NECESSARY AND SUFFICIENT CONDITIONS FOR CONTINUOUS DEPENDENCE FOR VOLTERRA OPERATOR EQUATIONS

BY RUBÉN FLORES AND ZDENEK VOREL

Let G be a non-empty open subset of the space $C([a, b], R^m)$ of continuous functions on a compact interval with sup norm and consider functions

$$T_n : G \to C([a, b], R^m)$$

for $n = 0, 1, 2, \cdots$, which have the following properties:

(1) $z_i \in G$ for $i = 1, 2$, implies $(T_n z_1)(a) = (T_n z_2)(a)$ and there exists $z \in G$ such that $(T_n z)(a) = z(a)$.
(2) $z_i \in G$ for $i = 1, 2, t \in [a, b), z_1(s) = z_2(s)$ for $a \leq s \leq t$, imply $(T_n z_1)(t) = (T_n z_2)(t)$ (causality).
(3) T_n is almost continuous in G (cf [1], i.e. for every $z \in G$ and $\epsilon > 0$ there is a $\delta > 0$ such that $x \in G$, $x(a) = z(a)$, $|x - z| < \delta$ imply $|T_n x - T_n z| < \epsilon$.
(4) T_n is almost locally compact, i.e. for every $z \in G$ there is a $\delta > 0$ such that if $S = \{x \in G : x(a) = z(a), |x - z| < \delta\}$ then TS is relatively compact in $C([a, b], R^n)$. The following result was proved in [2]:

THEOREM 1: *Let x_0 be a unique fixed point of T_0, let $r > 0$ be such that the closed ball $\overline{B}(x_0, r) \subset G$ and define a function $Z: C([a, b], R^m) \to C([a, b], R^m)$ as follows: $Zx = x$ if $x \in B(x_0, r)$ or if $|x(a) - x_0(a)| > r$; $(Zx)(t) = x(t)$ for $a \leq t \leq t_x$, where $t_x = \inf\{s \in [a, b]: |x(s) - x_0(s)| > r\}$; $(Zx)(t) = x_0(t) + x(t_x) - x_0(t_x)$ for $t_x \leq t \leq b$ if $x \in C([a, b], R^n)$, $|x(a) - x_0(a)| \leq r$, $x \in (\overline{B}(x_0, r))^C$. Assume that there is a natural p such that*

(5) $$\lim_{n\to\infty} (T_n - T_0)(ZT_n)^p x \to 0 \text{ uniformly for } x \in \overline{B}(x_0, r).$$

Then for every $\epsilon > 0$ there is a fixed point x_n of T_n in $B(x_0, \epsilon)$ for n sufficiently large.

The following example shows that condition (5) is not necessary for continuous dependence of fixed points.

Example 1. Consider a sequence of ordinary differential equations

(6) $$\dot{x} = f_n(x), \quad n = 1, 2, 3, \cdots,$$

$$\text{where} \quad f_n(x) = nx^{\frac{n-1}{n}} \quad \text{for} \quad x \geq 0$$

$$f_n(x) = 0 \quad \text{for} \quad x < 0$$

with initial conditions $x_n(0) = 0$, solutions x_n being considered on the interval $[0, 1]$. It is easy to see that for every natural n there is a continuum of solutions of (b) whose graphs lie all between the graphs of $y_1(t) = 0$ and $y_2(t) = t^n$, $0 \leq t \leq 1$.

Equations (6) generate operators T_n on $C([0, 1], R^1)$, $(T_n x)(t) = \int_0^t f_n(x(s))\,ds$ for every natural n, $0 \le t \le 1$, $x \in C([0, 1], R^1)$, which satisfy conditions (1) through (4). From the definition of f_n it follows that for every $r > 0$ there is an $x \in \bar{B}(0, r)$ such that $T_n(ZT_n)^p x(t) = n \int_0^t \{\min[g(s), r]\}^{1-1/n}\,ds$ where g depends on n, p, r, x, is continuous increasing in [0, 1], $g(0) = 0$, and, for n sufficiently large assumes the value r for arbitrarily small $t \in (0, 1]$. This shows that for every $r > 0$ there is an $x \in \bar{B}(0, r)$ such that the norm of left hand side of (5) tends to $+\infty$ with $n \to +\infty$.

The following plausible argument will lead to a sufficient and necessary condition for continuous dependence. In (5), assume that for every fixed natural n $Z \circ T_n$ is a contraction in $\bar{B}(x_0, r)$. Assume that (5) holds for some natural $p = p_1$. Clearly, (5) holds for every natural $p \ge p$, and $\lim_{p \to \infty}(Z \circ T_n)^p \bar{B}(x_0, r)$ consists of the unique fixed point x_n of T_n in $\bar{B}(x_0, r)$. It follows that

$$\lim_{n \to \infty}(T_n - T_0)x_n \to 0. \tag{9}$$

We shall prove in a more general setting that (9) is sufficient and necessary for $x_n \to x_0$.

THEOREM 2. *Let S_n be a function from a complete metric space M into itself for $n = 0, 1, 2, \cdots$. Let S_0 be continuous and compact (i.e. if A is a bounded set in M then $\overline{S_0(A)}$ is compact. Let x_0 be the unique fixed point of S_0 in M and let B be a closed ball in M with center at x_0. Assume for every n S_n has a fixed point x_n (not necessarily unique) in B. Then $x_n \to x_0$ iff the distance $d(x_n, S_0 x_n) \to 0$ as $n \to \infty$.*

Proof. If $d(x_n, x_0) \to 0$ then $d(x_n, S_0 x_n) \le d(x_n, x_0) + d(x_0, S_0 x_n)$ tends to 0 as $x_0 = S_0 x_0$ and S_0 is continuous.

If $d(x_n, S_0 x_n) \to 0$, then there is a subsequence $\{x_{n_k}\}$, such that $S_0 x_{n_k}$ converges to some element $y \in M$ as S_0 is compact and $\{x_{n_k}\}$ is bounded. But $d(x_{n_k}, y) \le d(x_{n_k}, S_0 x_{n_k}) + d(S_0 x_{n_k}, y) \to 0$ by the continuity of S_0.

$y = \lim_k S_0 x_{n_k} = S_0 y = x_0$, as x_0 is the unique fixed point of S_0. The same argument is valid if instead of the original sequence $\{x_n\}$ one considers its arbitrary subsequence. Thus an arbitrary subsequence of $\{x_n\}$ contains a subsequence converging to x_0, which proves that $x_n \to x_0$.

UNIVERSIDAD DE SONORA,
UNIVERSITY OF SOUTHERN CALIFORNIA AND CENTRO DE INVESTIGACION DEL IPN.

REFERENCES

[1] L. W. NEUSTADT. *On the solutions of certain integral-like operator equations.* Arch. Rational Mech. Anal., **38**, 2 (1970), 131–160.

[2] Z. VOREL, *On a theorem of L Neustadt,* in Notas de Matemática y Simposia, III Simposio México-Estados Unidos sobre Ecuaciones Diferenciales, Fondo de Cultura Económica, México, 1976.