FUNCTIONAL INTEGRAL EQUATIONS OF VOLTERRA TYPE

BY C. BOSCH, M. FALCONI, C. HERNÁNDEZ AND Z. VOREL

It was shown in [3] that a causal functional differential equation in $\mathbb{R}^n$ is equivalent to an ordinary differential equation in L_p, $p \geq 1$. A similar result was proved in [2] in a different way, which is used in this paper to extend the above mentioned equivalence result to Volterra's integral equations.

Let a and h be positive numbers, $x:[-h, a] \to \mathbb{R}^n$, $x_t(s) = x(s)$ if $-h \leq s \leq t$, $x_t(s) = x(t)$ if $\leq s \leq a$(x_t is the truncation of x at the point $t \in [-h, a]$).

LEMMA 1. *Let* $\psi:[0, a] \times [0, a] \to \mathbb{R}^n$, $\psi(\bullet, t)$ (Lebesgue) *measurable for every fixed* $t \in [0, a]$ *and let* $\psi(s, \bullet)$ *be right continuous in* $[0, a]$ *for every fixed* $s \in [0, a]$. *Then* ψ *is measurable in* $[0, a] \times [0, a]$.

Proof. For every natural n define the function $\psi_n(s, t) = \psi\left(s, \dfrac{i-1}{n}a\right)$ for $s \in [0, a]$, $t \in \left(\dfrac{i-1}{n}a, \dfrac{ia}{n}\right]$, $i = 1, 2, \cdots, n$, $\psi_n(s, 0) = \psi(s, 0)$. Now ψ_n is measurable in $[0, a] \times [0, a]$ as

$$(\psi_n)^{-1}(G) = \bigcup_{i=1}^{n} \left\{\left[\psi\left(\bullet, \frac{ia}{n}\right)\right]^{-1}(G) \times \left(\frac{i-1}{n}a, \frac{ia}{n}\right]\right\} \cup \{[\psi(\bullet, 0)]^{-1}(G) \times \{0\}\}$$

for every open $G \subset \mathbb{R}^n$. Also, $\psi_n(s, t) \to \psi(s, t)$ for every $(s, t) \in [0, a] \times [0, a]$ as ψ is right continuous.

LEMMA 2. *Let* $D = \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq a\}$ *and let* $\varphi: D \to \mathbb{R}^n$ *satisfy*

(*i*) $\varphi(s, \cdot)$ *is right continuous in* $[s, a]$ *for every fixed* $s \in [0, a]$.
(*ii*) $\varphi(\cdot, t)$ *is measurable for every fixed* $t \in [0, a]$.
(*iii*) $|\varphi(s, t)| \leq M(s)$ *in* D *where* $M \in L_p[0, a]$, $p \geq 1$.
(By usual abuse of language we do not always distinguish between equivalence classes in L_p *and their representatives). Then* $\varphi \in L_p(D)$.

Proof. Consider the extension $\tilde{\varphi}$ of φ to $[0, a] \times [0, a]$ defined by $\tilde{\varphi} = 0$ in D^c. By Lemma 1 $\tilde{\varphi}$ is measurable. But D is a measurable subset of $[0, a] \times [0, a]$, which implies that φ is measurable. (iii) implies that $\varphi \in L_p(D)$.

To be able to state Theorems 1 and 2, which contain the principal results of this paper, we shall assume that B is a subset of $L_p([-h, a], \mathbb{R}^n)$ with the following property: $x \in B$ implies that: (1) there is a function in the equivalence class x, which will again be denoted by x, whose restriction $x_{[0,a]}$ to the subinterval $[0, a]$ is continuous and (2) the truncation $x_t \in B$ for $t \in [0, a]$.

Let $f: B \times D \to \mathbb{R}^n$ have the following properties:

(1) If x and y are from B and their restrictions to $[-h, s]$ coincide for some $s \in [0, a]$ then $f(x, s, t) = f(y, s, t)$, $t \in [s, a]$. We say that such f is causal.

(2) $f(x, \cdot, t) \in L_p[0, t]$ for $x \in B$, $t \in [0, a]$.

(3) $f(x, s, \cdot)$ is continuous for every fixed $x \in B$, $s \in [0, a]$.

