ON COMPACT SPACES WHICH ARE NOT c-SPACES

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Abstract: A compact space K is not a c-space iff the space $[0, \Omega]$ of ordinals up to the first uncountable one, noted Ω, is homeomorphic to a subspace of a quotient of K (equivalently to a quotient of a subspace of K). βN, the Stone-Čech—compactification of N furnishes an example of a compact space that is not a c-space, but such that $[0, \Omega]$ is neither homeomorphic to a subspace nor to a quotient of βN.

In the second part we show that on every compact space that is not a c-space there exists a $\{0, 1\}$-valued σ-additive Borel-measure that is not outer regular (equivalently, not a Radon-measure), thus extending a famous example of such a measure on $[0, \Omega]$, constructed by J. Dieudonné.

§ 1

Following Archangel'skii [1], we call a topological space X a *c-space*, if every subset E of X that is countably closed (*i.e.* if $\{x_n\}_{n=1}^{\infty} \subseteq E$ and x is a cluster-point of $\{x_n\}_{n=1}^{\infty}$ then $x \in E$) is closed.

The prototype of a space not being a c-space is $[0, \Omega]$, the compact space of ordinals up to the first uncountable one, noted Ω, equipped with the order-topology, where $[0, \Omega[$ furnishes an example of a countably closed but non-closed subset.

The following proposition shows that compact non-c-spaces are closely related to $[0, \Omega]$.

PROPOSITION 1: *For a compact Hausdorff-space K the following are equivalent:*

(i) *K is not a c-space.*

(ii) *There exists a closed subspace of a quotient of K, homeomorphic to $[0, \Omega]$.*

(ii)$'$ *There exists a quotient of a closed subspace of K, homeomorphic to $[0, \Omega]$.*

Proof: The property of being a compact c-space is inherited by closed subspaces and quotients. This is completely trivial for subspaces; for quotient spaces (*i.e.* continuous images) assume that $\pi: K \to K_1$ is a continuous map of a compact c-space K onto a Hausdorff space K_1.

Let A_1 be a subset of K_1 and $x_1 \in \overline{A_1}$; then $\pi^{-1}(x_1) \cap \overline{\pi^{-1}(A_1)}$ is not empty in K. Indeed if it were so $\pi(\overline{\pi^{-1}(A_1)})$ would be a closed subset of K_1, containing A_1 but not containing x_1.

So by assumption there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in $\pi^{-1}(A_1)$ such that $\pi^{-1}(x_1) \cap \overline{\{y_n\}_{n=1}^{\infty}} \neq \phi$. But then $x_1 \in \overline{\{\pi(y_n)\}_{n=1}^{\infty}}$ which shows that K_1 is a c-space too.

So clearly we have (ii) $\Rightarrow$ (i) and (ii)$'$ $\Rightarrow$ (i) because $[0, \Omega]$ is not a c-space.

To show the other direction we apply an argument of [2]: suppose there exists a set E in K such that E is countably closed but not closed and let $y_0 \in \bar{E}\backslash E$. Countable closedness of E implies that E is semicompact (*i.e.* every countable open cover of E has a finite sub-cover). We define inductively a "long sequence" $\{x_\alpha\}_{\alpha<\Omega}$ of points of E and $\{f_\alpha\}_{\alpha<\Omega}$ of continuous functions of K, taking their values in $[0, 1]$ and such that $f_\alpha(y_0) = 1$.

Let x_0 be arbitrary in E and f_0: $K \to [0, 1]$ such that $f_0(x_0) = 0$ while $f_0(y_0) = 1$. Suppose $\{x_\gamma\}_{\gamma<\alpha}$ and $\{f_\gamma\}_{\gamma<\alpha}$ are chosen. Then let x_α be an element in E such that $f_\gamma(x_\alpha) = 1$ for every $\gamma < \alpha$. (This is possible because

$$\left\{f_\gamma^{-1}\left(\left[0, 1 - \frac{1}{n}\right]\right) : \gamma < \alpha,\, n \in N\right\}$$

is a countable family of open sets. If it would cover E then already a finite subfamily would cover E; but this is absurd, as $y_0 \in \bar{E}$, $f_\gamma(y_0) = 1\ \forall\gamma < \alpha$ and the f_γ are continuous.) Then choose f_α such that $f_\alpha(x_\gamma) = 0\ \forall\ \gamma \leq \alpha$ and $f_\alpha(y_0) = 1$. (This is possible by Tietze-Urysohn, because y_0 is not in the closure of $\{x_\gamma\}_{\gamma\leq\alpha}$).

