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e-ULTRAFILTERS AND THE STONE-ČECH COMPACTIFICATION

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1. Introduction

e-filters and e-ultrafilters were introduced in [3], p. 33. However, the construction of the Stone-Čech compactification as the space of e-ultrafilters is not given there. In the present paper we define e-filters in a slightly different way to that of [3], we define the space $\mathcal{M}_e$ of e-ultrafilters with the natural hull-kernel topology. We consider e-filters as lattice ideals of a certain lattice in order to be able to introduce prime e-filters and prove the Stone-Čech properties of $\mathcal{M}_e$. In Section 6 we indicate the connection between our e-filters and those of [3], which are a special kind of z-filter. We shall see that e-ultrafilters in the sense of [3] are distinguished among all the z-filters not by the topology on X, but by the uniformity of finite coverings of X by cozero-sets. All the necessary properties of e-filters for the proof of the Stone-Čech properties of $\mathcal{M}_e$ are of the lattice-theoretic type and are independent of their relations with z-filters.

Section 2 contains preliminary facts about lattices and definitions used repeatedly in the paper. In Section 3 we define e-filters and the Stone-Čech compactification $\mathcal{M}_e$ as the space of e-ultrafilters. Section 4 describes the Stone-Čech compactification of X as a completion $\hat{X}$ by using minimal Cauchy filters. For technical reasons it is preferable to use a uniformity of coverings. In Section 5 the correspondence between $\mathcal{M}_e$ and $\hat{X}$, which commutes with natural embeddings, is defined explicitly and shown to be a homeomorphism. In Section 6 the correspondence between $\mathcal{M}_e$ and the space $\mathcal{M}_z$ of z-ultrafilters is described (using $\hat{X}$ as a bridge). This elucidates the correspondence indicated in [3], p. 33, 2L15. Section 7 points out the connection between $\mathcal{M}_e$ and the maximal ideal space of $C^*(X)$.

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2. Notations and Definitions

By a *completely regular space* X we mean a topological T_1-space such that for every closed set $Y \subset X$ and for each $x \in X - Y$ there is a continuous $f{:}X \to [0, 1]$ such that $f(x) = 0, f \mid Y = 1$. A subset Y of X is called a *zero-set* (*z-set*, for brevity), if there is a continuous $f{:}X \to \mathbb{R}$ such that $Y = f^{-1}(0)$. Notice that f can be chosen as a bounded, continuous function. Define $C^*(X) = \{f{:}X \to \mathbb{R}{:}$ f is bounded, continuous$\}$. $C^*(X)$ is a commutative real Banach algebra with unit, if the addition, multiplication and multiplication by scalars are defined pointwise and if the norm is defined by $\| f \| = \sup\{|f(x)|{:}x \in X\}$. If Y is a compact Hausdorff space, put $C(Y) = \{f{:}\, Y \to \mathbb{R}{:}\, f$ is continuous$\}$. Then $C(Y) = C^*(Y)$.

We shall need some basic facts of lattice theory. Recall that a set L with a partial order $\leq$ is called a *lattice* if $x \vee y = \sup\{x, y\}$, $x \wedge y = \inf\{x, y\}$ exist and belong to L for each $x, y \in L$. A lattice L is called *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for each $x, y, z \in L$. A proper subset A of a lattice L is called a *lattice ideal* if:

(2.1) $$x \vee y \in A \quad \text{whenever} \quad x, y \in A,$$

(2.2) $$\text{if} \quad x \in A, y \in L, y \leq x, \quad \text{then} \quad y \in A.$$

A lattice ideal A is called *prime* if $x \wedge y \in A$ implies $x \in A$ or $y \in A$, for each $x, y \in L$. A lattice ideal is *maximal* if it is not properly contained in any other lattice ideal.

We shall use the following simple propositions (we include the proofs, which are adapted from [4], for the sake of completeness).

