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ON THE GROUP OF CONTACT DIFFEOMORPHISMS OF $\mathbb{R}^{2n+1}$

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1. Introduction and statement of the result

Let M be a smooth connected manifold of dimension $2n + 1$, $n \geq 1$, equipped with a contact form, i.e., a 1-form ω such that $\omega \wedge (d\omega)^n$ is everywhere nonzero. This form defines on M a contact structure [2]; the group $\mathrm{Diff}(M, \omega)$ of automorphisms of this structure, called here the group of contact diffeomorphisms, is the subgroup of the group $\mathrm{Diff}(M)$ of C^∞-diffeomorphisms of M whose elements preserve ω up to a positive function.

The Euclidian space $\mathbb{R}^{2n+1}$ has a contact form $\omega = \Sigma\, x_i dy_i + dz$ in coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$ on $\mathbb{R}^{2n+1}$.

Denote by $\mathrm{Diff}_+(\mathbb{R}^{2n+1}, \omega)$ the subgroup of $\mathrm{Diff}(\mathbb{R}^{2n+1}, \omega)$ consisting of those elements h isotopic to the identity through isotopies h_t belonging to $\mathrm{Diff}(\mathbb{R}^{2n+1}, \omega)$ and satisfying the following condition (x): there exist constants c_1 and c_2 such that: $c_1 \| x \| \leq \| h_t(x) \| \leq c_2 \| x \|$, for $t \in [0, 1]$ and $x \in \mathbb{R}^{2n+1}$ with $\| x \| \geq 1$, here $\| x \|^2 = x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 + z^2$.

In this paper, we prove the following

THEOREM: *The group* $\mathrm{Diff}_+(\mathbb{R}^{2n+1}; \omega)$ *is perfect.*

The key observation is that the proof McDuff [4] has given of the perfectness of the group $\mathrm{Diff}_+{}^r(\mathbb{R}^n)$, $r \geq 0$, of C^r orientation preserving diffeomorphisms of $\mathbb{R}^n$ can be adapted to our case. McDuff defines a convenient subgroup L of $\mathrm{Diff}_+{}^r(\mathbb{R}^n)$ and proves two facts about L that imply that $\mathrm{Diff}_+(\mathbb{R}^n)$ is perfect. Our contribution is to observe that these facts remain true for $\mathrm{Diff}_+(\mathbb{R}^{2n+1}, \omega)$.

The first named author wishes to thank McDuff for having explained to him the proof of [4] and for other helpful conversations, in particular, she has suggested the introduction of the condition (x) above. Discussions with Roberto Mariyón have been valuable to the second named author.

Finally let us point out that Ling [3] and Schweitzer (unpublished) have also proved that $\mathrm{Diff}_+{}^r(\mathbb{R}^n)$ is perfect.

2. Notation and proof of the result

From now on G will stand for $\mathrm{Diff}_+(\mathbb{R}^{2n+1}, \omega)$ and $(\bar{x}, \bar{y}, z)$ for $(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \in \mathbb{R}^{2n+1}$. Let $\alpha \in \mathbb{R}$, $\alpha > 0$, we define the "contact homothety" θ_α: $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ by: $(\bar{x}, \bar{y}, z) \mapsto (\alpha\bar{x}, \alpha\bar{y}, \alpha^2 z)$. Considering $\alpha_t = (1 - t) + t\alpha$, for $t \in [0, 1] = I$, we see that θ_{α_t} defines an isotopy in G from θ_α to the identity.

We wish to show that any σ in G is a product of commutators of elements of G. Let σ_t be an isotopy from σ to the identity in $\mathrm{Diff}(\mathbb{R}^{2n+1}, \omega)$ satisfying the

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condition (x) in the introduction with constants c_1 and c_2. See introduction. In the sequel, we will denote by θ and $\tilde{\theta}$, the homotheties θ_α and $\theta_{\sqrt{\alpha}}$, where $\alpha = (2\max(c_1, c_2 + 1))^2$.

If B is any subset of $\mathbb{R}^{2n+1}$, we denote by $\mathring{B}$, $\bar{B}$, and ∂B the topological interior, closure and boundary of B respectively.

