ON σ-UNICOHERENCE

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1. Introduction

In this paper we introduce the concepts of σ-unicoherent and locally σ-connected spaces. We give several equivalent formulations of σ-unicoherence in the class of connected and locally σ-connected spaces. We prove, incidentally, that every connected, locally σ-connected and σ-unicoherent space is unicoherent. John H. V. Hunt ([2]) exhibits an example of a Peano space (that is, a connected, locally connected and locally compact metric space) which is unicoherent but not σ-unicoherent. So, not every connected, locally σ-connected unicoherent space is σ-unicoherent. However, all Euclidean spaces R^n and all unicoherent Peano continua are σ-unicoherent.* If we drop the assumption of local connectednesss, it is possible to find a closed connected subset of R^3 which is σ-unicoherent but not unicoherent (see example 3.4 below).

2. Preliminary definitions

Let X be an arbitrary topological space. A sequence $C_1, C_2, \cdots$ of subsets of X is a *σ'-partition* of X if the C_i's are mutually disjoint, their union is X and for at least two different indices i, j, C_i and C_j are non-empty. A *σ-partition* of X is a σ'-partition $C_1, C_2, \cdots$ of X such that C_i is non-empty for every i. A σ'-partition is *closed* (resp. *compact*) if all of its elements are closed (resp. compact). X is *σ-connected* if it has no closed σ'-partition. X is *locally σ-connected* if for each $x \in X$ and each neighborhood V of x, there is a σ-connected neighborhood W of x contained in V. A subset A of X *σ-separates* X if $X - A$ is not σ-connected. Finally, X is *σ-unicoherent* if X is connected and for every pair H, K of closed σ-connected subspaces of X with union X, the set $H \cap K$ is σ-connected.

3. Equivalent formulations of σ-unicoherence

We start this section with a lemma:

3.1 *Let C be a closed and σ-connected subspace of a connected and locally σ-connected space X. If R is a component of $X - C$, then $X - R$ is σ-connected.*

Proof. Clearly, every region in a locally σ-connected space is σ-connected. If $\{R_\alpha \mid \alpha \in J\}$ are the components of $X - C$ different from R, then $\Phi \neq Fr\ R_\alpha \subset C$ for each $\alpha \in J$. By [1], 1.3, $X - R = C \cup (\bigcup_{\alpha \in J} R_\alpha^-)$ is σ-connected.

* For more general results, see [3].

We can now state the equivalence theorem:

3.2 *In a connected and locally σ-connected space X, the following statements are equivalent*:

1) *X is σ-unicoherent.*
2) *If $C \subset X$ is closed and σ-connected and R is a component of $X - C$, then $Fr\,R$ is σ-connected.*
3) *If V is an open set which σ-separates X, then there exists a component W of V which σ-separates X.*
4) *If S is a region in X and R is a component of $X - S^-$, then $Fr\,R$ is σ-connected.*
5) *If K is a closed set separating two points a, $b \in X$ and $K_1, K_2, \cdots$ is a closed σ'-partition of K, then some K_i separates a, b in X.*
6) *If K is a closed set separating X and $K_1, K_2, \cdots$ is a closed σ'-partition of K, then some K_i separates X.*
7) *Every closed set K in X separating a pair of points a, $b \in X$ irreducibly is σ-connected.*
8) *If L is a closed set separating X, then some σ-component of L separates X.*

Proof.

1) $\Rightarrow$ 2) By 3.1, $X - R$ is σ-connected. Hence R^- and $X - R$ are closed σ-connected sets with union X. By hypothesis, $Fr\,R = R^- \cap (X - R)$ is σ-connected.

2) $\Rightarrow$ 3) Let $\{V_\alpha \mid \alpha \in J\}$ be the components of V and assume, on the contrary, that each $X - V_\alpha$ is σ-connected. Let $A_1, A_2, \cdots$, be a closed σ'-partition of $X - V$. By 2), each $Fr\,V_\alpha$ is σ-connected. If

$$J_i = \{\alpha \in J \mid Fr\,V_\alpha \subset A_i\},$$

then $J = J_1 \cup J_2 \cup \cdots$. Defining $A_i^* = A_i \cup (\bigcup_{\alpha \in J_i} V_\alpha)$, we obtain a closed σ'-partition of X, contradicting the fact that X is σ-connected.