PROPOSITION 2.3. *Let L be a distributive lattice. A lattice ideal $A \subset L$ is maximal if and only if for each $x, y \in L$ such that $x \notin A$, $x \leq y$, there is $a \in A$ such that $y = x \vee a$.*

Proof. Suppose A is a maximal lattice ideal of L. Take $x, y \in L$ such that $x \notin A, y \leq x$. If $x = y$, then, taking arbitrary $a \in A$, we get $y = x \vee (x \wedge a)$ and $x \wedge a \in A$, by (2.2). Suppose that $x \leq y$, $x \neq y$ and that

(2.4) $$y \neq x \vee a, \quad \text{for each} \quad a \in A.$$

Then

(2.5) $$\text{there is no} \quad a \in A \quad \text{such that} \quad y \leq x \vee a.$$

For if there is $a \in A$ such that $y \leq x \vee a$, then $x \vee (y \wedge a) = (x \vee y) \wedge (x \vee a) = x \vee y = y$, and $y \wedge a \in A$. Define $B = \{z \in L\colon$ there is $a \in A$ such that $z \leq x \vee a\}$. B is a lattice ideal of L (B is proper, because $y \notin B$) such that $A \subset B$, $x \in B$, $x \notin A$. Since A is maximal, (2.4) cannot hold. Conversely, suppose that A is not maximal. Hence there is a lattice ideal B strictly bigger than A. Take $x \in B - A$, $z \in L - B$. Put $y = x \vee z$. Clearly, $y \geq x$, $y \notin B$ (because $y \geq z$, $z \notin B$). If there is $a \in A$ such that $y = x \vee a$, then $y \in B$, which is a contradiction. q.e.d.

PROPOSITION 2.6. *Let L be a distributive lattice. Every maximal lattice ideal is prime.*

Proof. Let A be a maximal lattice ideal of L. Take $x, y \in L - A$ and put $z = x \vee y$. By (2.3), there are $a_1, a_2 \in A$ such that $z = x \vee a_1$, $z = y \vee a_2$. Hence $(x \wedge y) \vee (a_1 \vee a_2) = (x \vee a_1 \vee a_2) \wedge (y \vee a_1 \vee a_2) = (z \vee a_2) \wedge (z \vee a_1) = z \vee (a_1 \wedge a_2) \geq z \geq x$. If $x \wedge y \in A$, then $x \in A$ (by (2.1), (2.2)). Hence $x \wedge y \notin A$. q.e.d.

Denote by $\mathscr{M}(L)$ the set of all maximl lattice ideals of a distributive lattice L. For $x \in L$ define $^\circ x = \{A \in \mathscr{M}(L) : x \in A\}$. By (2.6);

$$(2.7) \qquad {}^{\circ}(x \wedge y) = {}^{\circ}x \cup {}^{\circ}y.$$

Since maximal ideals are proper, $\bigcap_{x \in L} {}^{\circ}x = \emptyset$. Hence $\{{}^{\circ}x : x \in L\}$ is a base of closed sets in $\mathscr{M}(L)$, which defines a hull-kernel type topology on $\mathscr{M}(L)$. We denote this topology by $h - k$. Notice that:

$$(2.8) \qquad (\mathscr{M}(L), h - k) \quad \text{is a compact space.}$$

One can adapt the proof of [8] 11.6. But, in general, $(\mathscr{M}(L), h - k)$ need not be Hausdorff.

Let L, L' be two lattices. A $1 - 1$ mapping $T: L \to L'$ of L onto L' is called a *lattice isomorphism* if $T(x \wedge y) = (Tx) \wedge (Ty)$ and $T(x \vee y) = (Tx) \vee T(y)$ for each $x, y \in L$. One verifies immediately that:

PROPOSITION 2.9. *If L, L' are two distributive lattices and if $T: L \to L'$ is a lattice isomorphism, then the mapping $\mathscr{M}(L) \to \mathscr{M}(L')$ given by $A \mapsto T(A)$ is a homeomorphism.*

$(\mathscr{M}(L), h - k)$ will be called the *maximal lattice ideal space* of L.

Let $\mathscr{A}$ be a family of subsets of an arbitrary set Z. We define $\mathscr{A}^+ = \{B \subset Z$: there is $A \in \mathscr{A}$ such that $A \subset B\}$.

