A NOTE ON THE PERFECTION OF THE FUNDAMENTAL GROUP OF THE CLASSIFYING SPACE FOR CODIMENSION ONE REAL ANALYTIC FOLIATIONS

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Let $B\Gamma_1^\omega$ be the classifying space of the groupoid Γ_1^ω of germs of local real analytic orientation preserving homeomorphisms of $\mathscr{R}$. Then $B\Gamma_1^\omega$ is a $K(G, 1)$, [1], [2]. In this note we give a new proof of the result of A. Haefliger:

THEOREM. *$G = \pi_1(B\Gamma_1^\omega)$ is perfect. That is $H_1(B\Gamma_1^\omega, Z) = 0$.*

Our proof is in the spirit of the calculation of G made in [2]. From the presentation of G given there we give an explicit formula for every generator as a commutator. Recall G is the free group on the components of Γ_1^ω modulo "composition when defined". That is, the generators are maximally extended local orientation preserving homeomorphisms of $\mathscr{R}$. There is a relation $F \bullet G = F \circ G$, where $F \bullet G$ is the product in the free group and $F \circ G$ is the composition of F and G as functions, whenever the range of F intersects the domain of G. Moreover every generator of G represents a nontrivial element of G [2].

Note, we will write composition of homeomorphisms from *left to right* in the sequel.

Proof: Let S be the set of generators of G. Let $T_\alpha \in S$ correspond to (the maximal homeomorphism) translation by $\alpha \in \mathscr{R}$. Let $E_\beta \in S$ correspond to the exponential map $x \to e^{\beta x}$, $\beta > 0$ and to $x \to -e^{\beta x}$, $\beta < 0$. Let L_γ correspond to the linear map $x \to \gamma \bullet x$, $\gamma \in \mathscr{R}$, $\gamma > 0$. Let $A(p, q)$ correspond to the affine map $x \to px + q$, $p, q \in \mathscr{R}$, $p > 0$. Let $[Q, R]$ denote the commutator $Q \circ R \circ Q^{-1} \circ R^{-1}$, $Q, R \in S$. Let X^{-1} denote the inverse component of X, $X \in S$.

The following lemmas are trivial.

LEMMA 1: $T_\alpha = [A(a, 1), A(b, 1)]$ *where* $\dfrac{1}{a} - \dfrac{1}{b} = \alpha$, $a, b > 0$.

LEMMA 2: $L_\gamma = E_{\ell n(\gamma)}{}^{-1} \circ T_1 \circ E_{\ell n(\gamma)}$, $\gamma > 0$.

Lemma 3 is the well known linearization theorem of S. Sternberg formulated to fit our context [4].

LEMMA 3: *Let $Q \in S$ be a homeomorphism such that $Q(0) = 0$, $0 \in \mathscr{R}$, $Q'(0) = \gamma \neq 1$. Then there exists an element $K \in S$ such that $K(0) = 0$ and $K \circ Q \circ K^{-1} = L_\gamma$.*

To prove the theorem let $X \in S$. If $X = T_\alpha$ then we are done by lemma 1. Otherwise choose a point p in the domain of X such that $X'(p) = a \neq 1$. Let $q = X(p)$. Then

$$(1) \qquad Y = T_p \circ X \circ T_{-q}$$

is a component with $Y(0) = 0$, $Y'(0) = a$. By lemma 3 there is a $K \in S$ such that

$$K \circ Y \circ K^{-1} = L_a . \tag{2}$$

By lemma 2,

$$L_a = E_{\ell n(a)}{}^{-1} \circ T_1 \circ E_{\ell n(a)} . \tag{3}$$

Formulas (1), (2) and (3) together yield

$$X = T_{-p} \circ (E_{\ell n(a)} \circ K)^{-1} \circ T_1 \circ (E_{\ell n(a)} \circ K) \circ T_q .$$

Hence we obtain the following commutator formula: Consider $X \in S$ with $X(p) = q, X'(p) = a$ and let K be a conjugating homeomorphism of $T_p \circ X \circ T_{-q}$. Set $Q = E_{\ell n(a)} \circ K$. Then

$$X = [A(a_0, 1), A(a_1, 1)] \cdot [Q \circ A(b_0, 1) \circ Q^{-1}, Q \circ A(b_1, 1) \circ Q^{-1}][A(c_0, 1), A(c_1, 1)]$$

where

$$\frac{1}{a_0} - \frac{1}{a_1} = -p, \quad \frac{1}{b_0} - \frac{1}{b_1} = 1 \quad \text{and} \quad \frac{1}{c_0} - \frac{1}{c_1} = q.$$

We remark that in an analogous manner we can prove that the fundamental groups of the classifying spaces in codimension greater than 1 vanish, using Sternberg's linearization theorem in $\mathcal{R}^n$, $n > 1$, [3].

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