After this induction has been effected, define for $\alpha \in [0, \Omega]$ the sets

$$F_\alpha = \bigcap_{\beta<\alpha} \overline{\{x_\gamma\}_{\beta<\gamma\leq\alpha}},$$

and

$$F_\Omega = \bigcap_{\beta<\Omega} \overline{\{x_\gamma\}_{\beta<\gamma<\Omega}} = \bigcap_{\beta<\Omega} \overline{\{F_\gamma\}_{\beta<\gamma<\Omega}}.$$

Clearly the $\{F_\alpha\}_{\alpha\leq\Omega}$ are nonempty, compact, disjoint subsets of K and F_α is reduced to $\{x_\alpha\}$ if α has a predecessor.

Define $F = \bigcup_{\alpha\leq\Omega} F_\alpha$. F is closed: indeed let $\{Z_i\}_{i\in I} \to Z$ be a convergent net in K, such that $Z_i \in F$. Let, for every i be $\alpha(i)$ the unique index such that $i \in F_{\alpha(i)}$. From the definition of the $\{f_\alpha\}_{\alpha<\Omega}$ it is clear that $\{\alpha(i)\}_{i\in I}$ is a convergent net in $[0, \Omega]$, say it converges to a certain $\alpha_0 \in [0, \Omega]$.

This implies that for every $\beta<\alpha_0$, $\{Z_i\}_{i\in I}$ finally lies in the set $\bigcup_{\beta<\gamma\leq\alpha_0} F_\gamma = \bigcup_{\beta<\gamma\leq\alpha_0} \overline{\{x_\gamma\}}$. So $Z = \lim Z_i$ lies in F_{α_0} by the very definition of F_{α_0}.

Now it is clear how to construct the spaces as in (ii) and (ii)$'$: For (ii) define on K the equivalence relation $\mathscr{R}$: $x\ \mathscr{R}\ y \Leftrightarrow \exists\alpha \in [0, \Omega]$ such that $x \in F_\alpha$ and $y \in F_\alpha$.

From the definition of the F_α and f_α one immediately sees that the quotient $K/\mathscr{R}$ is Hausdorff. Clearly the image of F in $K/\mathscr{R}$ is closed and it is easily verified that is is homeomorphic to $[0, \Omega]$.

To show (i) $\Rightarrow$ (ii)$'$ let F be the closed subspace of K and define on F again the equivalence relation $\mathscr{R}$; then $F/\mathscr{R}$ again is homeomorphic to $[0, \Omega]$.

This completes the proof of Proposition 1.

Example: βN, the Stone-Čech-compactification of N

(a) βN is not a c-space. Indeed $\{0, 1\}^R$ is separable, so it is a continuous image

of βN. Also $\{0, 1\}^{R}$ contains a copy of $[0, \Omega]$ (take a collection $\{f_\alpha\}_{\alpha<\Omega}$ as in the proof of Proposition 1 to define such an embedding).
(b) βN does not have a quotient isomorphic to $[0, \Omega]$. Indeed every continuous image of βN is separable, while $[0, \Omega]$ is not.
(c) βN does not contain a copy of $[0, \Omega]$; in fact it does not even contain a copy of $[0, \omega]$, ω being the first infinite ordinal (*i.e.* a non-trivial convergent sequence).
Indeed suppose $\{x_n\}_{n=1}^{\infty}$ to be a convergent sequence (to x_0 say) in βN such that $x_n \neq x_m$ if $n \neq m$. In functional-analytic language this may be stated as follows. The Dirac-measures $\delta_{\{x_n\}}$, which define simply additive measures on the σ- algebra $P(N)$ of all subsets of N, converge to $\delta_{\{x_0\}}$ on every member of $P(N)$. Whence by the Vitali-Hahn-Saks-theorem, in its form for simply additive measures (see for example [3]: Theorem 1.4.8), $\{\delta_{\{x_n\}}\}_{n=1}^{\infty}$ would be uniformly strongly additive, which is evidently absurd. *q. e. d.*

