J. F. ADAMS, *On the groups $J(X)$—II*, Topology **3** (1965), 137–171.
[2] ———, *Vector fields on spheres,* Ann. of Math. **75** (1962), 603–632.
[3] ———, Stable Homotopy Theory, Springer-Verlag, Berlin, 1964.
[4] M. F. ATIYAH, *Thom complexes,* Proc. London Math. Soc. **11** (1961), 291–310.
[5] G. E. BREDON, *Equivariant stable stems,* Bull. Amer. Math. Soc. **73** (1967), 269–273.
[6] ———, *Equivariant homotopy,* Proc. Conf. on Transformation Groups, New Orleans, 1967, 281–292.
[7] P. E. CONNER AND E. E. FLOYD, *Fixed point free involutions and equivariant maps II,* Trans. Amer. Math. Soc. **105** (1962), 222–228.
[8] C. S. HOO AND M. MAHOWALD, *Some homotopy groups of Stiefel manifolds,* Bull. Amer. Math. Soc. **71** (1965), 661–667.
[9] D. HUSEMOLLER, Fibre Bundles, McGraw Hill, New York, 1966.
[10] M. MAHOWALD, *The metastable homotopy of S^n*, Mem. Amer. Math. Soc. **72,** 1967.
[11] B. MORIN, *Champs de vecteurs sur les sphères d'après J. F. Adams,* Seminaire Bourbaki, 233, 1962.
[12] E. REES, *Symmetric maps,* J. London Math Soc. **3** (1971), 267–272.
[13] E. H. SPANIER, Algebraic Topology, McGraw Hill, New York, 1966.
[14] ———, *Function spaces and duality,* Ann. of Math. **70** (1959), 338–378.
[15] J. STRUTT, *Projective homotopy classes of spheres in the stable range,* Bol. Soc. Mat. Mexicana **16** (1971), 15–25.
[16] ———, *Projective homotopy classes of Stiefel manifolds,* Canadian J. Math. **3** (1972), 465–476.
[17] H. TODA, Composition Methods in the Stable Homotopy of Spheres, Princeton University Press, 1962.
[18] J. H. C. WHITEHEAD, *On the groups $\pi_r(V_{n,m})$ and sphere bundles,* Proc. London Math. Soc. **48** (1944), 243–291.
[19] P. ZVENGROWSKI, *Skew linear vector fields on spheres,* J. London Math. Soc. **3** (1971), 625–632.
[20] D. S. KAHN AND S. B. PRIDDY, *Applications of the transfer to stable homotopy theory,* Bull. Amer. Math. Soc. **78** (1972), 981–987.
[21] D. RANDALL, *Projectivity of the Whitehead square,* Proc. Amer. Math. Soc. **40** (1973), 609–611.
[22] M. MIMURA, J. MUKAI AND G. NISHIDA, *Representing elements of stable homotopy groups by symmetric maps,* Osaka J. Math. **11** (1974), 105–111.
[23] S. FEDER, S. GITLER AND K. Y. LAM, *Composition properties of projective homotopy classes,* Pacific J. Math. **68** (1977), 47–61.