3) $\Rightarrow$ 4) Let $V = X - Fr\,R = R \cup (X - R^-)$. According to 3.1, no component of V σ-separates X. Therefore, $X - V = Fr\,R$ is σ-connected.

4) $\Rightarrow$ 5) Let S be the component of $X - K$ containing a and let R be the component of $X - S^-$ that contains b. Our hypothesis implies that $Fr\,R$ is σ-connected. Hence $Fr\,R \subset K_i$ for some i. Since $Fr\,R$ separates a, b in X, the same holds for K_i.

5) $\Rightarrow$ 6) This implication is obvious.

6) $\Rightarrow$ 7) Let C_a, C_b be the components of $X - K$ containing a, b, resp., and let $\{V_\alpha \mid \alpha \in J\}$ be the components of $X - K$ other than C_a and C_b. The irreducibility of K implies $Fr\,C_a = Fr\,C_b = K$. Proceeding by contradiction, let $K_1, K_2, \cdots$, be a closed σ'-partition of K. For each i, let $K_i^* = K_i \cup (\bigcup \{V_\alpha \mid Fr\,V_\alpha \subset K_i\})$. Each $X - K_i^*$ is connected, since

$$X - K_i^* = C_a \cup C_b \cup (\cup_{j \neq i} K_j) \cup (\cup \{V_\alpha | (X - K_i) \cap Fr\, V_\alpha \neq \Phi\}).$$

The union of the first three sets is connected and each V_α satisfying $(X - K_i) \cap Fr\, V_\alpha \neq \Phi$ has limit points in $\cup_{j \neq i} K_j$. Our hypothesis implies that $K^* = K_1^* \cup K_2^* \cup \cdots$ does not separate X. However, C_a and C_b are different components of $X - K^*$, a contradiction.

7) $\Rightarrow$ 8) Let a, b points in different components of $X - L$ and let K be a closed subset of L separating a, b irreducibly (this is always possible in a connected and locally connected space). By hypothesis, K is σ-connected. Hence the σ-component of L containing K separates a, b in X.

8) $\Rightarrow$ 1) Assume X is not σ-unicoherent. Then there exist two closed σ-connected sets H, K such that $X = H \cup K$ and a closed σ'-partition $A_1, A_2, \cdots$ of $H \cap K$. Since H is σ-connected, there exists a component C of $H - K = X - K$ whose frontier intersects more than one set A_i. Let $P_i = A_i \cap Fr\, C$ $(i = 1, 2, \cdots)$ and let $\{V_\alpha | \alpha \in J\}$ be the components of $X - C^-$. Let us define

$$P_i^* = P_i \cup (\cup \{V_\alpha | Fr\, V_\alpha \subset P_i\}).$$

For some $\alpha \in J$ and different indexes i, j we must have $P_i \cap Fr\, V_\alpha \neq \Phi \neq P_j \cap Fr\, V_\alpha$, for otherwise,

$$X - C = P_1^* \cup P_2^* \cup \cdots$$

contradicting the fact that $X - C$ is σ-connected (lemma 3.1). Let $L = P_1^* \cup P_2^* \cup \cdots$ L is then a closed set separating X because C and any V_α with $Fr\, V_\alpha \cap (X - P_i) \neq \Phi$ for every i are different components of $X - L$. Observe also that each $X - P_i^*$ is connected. 8) implies the existence of a σ-component D of L which separates X. Necessarily $D \subset P_i^*$ for some i, say $D \subset P_1^*$. Then $X - D$ is connected, for if

$$X - D = A \cup B$$

were a separation of $X - D$, where $X - P_1^* \subset A$, then $D \cup B$ would be a σ-connected set in P_1^* containing D properly. This contradiction completes the proof.

3.3 COROLLARY. *Every locally σ-connected σ-unicoherent space is unicoherent.*

3.4 EXAMPLE. *A closed connected subset of R^3 which is σ-unicoherent but not unicoherent.*

Let $X = K_1 \cup K_2 \cup \cdots$ be the example described in [1], 4.3. Since X is connected but not σ-connected, X is necessarily σ-unicoherent. Let $H = K_1$ and $K = p(K_1) \cup K_2 \cup K_3 \cup \cdots$, where $p(K_1)$ is the projection of K_1 on the plane $z = 0$. Clearly $X = H \cup K$, H, K are both closed and connected and $H \cap K = p(K_1)$. This proves X is not unicoherent.

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