(4) For every $x \in B$ there exists an $M_x(\cdot) \in L_p[0, a]$ such that $|f(x, s, t)| \leq M_x(s)$ for $(s, t) \in D$.

Our aim is to show that a causal functional integral equation of Volterra type in $\mathbb{R}^n$

(5) $x(t) = z(t) + \int_0^v f(x, s, v)ds$, where $v = \max(0, t)$, $z \in B$, $t \in [-h, a]$, is a special case of an ordinary Volterra equation in L_p. To construct such an equation define a function $F: B \times D \to L_p[-h, a]$ by

$$F(x, s, t)(\tau) = \begin{cases} 0 & \text{for} \quad -h \leq \tau < s \leq t \\ f(x, s, \tau) & \text{for} \quad s \leq \tau \leq t \\ f(x, s, t) & \text{for} \quad t \leq \tau \leq a \end{cases} \tag{6}$$

if $x \in B$, $(s, t) \in D$. Denote for $z \in B$, $z_t = w(t)$, $t \in [0, a]$.

THEOREM 1. *If $x \in B$ is a solution of* (5) *then the function $y:[0, a] \to B$ defined by $y(t) = x_t$ for $t \in [0, a]$ is a solution of*

$$y(t) = w(t) + \int_0^t F(y(s), s, t)\, ds \text{ in } [0, a]. \tag{7}$$

THEOREM 2. *If $y:[0, a] \to B$ is a solution of* (7) *then there exists a unique $x \in B$ such that $y(t) = x_t$ for $t \in [0, a]$ and this x is a solution of* (5).

Remark 1. Theorems 1 and 2 are generalizations of similar equivalence results for ordinary differential equations in [2] and [3]. They form a basis for a unified theory of ordinary and causal functional equations of Volterra type. A unified approach to existence and continuous dependence theorems for ordinary and functional differential equations was presented in [4].

Theorems 1 and 2 will be proved by means of the following lemmas.

LEMMA 3. *Assume φ as in Lemma* 2 *except that the right continuity in* (i) *is replaced by continuity. If $\Phi: D \to L_p([-h, a], \mathbb{R}^n)$* is defined by

$$\Phi(s, t)(\tau) = \begin{cases} 0 & \text{for} \quad -h \leq \tau < s \\ \varphi(s, \tau) & \text{for} \quad s \leq \tau \leq t \\ \varphi(s, t) & \text{for} \quad t \leq \tau \leq a \end{cases}$$

then $\int_0^t \Phi(s, t)\, ds$ exists (in the sense of Bochner [1]) for $t \in [0, a]$ and its equivalence class in $L_p([-h, a], \mathbb{R}^n)$ contains the function $\tau \to \int_0^t \Phi(s, t)(\tau)\, ds$, $t \in [-h, a]$.

Proof. Let $t_1 \in [0, a]$ and define

$$D_{t_1} = \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq t_1\}.$$

By Lemma 2 $\varphi \in L_p(D_{t_1})$ and consequently (see [5]), there exist sequences $\{\tilde{\varphi}_n\}$, $\{\hat{\varphi}_n\}$ of continuous functions such that $\tilde{\varphi}_n \to \varphi$ in $L_p(D_{t_1})$ and $\hat{\varphi}_n \to \varphi(\cdot,$

t_1) in $L_p[0, t_1]$. For each n there exist sequences $\{\tilde{\varphi}_n^i\}_{i=1}^\infty$, $\{\hat{\varphi}_n^i\}_{i=1}^\infty$ of step functions such that $\tilde{\varphi}_n^i(\cdot, t)$ and $\hat{\varphi}_n^i$ is constant in every interval $\left[\frac{kt_1}{2^i}, \frac{(k+1)t_1}{2^i}\right)$ for $k = 0, 1, \cdots, 2^i - 1$ and every fixed $t \in [0, t_1]$. For each n let i_n be such that

$$|\tilde{\varphi}_n^{i_n} - \varphi|_{L_p(D_{t_1})} < \frac{1}{n}$$

and

$$|\hat{\varphi}_n^{i_n} - \varphi(\cdot, t_1)|_{L_p[0, t_1]} < \frac{1}{n}.$$

Denoting

$$\varphi_n(s, t) = \tilde{\varphi}_n^{i_n}(s, t), \ (s, t) \in D_{t_1}, t \le t_1$$