3. *e*-Ultrafilters as Maximal Lattice Ideals

Let X be a completely regular space. For $f, g \in C^*(X)$ put $f \vee g = \max(f, g)$, $f \wedge g = \min(f, g)$. If $f \in C^*(X)$, $\alpha > 0$, define $Z_f(\alpha) = \{x \in X : |f(x)| \leq \alpha\}$ and put $Z_f = \{(\alpha, Z_f(\alpha)) : \alpha > 0\}$. Z_f is a function from the open interval $(0, \infty)$ to the collection of all z-sets in X. Denote by Im Z_f the image $\{Z_f(\alpha) : \alpha > 0\}$ of $(0, \infty)$ under Z_f.

For $f, g \in C^*(X)$ define:

$$(3.1) \qquad Z_f \leq Z_g \quad \text{is and only if} \quad Z_g(\alpha) \subset Z_f(\alpha) \quad \text{for each} \quad \alpha > 0,$$

$$(3.2) \qquad Z_f \vee Z_g = \{(\alpha, Z_f(\alpha) \cap Z_g(\alpha)) : \alpha > 0\},$$

$$(3.3) \qquad Z_f \wedge Z_g = \{(\alpha, Z_f(\alpha) \cup Z_g(\alpha)) : \alpha > 0\}.$$

Observe that for each $f, g \in C^*(X)$, $f, g > 0$:

$$(3.4) \qquad Z_f \leq Z_g \quad \text{if and only if} \quad f \leq g,$$

$$(3.5) \qquad Z_f \vee Z_g = Z_{f \vee g},$$

$$(3.6) \qquad Z_f \wedge Z_g = Z_{f \wedge g}.$$

Moreover,

$$(3.7) \qquad Z_f = Z_{|f|} \quad \text{for each} \quad f \in C^*(X),$$

$$(3.8) \qquad Z_0 \leq Z_f \quad \text{for each} \quad f \in C^*(X).$$

Define $\mathscr{C} = \{Z_f : f \in C^*(X)\}$. By (3.7), $\mathscr{C} = \{Z_f : f \in C^*(X), f \geq 0\}$. The relation $\leq$ (defined in (3.1)) is a partial order in $\mathscr{C}$. It follows from (3.4), (3.5), (3.6) that $(\mathscr{C}, \leq, \vee, \wedge)$ is a distributive lattice.

A non-empty proper subset ξ of $\mathscr{C}$ is called an *e-filter* if:

(3.9) $Z_f \vee Z_g \in \xi$ whenever $Z_f, Z_g \in \xi$,

(3.10) if $Z_f \in \mathscr{C}$ is such that $Z_f \leq Z_g$ for some $Z_g \in \xi$, then $Z_f \in \xi$,

(3.11) $Z_{\alpha f} \in \xi$ whenever $\alpha \geq 0, Z_f \in \xi$.

We have defined an e-filter to be a lattice ideal of $\mathscr{C}$ ((3.9), (3.10)) which satisfies additionally the "homogeneity" condition (3.11). An e-filter ξ is called *prime* if it is prime as a lattice ideal of $\mathscr{C}$, i.e. if $Z_f \wedge Z_g \in \xi$ implies $Z_f \in \xi$ or $Z_g \in \xi$. Notice that Z_0 belongs to each e-filter (by (3.8)).

Observe that:

(3.12) A non-empty subset ξ of $\mathscr{C}$ satisfying (3.9), (3.10), (3.11) is an e-filter (i.e. $\xi \neq \mathscr{C}$) if and only if ξ does not contain any $Z_f \in \mathscr{C}$ such that $\varnothing \in \operatorname{Im} Z_f$.

Proof: Suppose there is $Z_f \in \mathscr{C}$ such that $\varnothing \in \operatorname{Im} Z_f$ and $Z_f \in \xi$ (we may assume $f \geq 0$). Hence $f \geq \alpha$; thus by (3.4), $Z_\alpha \leq Z_f$. By (3.10), $Z_\alpha \in \xi$ and by (3.11), $Z_r \in \xi$ for each constant function $r \geq 0$. Take arbitrary $g \in C^*(X)$, $g \geq 0$. Then there is $r \geq 0$ such that $g \leq r$. By (3.4), $Z_g \leq Z_r$ and $Z_r \in \xi$. By (3.10), $Z_g \in \xi$. Thus $\xi = \mathscr{C}$. q.e.d.