Hence it is really necessary in the above proposition to speak about "subspaces of quotients" (resp. "quotients of subspaces"), to get a characterization of compact non-c-spaces.

§ 2

Our last result in this paper is to show that on a compact non-c-space K one may always construct a $\{0, 1\}$-valued σ-additive Borel-measure which is not outer regular, in a similar fashion as J. Dieudonné did on $[0, \Omega]$ (See, for example [5], exercise 52.10).

Although the result is, of course, related to the above characterization of compact non-c-spaces, the proof is completely independent of it.

PROPOSITION 2: *Let K be a compact Hausdorff-space which is not a c-space. Then there is a σ-additive, $\{0, 1\}$-valued Borel-measure μ on K which is not a Radon-measure.*

Proof: Let E be countably closed but not closed in K and let $x_0 \in \bar{E}\backslash E$. Let $\mathscr{A}$ be an ultra-filter of closed sets in E (in its induced topology), that converges to x_0. Note again that the countable closedness of E implies that E is semicompact, *i.e.* every countable open cover of E has a finite subcover. So $\mathscr{A}$ has the countable intersection property (*i.e.* if $\{A_n\}_{n=1}^{\infty} \in \mathscr{A}$ then $\bigcap_{n=1}^{\infty} A_n \neq \phi$).

Let F be any closed subset of K. Then $\mathscr{A}$ lies finally in F or in its complement F. Indeed suppose $F \cap A \neq \phi$ and $F \cap A \neq \phi$ for every $A \in \mathscr{A}$, then $\{F \cap A\}_{A \in \mathscr{A}}$ is a filter of closed subsets of E strictly finer than $\mathscr{A}$.

Clearly also for every open set G and for every set of the form $\bigcup_{i=1}^{n} \bigcap_{j=1}^{m(i)} H_{i,j}$ where $H_{i,j}$ are either open or closed subsets of K, we have the same property that $\mathscr{A}$ finally lies in it or in its complement. Note that the family of the latter sets forms an algebra. Define μ on the sets B of this algebra by

$$\mu(B) = 1 \text{ if } A \text{ lies finally in } B$$

$$\mu(B) = 0 \text{ if not.}$$

Clearly μ is additive and if B_n is a decreasing sequence in this algebra such that $\mu(B_n) = 1$ for every n, then there are $A_n \in \mathscr{A}$ such that $A_n \subseteq B_n$; as $\bigcap_{n=1}^{\infty} A_n \neq \phi$ we get $\bigcap_{n=1}^{\infty} B_n \neq \phi$, which readily shows the σ-additivity of μ. By the Carathéodory-procedure μ has a σ-additive extension to the Borel-algebra $\mathscr{B}$ of K, which clearly is 0-1-valued too and will also be denoted μ.

But μ is not a Radon-measure: indeed $\mu(\{x_0\}) = 0$ while for every open neighborhood U of x_0 we have $\mu(U) = 1$; so μ is not outer regular.

The last proposition shows that every compact Radon-space (*i.e.* where every σ-additive finite Borel-measure is a Radon-measure) is a c-space. Conversely it was shown by the author [6], that every Eberlein-compact, satisfying a mild cardinality restriction, is a Radon-space, a fact which also follows from the independent work of G. Edgar [4].

But the problem to characterize topologically the class of (compact) Radon-spaces seems very hard, as is also indicated by the recent example of M. Wage [7], showing that this class is not stable under forming finite products.

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