$$\varphi_n(s, t_1) = \hat{\varphi}_n^{i_n}(s, t_1)$$

for $s \in [0, t_1]$ and replacing $\{\varphi_n\}$ by an appropriate subsequence, which is again denoted by $\{\varphi_n\}$, one obtains

$$(8) \quad \varphi_n \to \varphi \quad \text{in} \quad L_p(D_{t_1}), \varphi_n(\cdot, t_1) \to \varphi(\cdot, t_1) \quad \text{in} \quad L_p[0, t_1]$$

$$\text{and} \quad \varphi_n(\cdot, t_1) \to \varphi(\cdot, t_1) \quad \text{a.e. in} \quad [0, t_1].$$

By Fubini's theorem one obtains from (8): There exists $\{n_k\} \subset \{n\}$ such that

$$(9) \qquad \varphi_{n_k}(s, \cdot) \to \varphi(s, \cdot) \quad \text{in} \quad L_p[s, t_1] \quad \text{for a.e.} \quad s \in [0, t_1]$$

and

$$(10) \qquad \varphi_{n_k}(\cdot, t) \to \varphi(\cdot, t) \quad \text{in} \quad L_p[0, t] \quad \text{for a.e.} \quad t \in [0, t_1].$$

Denote $\{\varphi_{n_k}\}$ again by $\{\varphi_n\}$, let $\frac{kt_1}{2^{i_n}} = s_n^k$ and fix a natural number n.

For $s \in [s_n^k, s_n^{k+1})$, $k = 0, \cdots, 2^{i_n} - 1$ let

$$\Phi_n(s, t_1)(\tau) = \begin{cases} 0 & \text{if} \quad -h \le \tau < s_n^{k+1} \\ \varphi_n(s, \tau) & \text{if} \quad s_n^{k+1} \le \tau < t_1 \\ \varphi_n(s, t_1) & \text{if} \quad t_1 \le \tau \le a \end{cases}$$

Clearly for every n, $\Phi_n(\cdot, t_1)$ is a step function in $[0, t_1]$ whose values are in $L_p[-h, a]$. We shall show that

$$(11) \qquad \Phi_n(s, t_1) \to \Phi(s, t_1) \quad \text{for a.e.} \quad s \in [0, t_1].$$

From the definitions of Φ_n and Φ it follows for a fixed $s \in [s_n^k, s_n^{k+1})$, (k may vary with n)

$$\lim_n |\Phi_n(s, t_1) - \Phi(s, t_1)|_{L_p([-h,a],R^n)}^p = \lim_n \{\textstyle\int_s^{s_n^{k+1}} |\varphi(s,\tau)|^p d\tau + \int_{s_n^{k+1}}^{t_1} |\varphi_n(s,\tau) - \varphi(s,\tau)|^p d\tau + \int_{t_1}^{a} |\varphi_n(s,t_1) - \varphi(s,t_1)|^p d\tau\}.$$

Each of the last three integrals tends to 0 because of (iii) in Lemma 2, (9) and (8), which proves (11).

Next we shall show that $\Phi(\cdot, t_1)$ is p-integrable in $[0, t_1]$ and that

$$\lim_n \textstyle\int_0^{t_1} |\Phi_n(s, t_1) - \Phi(s, t_1)|^p ds = 0. \tag{12}$$

By (11) $\Phi(\cdot, t_1)$ is measurable as an a.e. limit of measurable functions. Also

$$\begin{aligned}\lim_n \textstyle\int_0^{t_1} |\Phi_n(s, t_1) - \Phi(s, t_1)|^p ds &= \lim_n \textstyle\int_0^{t_1}\int_{-h}^{a} |[\Phi_n(s, t_1) - \Phi(s, t_1)](\tau)|^p d\tau\, ds \\ &= \lim_n (\textstyle\int_0^{t_1}\int_s^{s_n^{k+1}} |\varphi(s,\tau)|^p d\tau\, ds \\ &\quad + \textstyle\int_0^{t_1}\int_{s_n^{k+1}}^{t_1} |\varphi_n(s,\tau) - \varphi(s,\tau)|^p d\tau\, ds \\ &\quad + \textstyle\int_0^{t_1}\int_{t_1}^{a} |\varphi_n(s,t_1) - \varphi(s,t_1)|^p d\tau\, ds).\end{aligned}$$

The first of the last three integrals tends to zero because of (iii), the second because of (8) and Fubini's theorem and the third because of (8), which proves (12).