(3.13) Let η be a lattice ideal of $\mathscr{C}$. Then $\xi = \{Z_h \in \mathscr{C}$: there is $\alpha > 0$, $Z_f \in \eta$ such that $Z_h \leq Z_{\alpha f}\}$ is an e-filter containing η.

The proof is straightforward (observe $Z_{\alpha f} \vee Z_{\alpha g} = Z_{\alpha(f \vee g)}$, $\alpha > 0$, $f, g \geq 0$).

(3.14) Every maximal lattice ideal of $\mathscr{C}$ is an e-filter (this is a consequence of (3.13)).

(3.14) justifies the following definition: An *e-ultrafilter* is a maximal lattice ideal of $\mathscr{C}$.

PROPOSITION 3.15.
(a) *Every e-ultrafilter is prime.*
(b) *An e-filter ξ is an e-ultrafilter if and only if each $Z_f \in \mathscr{C}$ such that $\varnothing \notin \operatorname{Im}(Z_f \vee Z_g)$ for all $Z_g \in \xi$, belongs to ξ.*

Proof:
(a) has a purely lattice-theoretic character (see (2.6)).
(b) Suppose ξ is an e-ultrafilter and take $Z_f \in \mathscr{C}$ such that $\varnothing \notin \operatorname{Im}(Z_f \vee Z_g)$ for all $Z_g \in \xi$. Then $\eta = \{Z_h \in \mathscr{C}$: there is $Z_g \in \xi$ such that $Z_h \leq Z_f \vee Z_g\}$ is a lattice ideal of $\mathscr{C}$ such that $Z_f \in \eta$, $\xi \subset \eta$. Since ξ is maximal, $\eta = \xi$ and $Z_f \in \xi$.
Conversely suppose that ξ is contained in an e-ultrafilter η and that ξ satisfies the assumption. Take $Z_f \in \eta$. Then $Z_f \vee Z_g \in \eta$ for each $Z_g \in \xi$ (by (3.9)). By (3.12) and (3.14), $\varnothing \notin \operatorname{Im}(Z_f \vee Z_g)$, for each $Z_g \in \xi$. Hence $Z_f \in \xi$ and $\eta = \xi$. q.e.d.

Denote by $\mathscr{M}_e$ the set of all e-ultrafilters on X. By the results of Section 2, $\mathscr{M}_e$ has a natural $h - k$ topology. We shall show that $(\mathscr{M}_e, h - k)$ is the Stone-Čech compactification of X.

PROPOSITION 3.16. *$(\mathcal{M}_e, h - k)$ is a compact Hausdorff space.*

Proof: $(\mathcal{M}_e, h - k)$, being the maximal lattice ideal space of $\mathscr{C}$, is compact (2.8). To prove that $(\mathcal{M}_e, h - k)$ is Hausdorff, take $\xi, \eta \in \mathcal{M}_e$, $\xi \neq \eta$. Let $Z_f \in \xi - \eta$. By (3.15) (b), there is $g \in \eta$ such that $Z_f(\alpha) \cap Z_g(\alpha) = \varnothing$ for some $\alpha > 0$. Again by (3.15) (b), $Z_g \notin \xi$. We may assume that $f, g \geq 0$. Since $Z_f(\alpha), Z_g(\alpha)$ are z-sets, we find two continuous functions $f_1, g_1 : X \to [0, 1]$ such that

$$f_1^{-1}(0) \supset f^{-1}([\alpha, \infty)),\ g_1^{-1}(0) \supset g^{-1}([\alpha, \infty)),\ f_1^{-1}(1) \supset f^{-1}(0),\ g_1^{-1}(1) \supset g^{-1}(0).$$