The unit closed ball in $\mathbb{R}^{2n+1}$ will be denoted by D. Write $S = \partial D$; $D_i = \theta^i(D)$, $\tilde{D}_i = \tilde{\theta}^i(D)$ for $i \geq -1$. We fix V a "small" neighbourhood of S such that

$$\bar{\tilde{V}}_i \cap \bar{\tilde{V}}_j = \varnothing, \quad \text{when} \quad i \neq j, \quad \text{where} \quad \tilde{V} = \tilde{\theta}^i(V), \quad i \geq -1.$$

We consider also $V_i = \theta^i(V)$, $i \geq -1$. By construction: $\tilde{V}_{2i} = V_i$, $\tilde{V}_{2i+1} = \tilde{\theta}(V_i)$ and $D_{-1} = \theta^{-1}(D) \subsetneq \tilde{D}_{-1} = \tilde{\theta}^{-1}(D) \subsetneq D$.

Now we define L and state the analogues of McDuff's two facts. See introduction.

$$L = \{h \in G \mid \operatorname{supp}(h) \subset \mathbb{R}^{2n+1} - (D_{-1} \cup \textstyle\bigcup_{i=0}^{\infty} V_i)\};$$

here $\operatorname{supp}(h)$ denotes the support of h, i.e., the closure of $\{x \in \mathbb{R}^{2n+1} \mid h(x) \neq x\}$.

FACT I. *Let $[L, G]$ be the subgroup generated by the commutators $lgl^{-1}g^{-1}$ where $l \in L$ and $g \in G$, then $L \subseteq [L, G]$.*

FACT II. *There exist $f, h \in G$, $h \in L$ and f conjugate to an element of L such that $\sigma = h \cdot f$.*

Clearly these two facts imply the result.

Let A_i denote the annulus $D_i - \mathring{D}_{i-1}$, $i \geq 0$ and $A = A_1$. *We write* $\theta^{-1}|_A = \psi_0$: $A \to A_0$ and inductively, $\psi_i = \theta\psi_{i-1} = \theta^i\psi_0$: $A \to A_i$, $i \geq 1$.

If $g \in L$, let $g^i = g|_{A_i}: A_i \to A_i$ and $g_i = \psi_i^{-1} \cdot g^i \cdot \psi_i : A \to A$. So we can write the two sequences $(g^0, g^1, \cdots, g^n, \cdots)$ *or* $(g_0, g_1, \cdots, g_n, \cdots)$ each of them completely defining $g \in L$.

ASSERTION 1. *If $g = (g_0, g_1, \cdots, g_n, \cdots) \in L$, then*:

$$\theta g \theta^{-1} = (e, g_0, g_1, \cdots, g_n, \cdots)$$

where $e = \psi_1$ is the identity diffeomorphism of A.

This was proved by McDuff; for completeness, we give again her proof. To do this, we compute $(\theta g \theta^{-1})|_{A_{i+1}}$; if $x \in A_{i+1}$, then $\theta^{-1}(x) \in A_i$, hence $g\theta^{-1}(x) = g^i\theta^{-1}(x)$. Thus: $g\theta^{-1}(x) = (\psi_i \cdot g_i \cdot \psi_i^{-1})(\theta^{-1}(x))$. *So*

$$(\theta g \theta^{-1})(x) = [(\theta\psi_i) g_i (\theta\psi_i)^{-1}](x) = (\psi_{i+1} g_i \psi_{i+1}{}^{-1})(x),$$

i.e., $(\theta g \theta^{-1})^{i+1} = \psi_{i+1} g_i \psi_{i+1}{}^{-1}$. Thus $(\theta g \theta^{-1})_{i+1} = g_i$, for $i \geq 0$. Since $\theta^{-1}(A_0) \subset D_{-1}$, $(\theta g \theta^{-1})|_{A_0} =$ identity and $(\theta g \theta^{-1})_0 = e$. ■

Using the assertion 1 above we easily prove the fact *I*. Let $g \in L$, $g = (g_0, \cdots, g_n, \cdots)$. If $\bar{g} = (g_0, g_1 g_0, \cdots, g_n g_{n-1} \cdots g_0, \cdots)$, then: $\bar{g}\theta\bar{g}^{-1}\theta^{-1} = g$. Hence $L \subseteq [L, G]$. ■

Now we recall some results about contact manifolds that can be found in [2] and we prove some facts we need for the proof of Fact II.