Finally, we will show that for a.e. $\tau \in [-h, a]$

$$\textstyle\int_0^{t_1} \Phi(s, t_1)(\tau)\, ds = [\int_0^{t_1} \Phi(s, t_1)\, ds](\tau)$$

where $\int_0^{t_1} \Phi(s, t_1)\, ds$ is a suitable representative of its equivalence class.

It follows from (12) that there is a subsequence of $\{\int_0^{t_1} \Phi_n(s, t_1)\, ds\}_{n=1}^{\infty}$ which will again be denoted $\{\int_0^{t_1} \Phi_n(s, t_1)\, ds\}_{n=1}^{\infty}$ converging a.e. in $[-h, a]$ to $\int_0^{t_1} \Phi(s, t_1)\, ds$ i.e.

$$\lim_n [(\textstyle\int_0^{t_1} \Phi_n(s, t_1)\, ds)(\tau) = (\int_0^{t_1} \Phi(s, t_1)\, ds)(\tau) \tag{13}$$

for a.e. $\tau \in [-h, a]$.

As every $\Phi_n(\cdot, t_1)$ is a step function

$$\lim_n [(\textstyle\int_0^{t_1} \Phi_n(s, t_1)\, ds)(\tau)] = \lim_n \int_0^{t_1} \Phi_n(s, t_1)(\tau)\, ds \tag{14}$$

for a.e. $\tau \in [-h, a]$.

To finish the proof it is sufficient to show that

$$\lim_n \textstyle\int_0^{t_1} \Phi_n(s, t_1)(\tau)\, ds = \int_0^{t_1} \Phi(s, t_1)(\tau)\, ds \tag{15}$$

for a.e. $\tau \in [-h, a]$.

For $\tau \in [-h, 0]$ (15) is clear. If $\tau \in [0, t_1]$ for every n there exists of non-negative integer $k < 2^{i_n}$ such that $\tau \in \left[\frac{kt_1}{2^{i_n}}, \frac{k+1}{2^{i_n}} t_1\right)$. By the definition of Φ

and Φ_n one gets

$$\lim_n \int_0^{t_1} |[\Phi_n(s, t_1) - \Phi(s, t_1)](\tau)|\, ds$$

$$\leq \lim_n \left[\int_0^{s_n^k} |\varphi_n(s, \tau) - \varphi(s, \tau)|\, ds + \int_{s_n^k}^{\tau_k} |\varphi(s, \tau)|\, ds\right] = 0$$

in view of (10), Schwarz' inequality, (iii) and in view of the fact that $s_n^k \to \tau$ as $n \to \infty$. The case $\tau \in [t_1, a]$ follows from (8). Lemma 3 is proved.

LEMMA 4. *Let $x \in B$, $t \in [0, a]$ and let $f: B \times D \to \mathbb{R}^n$ have the properties (1) through (4). If F is defined by (6) then*

$$\int_0^t F(x, s, t]\, ds \tag{16}$$

exists and its class in $L_p[-h, a]$ contains the continuous function α:

$$\alpha(\tau) = 0 \quad \text{if} \quad \tau \in [-h, 0],$$

$$\alpha(\tau) = \int_0^m f(x, s, m)\, ds \quad \text{if} \quad \tau \in [0, a], \quad m = \min(\tau, t).$$

Proof. For a fixed $x \in B$ the function $F(x, ., .)$ has all the properties of Φ in Lemma 3 and it follows that the integral (16) exists as an element of $L_p[-h, a]$ whose equivalence class contains a function again denoted by $\int_0^t F(x, s, t)\, ds$ such that

$$\left(\int_0^t F(x, s, t)\, ds\right)(\tau) = \int_0^t F(x, s, t)(\tau)\, ds = \alpha(\tau)$$

for $\tau \in [-h, a]$ in view of (6).