Hence $f_1 \wedge g_1 = 0$, $f_1 \vee f \geq \beta$, $g_1 \vee g \geq \gamma$ for some constants β, γ. Put $f_2 = f \wedge g_1$, $g_2 = g \wedge f_1$. Hence ${}^\circ(Z_{f_2}) \cup {}^\circ(Z_{g_2}) = {}^\circ(Z_{f_2} \wedge Z_{g_2}) = {}^\circ(Z_{f_2 \wedge g_2}) = {}^\circ(Z_0) = \mathcal{M}_e$. Moreover,

$$\xi \in {}^\circ(Z_{f_2}),\ \eta \in {}^\circ(Z_{g_2}) \quad \text{and} \quad {}^\circ(Z_{f_2}),\ {}^\circ(Z_{g_2})$$

are closed. It remains to show that $\xi \notin {}^\circ(Z_{g_2})$, $\eta \notin {}^\circ(Z_{f_2})$. Suppose $\xi \in {}^\circ(Z_{g_2})$. Hence $Z_{g_2} = Z_{f_1} \wedge Z_g \in \xi$. Since ξ is prime ((3.15) (a)), $Z_{f_1} \in \xi$ or $Z_g \in \xi$. But we know that $Z_g \notin \xi$. Hence $Z_{f_1} \in \xi$. Thus $Z_{f_1} \vee Z_f \in \xi$ (by (3.9)). But this is impossible, because $f_1 \vee f \geq \beta$, hence $Z_{f_1}(\beta) \cap Z_f(\beta) = \varnothing$ (apply (3.15) (b)). The proof that $\eta \notin {}^\circ(Z_{f_2})$ is similar. q.e.d.

PROPOSITION 3.17. *The mapping $x \mapsto \xi_x$, where $\xi_x = \{Z_f \in \mathscr{C} : f(x) = 0\}$ is a dense embedding of X into $\mathcal{M}_e$.*

Proof: It is clear that ξ_x is an e-filter. To see that ξ_x is an e-ultrafilter, take $Z_f \in \mathscr{C}$ such that $f(x) \neq 0$. Then there is $\alpha > 0$ such that $x \notin Z_f(\alpha)$. Hence there is a continuous $g : X \to [0, 1]$ such that $g(x) = 0$, $g \mid Z_f(\alpha) = 1$. Thus $Z_g \in \xi_x$ and $Z_g(\beta) \cap Z_f(\beta) = \varnothing$ for a sufficiently small $\beta < \alpha$. By (3.15) (b), ξ_x is an e-ultrafilter. The proof that $x \mapsto \xi_x$ is a homeomorphism onto its image is straightforward. Since $\bigcap_{x \in X} \xi_x = Z_0$, $\{\xi_x : x \in X\}$ is dense in $\mathcal{M}_e$. q.e.d.

For an e-filter ξ on X we define the *trace* of ξ by $\operatorname{Tr} \xi = \{Z_f(\alpha) : Z_f \in \xi, \alpha > 0\}$. $\operatorname{Tr} \xi$ is a filter-base. For if $Z_f(\alpha), Z_g(\beta) \in \operatorname{Tr} \xi$, then, assuming $\alpha \leq \beta$, we get $Z_f(\alpha) \cap Z_g(\alpha) \subset Z_f(\alpha) \cap Z_g(\beta)$. Since $Z_f \vee Z_g \in \xi$, it follows that $Z_f(\alpha) \wedge Z_g(\alpha) \in \operatorname{Tr} \xi$. Hence $(\operatorname{Tr} \xi)^+$ is a filter on X.

Our next purpose is to prove the universal extension property of $(\mathcal{M}_e, h - k)$. We need the following:

PROPOSITION 3.18. *Let ξ be a prime e-filter on a completely regular space X. If $x \in X$ is a point of accumulation of* $(\operatorname{Tr} \xi)^+$, *then* $(\operatorname{Tr} \xi)^+$ *converges to x.*

Proof: Take an open neighbourhood U of x. By the complete regularity of X, there is a continuous $f : X \to [0, 1]$ such that $f(x) = 0$, $f(X - U) = \{1\}$. Define two functions:

$$h = \begin{cases} 0 & \text{on} \quad X - f^{-1}([0, \tfrac{1}{2})) \\ \tfrac{1}{2} - f & \text{on} \quad f^{-1}([0, \tfrac{1}{2}]) \end{cases},$$

$$g = \begin{cases} f - \tfrac{1}{2} & \text{on} \quad X - f^{-1}([0, \tfrac{1}{2})) \\ 0 & \text{on} \quad f^{-1}([0, \tfrac{1}{2}]) \end{cases}.$$

Then $h \wedge g = 0$ and $Z_h \wedge Z_g = Z_0 \in \xi$. Since $x \in \bigcap \mathrm{Tr}\, \xi$, Z_h cannot belong to ξ. Since ξ is prime, $Z_g \in \xi$. Thus $Z_g(\alpha) \in \mathrm{Tr}\, \xi$ for each $\alpha > 0$. But $Z_g(\frac{1}{4}) \subset U$, thus $U \in (\mathrm{Tr}\, \xi)^+$. q.e.d.

