A SPECIAL CLASS OF RINGS OF FUNCTIONS AND THEIR FINITELY SUBCOVERABLE EXTENSIONS

By Alexander Abian

1. Introduction. We consider a special class of rings R of functions (pointwise addition and multiplication) from a set X into a ring Y with the special property that: any ideal of R which misses an element f of R is contained in an ideal J of R which also misses f and for which R/J is isomorphic to a subring of Y. We show that X can be extended to a set X' and every element f of R can be extended to a function f' from X' into Y such that the resulting set $R' = \{f' \mid f \in R\}$ is a ring of functions f' from X' into Y and the correspondence $f \to f'$ is a ring isomorphism from R onto R' and if an element f' of R' is covered by the elements of a subset E' of R' (i.e., for every $z \in X'$ if $f'(z) \neq 0$ then $g'(z) \neq 0$ for some $g' \in E'$) then f' is already covered by finitely many elements of E'. In particular, let R have a unit and let R satisfy the weaker property that: any proper ideal of R is contained in a proper ideal J of R for which R/J is isomorphic to a subring of Y. Let X (and X') be topologized by an open basis whose members are those subsets B (and B') of X (and X') for which there exists an element f (and f') of R (and R') such that B (and B') is the set of all those points of X (and X') on each of which f (and f') does not vanish. Then our result yields a compactification X' of X which includes as special cases many of the known compactifications.

2. In what follows all the ring-theoretical statements which are made in connection with a set of functions (with a common domain) refer to the pointwise addition and multiplication of the elements of that set.

Also, in what follows, we let R stand for a ring of functions from a set X (*i.e.*, X is the common domain of the elements of R) into a ring Y such that R satisfies the following condition:

For every ideal I of R and every element f of R if

(1) *$f \notin I$ then there exists an ideal J of R such that $I \subseteq J$ and $f \notin J$ and R/J is isomorphic to a subring of Y.*

i.e., R is such that any ideal of R which misses an element f of R is contained in an ideal J of R which also misses f and for which R/J is isomorphic to a subring of Y.

We observe that, except for the zero function, no other constant function need be an element of R. In particular, R need not have a unit.

Let E' be a set of functions from a set X' into a ring Y (with zero 0). Then a function f' from X' into Y is said to be *covered* by the elements of E' if and only if for every $z \in X'$

(2) $$f'(z) \neq 0 \quad \textit{implies} \quad g'(z) \neq 0 \quad \textit{for some} \quad g' \in E'.$$

i.e., f' is covered by elements of E' if and only if whenever f' does not vanish at a point so does an element of E' at that point.

LEMMA. *Let R' be a ring of functions from a set X' into a ring Y and let E' be a subset of R'. If an element f' of R' is covered by no finite number of elements of E' then f' is not an element of the ideal I' (of R') generated by E'.*

Proof. Since f' is covered by no finite number of elements of E', in view of (2), we see that for every finite number of elements $e_0', \cdots, e_n'$ of E' there exists $z \in X'$ such that $e_i'(z) = 0$ for every $i \leq n$ whereas $f'(z) \neq 0$. Thus, $f' \notin I'$ as desired.

Based on the above notions we prove:

THEOREM. *Let R be a ring of functions from a set X into a ring Y such that R satisfies (1). Then X can be extended to a set X' and every element f of R can be extended to a function f' from X' into Y such that:*

(i) *the resulting set $R' = \{f' \mid f \in R\}$ is a ring of functions f' from X' into Y.*
(ii) *the correspondence $f \to f'$ is a ring isomorphism from R onto R'.*
(iii) *if an element f' of R' is covered by the elements of a subset E' of R' then f' is already covered by finitely many elements of E'.*

Proof. Clearly, the set

$$J_x = \{f \mid (f \in R) \quad \text{and} \quad f(x) = 0\} \qquad \text{for every } x \in X \tag{3}$$

is an ideal of R such that R/J_x is isomorphic to a subring of Y.

Let

$$\{J_v \mid v \in V\} \tag{4}$$

be the set of all the ideals J_v of R such that R/J_v is isomorphic to a subring of Y and such that $J_v \notin \{J_x \mid x \in X\}$ where J_x is given by (3).

With every J_v which appears in (4) we associate a unique isomorphism i_v from R/J_v onto a subring of Y. Thus, for every $f \in R$ and every $v \in V$, we have:

$$i_v([f]) \text{ is a unique element of } Y \tag{5}$$

where $[f]$ is the coset of R/J_v of which f is an element.