Next it will be proved that α is continuous at every point $\tau_0 \in [-h, a]$. This is obvious if $\tau_0 \in [-h, 0) \cup (t, a]$. If $\tau_0 \in [0, t]$, $\tau \in (0, t)$ one has

$$\alpha(\tau) - \alpha(\tau_0) = \int_0^\tau f(x, s, \tau)\, ds - \int_0^{\tau_0} f(x, s, \tau_0)\, ds$$

$$= \int_0^{\tau_0} [f(x, s, \tau) - f(x, s, \tau_0)]\, ds$$

$$+ \int_{\tau_0}^{\tau} f(x, s, \tau)\, ds.$$

The first of the last two integrals tends to zero with $\tau \to \tau_0$ because of (3), (4) and the Lebesque dominated convergence theorem. The second does too in view of (4) and the absolute continuity of $\int M_x(\cdot)$.

Proof of Theorem 1. Let $x \in B$ be a solution of (5) in $[-h, a]$. Let $t \in [0, a]$, $\tau \in [-h, a]$, $m = \min(\tau, t)$, $v = \max(0, m)$. Then

$$y(t)(\tau) = x_t(\tau) = x(m) = z(m) + \int_0^v f(x, s, v)\, ds$$

$$= z_t(\tau) + \alpha(m) = w(t)(\tau) + \left[\int_0^t F(y(s), s, t)\, ds\right](\tau)$$

by Lemma 4 and by the fact that $F(x, s, t) = F(x_s, s, t)$. The last equality is a consequence of (1).

Proof of Theorem 2. Let y be a solution of (7). Obviously, there exists a (unique) $x \in B$ such that $y(t) = x_t$ for $t \in [0, a]$ iff

(17) $y(t)(\tau) = y(t)(t) = x(t)$ for $\tau \in [t, a]$

(18) $y(t)(\tau) = y(\tau)(\tau) = x(\tau)$ for $\tau \in [0, t]$ and

(19) $y(0)(\tau) = x(\tau)$ for $t \in [-h, 0)$.

(17) holds because

$$[\textstyle\int_0^t F(y(s), s, t)\, ds]_{[t,a]} = \int_0^t [F(y(s), s, t)]_{[t,a]}\, ds$$

is a constant function in $L_1([t, a], R^n)$. This follows from (7) and from the fact that the function $x \to x_I$ where $x \in L_1([-h, a], R^n)$ and $x_I \in L_1(I, R^n)$ is the restriction of x to a subinterval $I \subset [-h, a]$, is linear and bounded (cf. [1]).

To prove (18) use a similar argument and the fact that $[F(y(s), s, t) - F(y(s), s, \tau)]_{[-h,\tau]} = 0 \in L_1[-h, \tau]$. To finish the proof of Theorem 2 use (1) and Lemma 4 to obtain for

$$t \in [0, a], \quad \tau \in [-h, a], \quad m = \min(\tau, t), \quad v = \max(0, m)$$

$$x(m) = x_t(\tau) = y(t)\,(\tau) = [z_t + \textstyle\int_0^t F(x_s, s, t)\, ds](\tau)$$

$$= z(m) + \textstyle\int_0^v f(x, s, v)\, ds.$$

Remark 2. A similar result could be obtained if the continuity in (3) were replaced by right continuity.

Remark 3. If in (5) $z(t) = x(0)$ and $f: B \times [0, a] \to R^n$ one has a functional differential equation, which is equivalent to an ordinary differential equation with the right side $F(x, s) \in L_p[-h, a]$,

$$F(x, s)\,(\tau) = \begin{cases} 0 & \text{for } \tau \in [-h, s) \\ f(x, s) & \text{for } \tau \in [s, a] \end{cases}$$

(cf. [2], [3]).

Remark 4. Theorems 1 and 2 could also be proved by the method of [3].

UNIVERSIDAD NACIONAL AUTONOMA DE MEXICO
UNIVERSITY OF SOUTHERN CALIFORNIA, U.S.A.

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