Let Y be a compact Hausdorff space and let $\phi: X \to Y$ be a continuous function. We shall define its "extension" to $\mathcal{M}_e$. For $\xi \in \mathcal{M}_e$ put $\phi_*(\xi) = \{Z_f \in \mathscr{C}_Y : Z_{f\circ\phi} \in \xi\}$. Here $\mathscr{C}_Y$ denotes the collection of all Z_f, $f \in C(Y)$. Since $(f \wedge g) \circ\phi = (f\circ\phi) \wedge (g\circ\phi)$ for each $f, g \in C(Y)$, one can show that $\phi_*(\xi)$ is a prime e-filter on Y. Since Y is compact, $\mathrm{Tr}\, \phi_*(\xi)$ converges to a point $\bar{\phi}(\xi)$. Define $\bar{\phi}: \mathcal{M}_e \to Y$ by $\xi \mapsto \bar{\phi}(\xi)$. We claim that $\bar{\phi}$ is the "extension" of ϕ to $\mathcal{M}_e$. For take $x \in X$. Then $\phi_*(\xi_x) = \{Z_f \in \mathscr{C}_Y : f(\phi(x)) = 0\}$. By the definition, $\{\bar{\phi}(\xi_x)\} = \bigcap \mathrm{Tr}\, \phi_*(\xi)$. Hence, for each $f \in C(Y)$, $f(\phi(x)) = 0$ implies $f(\bar{\phi}(\xi_x)) = 0$. Since Y is completely regular, $\phi(x) = \bar{\phi}(\xi_x)$. It remains only to prove that $\bar{\phi}$ is continuous. Take $\xi \in \mathcal{M}_e$ and an open neighbourhood U of $\bar{\phi}(\xi)$. Then there is a continuous function $f: Y \to [0, 1]$ such that $f(\bar{\phi}(\xi)) = 0$, $f(Y - U) = \{1\}$. Defining, as in the proof of (3.18), two functions:

$$h = \begin{cases} 0 & \text{on} \quad Y - f^{-1}([0, \frac{1}{2})), \\ \frac{1}{2} - f & \text{on} \quad f^{-1}([0, \frac{1}{2}]), \end{cases}$$

$$g = \begin{cases} f - \frac{1}{2} & \text{on} \quad Y - f^{-1}([0, \frac{1}{2})), \\ 0 & \text{on} \quad f^{-1}([0, \frac{1}{2}]), \end{cases}$$

we get $h \wedge g = 0$ and $Z_{h\circ\phi} \notin \xi$. Hence $\xi \notin {}^\circ(Z_{h\circ\phi})$. Now take $\eta \in \mathcal{M}_e - {}^\circ(Z_{h\circ\phi})$. Hence $Z_{h\circ\phi} \notin \eta$, consequently, $Z_h \notin \phi_*(\eta)$. However $Z_g \wedge Z_h = Z_0 \in \phi_*(\eta)$. Since $\phi_*(\eta)$ is a prime e-filter on Y, it follows that $Z_g \in \phi_*(\eta)$. Therefore $Z_g(\frac{1}{4}) \in \mathrm{Tr}\, \phi_*(\eta)$. Hence $\{\bar{\phi}(\eta)\} = \bigcup \mathrm{Tr}\, \phi_*(\eta) \subset Z_g(\frac{1}{4}) \subset U$, which proves that $\bar{\phi}$ is continuous. q.e.d.

4. The Stone-Čech Compactification as a Completion

In this section we describe how to construct the Stone-Čech compactification of a completely regular space X as the completion of the uniform space $(X, \{\mathscr{U}_\lambda\}_\lambda)$, where $\{\mathscr{U}_\lambda\}_\lambda$ is the family of all finite coverings of X by cozero-sets. Although it is well-known that the Stone-Čech compactification of X can be obtained as a completion (see, for example, [2]), there is no readily available account of the construction using the minimal Cauchy filters on X and the finite coverings of X by cozero-sets. Thus we shall describe it briefly, referring to Section 5, Chapter II of [5] for the notation on coverings.