Let M be a smooth manifold equipped with a contact form ω. There is on M a unique vector field E called the characteristic vector field of ω satisfying: $i(E)\omega = 1$, $i(E)d\omega = 0$, where $i(\ )$ is the interior product operation. If X is any vector field on M, then we can write in a unique way:

$$X = f \cdot E + H(X)$$

where f is the function $i(X))\omega$ and $H(X)$ is "horizontal", i.e., $i(H(X)\omega = 0$. The vector field fE is called the vertical component of X.

It has been observed in [2] that the map $H \overset{\bar{\alpha}}{\mapsto} i(H)\, d\omega$ is an isomorphism from the horizontal vector fields into the semi-basic 1-forms.

If f is a C^∞ real valued map, then one can associate to it a unique contact vector field

$$\beta(f) = f \cdot E + \lambda(f)$$

where $\lambda(f) = \bar{\alpha}^{-1}((i(E)\, df)\omega - df)$.

Recall that X is said to be a contact vector field if $L_X\omega = \rho \cdot \omega$, where L_X stands for the Lie derivative in the direction X, and ρ is some non negative function on M.

PROPOSITION. *The vertical component of a contact vector field completely determines it; i.e., if X_1 and X_2 are two contact vector fields with the same vertical component, then they are equal.*

Proof. $X_i = f \cdot E + H(X_i)$, $i = 1, 2$, we want to prove that $H(X_1) = H(X_2)$. We have:

$$L_{X_i}\omega = \rho_i \cdot \omega \qquad \text{for some functions} \qquad \rho_i.$$

Using the classical Cartan's formula, we have

$$\begin{aligned} L_{X_i}\omega &= di(X_i)\omega + i(X_i)\, d\omega \\ &= di(fE)\omega + i(X_i)\, d\omega \\ &= df + i(X_i)\, d\omega. \end{aligned}$$

Since $i(E)i(X_i)\, d\omega = -i(X_i)i(E)\, d\omega = 0$, we get

$$\begin{aligned} i(E)L_{X_i}\omega &= i(E)\, df \\ &= i(E)(\rho_i \cdot \omega) = \rho_i. \end{aligned}$$

Now $\bar{\alpha}(H(X_i)) = i(H(X_i))\, d\omega = L_{H(X_i)}\omega - di(H(X_i))\omega = L_{H(X_i)}\omega$. But:

$$\begin{aligned} L_{H(X_i)}\omega &= L_{X_i}\omega - L_{fE}\omega = \rho_i \cdot \omega - df \\ &= (i(E)df)\omega - df. \end{aligned}$$

So: $\bar{\alpha}(H(X_1)) = \bar{\alpha}(H(X_2))$. Thus $H(X_1) = H(X_2)$. ■

If U is an open set in $\mathbb{R}^{2n+1}$ and X is a vector field satisfying the contact condition in U, then X is determined in U by its vertical component.

The following is the "contact" analogue of the Palais-Cerf lemma.

LEMMA. *Let M be a smooth manifold equipped with a contact form ω. Let $h_t \in \mathrm{Diff}_+(M, \omega)$ be a contact isotopy; let $F \subset M$ be any closed subset of M, $U \subseteq \bar{U} \subseteq W$ two open subsets such that $\bigcup_{t\in I} h_t(F) \subset U$. Then there exists a contact isotopy $\bar{h}_t$ such that $\bar{h}_t = h_t$ on F and* $\mathrm{supp}(\bar{h}_t) \subset W$.

Proof. First define a family of vector fields $\dot{h}_t$ on M by:

$$\dot{h}_t(x) = \frac{dh_t}{dt}(h_t^{-1}(x)), \qquad x \in M.$$

On $M \times I$, consider the vector field

$$\eta(x, t) = \dot{h}_t(x) + \partial/\partial t.$$

Let φ be a bump function on $M \times I$ with $\varphi = 1$ on $U \times I$ and $\varphi = 0$ outside $W \times I$ and define

$$\tilde{\eta}(x, t) = \varphi(x, t)\dot{h}_t(x) + \partial/\partial t.$$

This defines an isotopy $\tilde{h}_t$ on M. Let $u_t = i(\dot{\tilde{h}}_t)\omega$, following the notations above, we associate to u_t a family of contact vector fields $\beta(u_t)$ that defines, in turn, a contact isotopy $\bar{h}_t$. By construction, $\dot{\bar{h}}_t = \beta(u_t)$. Since $\beta(u_t)$ vanishes outside W, $\mathrm{supp}(\bar{h}_t) \subset W$ and by our proposition above, we have: $\dot{\bar{h}}_t = \beta(u_t) = \dot{\tilde{h}}_t$ on W. So $\bar{h}_t|_U = \tilde{h}_t|_U = h_t|_U$. ■