Now, let us consider the set X' given by:

$$X' = X \cup V \tag{6}$$

where V is as in (4).

Clearly, X' is an extension of X. Also, by (3), (4), (6) we see that

$$\{J_z \mid z \in X'\} \tag{7}$$

is the set of *all* the ideals J of R such that R/J is isomorphic to a subring of Y.

In view of (5), to every $f \in R$ we correspond a function f' from X' into Y

defined as:

(8) $f' = f$ *on* X *and* $f'(v) = i_v([f])$ *for every* $v \in V$.

Clearly, from (4), (6), (8) it follows that f' is an extension of f to X'.

Again, from (4) and (8) we see that

$$J_v = \{f \mid (f \in R) \text{ and } f'(v) = 0\} \quad \text{for every } v \in V$$

which by (3) and (7) implies that

(9) $J_z = \{f \mid (f \in R)$ *and* $f'(z) = 0\}$ *for every* $z \in X'$.

From (8) it follows that the correspondence $f \rightarrow f'$ is one-to-one and that

(10) $f' + g' = (f + g)'$ *and* $f' \cdot g' = (f \cdot g)'$

for every element f and g of R.

Thus, from (10) we see that the set R' (of functions from X' into the ring Y) given by:

(11) $R' = \{f' \mid f \in R\}$

is an isomorphic image of the ring R (of functions f from X into the ring Y).

Clearly, (10) and (11) establish (i) and (ii).

Next, let an element f' of R' be covered by the elements of a subset E' of R' (see (2)). To prove (iii) we must show that f' is already covered by some finitely many elements of E'. Let us assume to the contrary that f' is covered by no finite number of elements of E'. Thus, by the Lemma the ideal I' (of R') generated by E' is such that

(12) $f' \notin I'$.

Obviously,

(13) $E' \subseteq I'$.

Since $f \rightarrow f'$ is a ring isomorphism, we see that the subset I of R given by:

(14) $I = \{g \mid g' \in I'\}$

is an ideal of R and by (12) we have $f \notin I$. But then from (1) it follows that there exists an ideal J of R satisfying (1). However, by (7) every such J ideal of R must be equal to an ideal J_z (of R) for some $z \in X'$. Thus,

$$I \subseteq J_z \text{ and } f \notin J_z \text{ for some } z \in X'$$

which by (9) and (14) implies

$$f'(z) \neq 0 \text{ and } g'(z) = 0 \text{ for every } g' \in I'$$

which in turn, in view of (13) and (2) contradicts that f' is covered by the elements of E'. Hence our assumption is false and (iii) is also established.

Remark 1. Let the ring R instead of satisfying condition (1) satisfies the weaker condition (1*) given by:

(1*) *for every proper ideal I of R there exists a proper ideal J of R such that $I \subseteq J$ and R/J is isomorphic to a subring of Y.*

i.e., R is such that any proper ideal of R is contained in a proper ideal J of R for which R/J is isomorphic to a subring of Y.

Let the ring R also have a *unit* u. With these hypotheses the proof of our Theorem will establish the validity of (i) and (ii) and the following version (iii*) of (iii):

(*iii**) *if the image (under the isomorphism $f \to f'$) u' of the unit u of R is covered by the elements of a subset E' of R' then u' is already covered by finitely many elements of E'.*

Remark 2. In Topology, very often, the (common) domain X of a ring of functions (from X) into a ring Y is topologized by an open basis whose members consist of those subsets B of X for which there exists an element f of R such that B is the set of all those points of X on each of which f does not vanish. Let us call such a topology the *Characteristic* topology of R on X. Then in view of Remark 1 our Theorem obviously implies the following:

COROLLARY. *Let R be a ring of functions from a set X into a ring Y such that R has a unit and R satisfies* (1*). *Then X can be extended to a set X' and every element f of R can be extended to a function f' from X' into Y such that the resulting set $R' = \{f' \mid f \in R\}$ is a ring of functions f' from X' into Y and the correspondence $f \to f'$ is a ring isomorphism from R into R'. Moreover, the Characteristic topology of R' on X' (whose restriction to X is the Characteristic topology of R on X) is compact.*

The Corollary shows how X endowed with a Characteristic topology can be compactified by extending it to X' which is endowed by the corresponding Characteristic topology. Many of the known compactifications in Topology fall into the pattern of the compactification described above.

UNIVERSITY OF OXFORD, ENGLAND AND
IOWA STATE UNIVERSITY, AMES, IOWA, U.S.A.