As a first application of this lemma, we prove:

ASSERTION 2. *If $f \in G$ is the identity on $N \cup (\bigcup_{i=0}^{\infty} V_i)$, with N an open neighbourhood of the origin, then f is conjugate to an element of L.*

Proof. Let N_0 be a small ball such that $N_0 \subset N \cap D_{-1}$. Let $\gamma \geq 1$, such that: $D_{-1} \subset \theta_\gamma(N_0) \subset D - V$. Considering the isotopy θ_{γ_t}, $\gamma_t = (1 - t) + t\gamma$, we see that

$$\bigcup_{t\in I} \theta_{\gamma_t}(N_0) \subset \theta_\gamma(N_0).$$

Thus by the lemma, there exists $\mu \in G$, $\mu = \theta_\gamma$ on N_0, $\mathrm{supp}(\mu) \subset D$. It follows that $\mu f \cdot \mu^{-1} \in \mathrm{L}$. ■

We are now in a position to give a

Proof of Fact II. Our condition (x) implies that if V is "small" enough, there exist open subsets U, U_i, $i \geq 0$, with:

1) $\bigcup_{t\in I} \sigma_t(D_{-1}) \subset U \subsetneq \tilde{D}_{-1} - \tilde{V}_{-1}$

2) $\bigcup_{t\in I} \sigma_t(\tilde{V}_{2i}) \subset U_i \subsetneq K_i = (\tilde{D}_{2i+1} - \tilde{D}_{2i-1}) - (\tilde{V}_{2i+1} \cup \tilde{V}_{2i-1})$.

By the lemma, there exist f_i, $\varphi \in G$ such that:

$$\mathrm{supp}(\varphi) \subset \tilde{D}_{-1} - \tilde{V}_{-1}; \quad \varphi = \sigma \text{ on } D_{-1}$$

$$\mathrm{supp}(f_i) \subset K_i; \qquad f_i = \sigma \text{ on } \tilde{V}_{2i}, \quad i \geq 0.$$

Let $\bar{f}$ denote the element of G whose restriction to K_i is f_i. Then $f = \varphi \cdot \bar{f} \in G$ has the following properties:

$$f = \begin{cases} \text{identity on } \tilde{V}_i, & \text{for} \quad i \quad \text{odd} \\ \sigma \text{ on } D_{-1} \cup \bigcup_{k=0}^{\infty} \tilde{V}_{2k}. \end{cases}$$

To see that $f = \sigma$ on D_{-1}, it is enough to observe that $D_{-1} \subsetneq \tilde{D}_{-1}$.

Define now $h \in G$ to be $\sigma \cdot f^{-1}$. Thus,

$$h = \begin{cases} \text{identity on } D_{-1} \cup (\bigcup_{i=0}^{\infty} \tilde{V}_{2i}) \\ \sigma \text{ on } \tilde{V}_{2i+1}, \quad i \geq 0. \end{cases}$$

Since $\tilde{V}_{2k} = V_k$, $k \geq 0$, it follows that $h \in L$.

Choose a point y in $\tilde{V}_{-1}$ and an open neighbourhood N_y of y in $\tilde{V}_{-1}$. Thus N_y lies outside the support of f. By the transitivity theorem of Boothby [1], there exists a $\rho \in G$, $\text{supp}(\rho) \subset D - V$ such that $\rho(y) = 0 \in \mathbb{R}^{2n+1}$. Thus $N_0 = (\bar{\theta}^{-1} \cdot \rho)(N_y \cap \text{supp}(\rho))$ is a neighbourhood of $0 \in \mathbb{R}^{2n+1}$. It is clear that $\mu_0 f \mu_0^{-1}$, where $\mu_0 = \bar{\theta}^{-1} \cdot \rho$, is the identity on $N_0 \cup \bigcup_{i=0}^{\infty} \tilde{V}_{2i} = N_0 \cup \bigcup_{i=0}^{\infty} V_i$. By our assertion 2, $\mu_0 f \mu_0^{-1}$ is conjugate to an element in L. ■

COROLLARY. *The normal subgroup generated by G in* Diff$(\mathbb{R}^{2n+1}, \omega)$ *is a perfect group.*

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