Zero-sets, or z-sets, in a space X are defined in Section 2. Their complements are called *cozero-sets*. Zero-sets and cozero-sets have the following properties in an arbitrary space X:

(4.1) if A, B are zero-sets, so are $A \cap B, A \cup B$,

(4.2) if U, V are cozero-sets, so are $U \cup V, U \cap V$,

(4.3) if A, B are disjoint zero-sets, then there is a continuous function

$$f: X \to [0, 1] \quad \text{such that} \quad A = f^{-1}(0), \quad B = f^{-1}(1),$$

(4.4) if $U_1, U_2, \cdots, U_n$ is a covering of X by cozero-sets, then there is a covering $A_1, A_2, \cdots, A_n$ of X by zero-sets such that $A_i \subset U_i$ for each i.

For the remainder of this section, let X be a completely regular space and let $\{\mathcal{U}_\lambda\}_\lambda$ be the family of all finite coverings of X by cozero-sets. It is easily shown that $\{\mathcal{U}_\lambda\}_\lambda$ is a uniformity compatible with the topology on X; i.e., the following properties are satisfied,

$$\forall\, \lambda, \mu, \exists \nu : \mathcal{U}_\nu < \mathcal{U}_\lambda \wedge \mathcal{U}_\mu, \tag{4.5}$$

$$\forall\, \lambda, \exists \mu : \mathcal{U}_\mu{}^* < \mathcal{U}_\lambda, \tag{4.6}$$

and, furthermore, $\{\{\mathrm{St}(x, \mathcal{U}_\lambda)\}_\lambda : x \in X\}$ is a local neighbourhood system for the topology of X. In order to verify (4.6) one uses (4.4) to find a delta-refinement $\mathcal{U}_\nu$ of the given covering $\mathcal{U}_\lambda$. One then repeats the process to find a delta-refinement $\mathcal{U}_\mu$ of $\mathcal{U}_\nu$, and $\mathcal{U}_\mu$ is automatically a star-refinement of $\mathcal{U}_\lambda$.

A *Cauchy filter* η (in X) is a filter in X having the property that $\eta \cap \mathcal{U}_\lambda \neq \emptyset$ for each λ. The Cauchy filters η, η' are *equivalent* if $\eta \cap \eta'$ is also a Cauchy filter. The equivalence class of η is denoted by $[\eta]$. It is easily shown that $[\eta]$ has a canonical representative, namely $\bigcap [\eta]$. That is, $\bigcap [\eta]$ is a Cauchy filter equivalent to η. A *minimal Cauchy filter* (in X) is a Cauchy filter which contains no other Cauchy filter. Thus the minimal Cauchy filters are precisely those of the form $\bigcap [\eta]$. In [5] the completion of X is formed from the collection of equivalence classes of Cauchy filters. For our purposes it is necessary to form it from the collection of minimal Cauchy filters. We now describe this.

We denote by $\hat{X}$ the collection of all minimal Cauchy filters in X. We define $\hat{U} = \{\eta \in X : U \in \eta\}$ for each cozero-set U in X, and $\hat{\mathcal{U}}_\lambda = \{\hat{U} : U \in \mathcal{U}_\lambda\}$ for each $\mathcal{U}_\lambda$. Since each minimal Cauchy filter contains an element of $\mathcal{U}_\lambda$, it follows that $\hat{\mathcal{U}}_\lambda$ is a covering of $\hat{X}$. Also it is easily shown that

$$\hat{\mathcal{U}}_\lambda \wedge \hat{\mathcal{U}}_\mu = (\mathcal{U}_\lambda \wedge \mathcal{U}_\mu)\hat{}\,, \tag{4.7}$$

$$\text{if} \quad \mathcal{U}_\mu{}^* < \mathcal{U}_\lambda \quad \text{then} \quad (\hat{\mathcal{U}}_\mu)^* < \hat{\mathcal{U}}_\lambda. \tag{4.8}$$

Thus $\{\hat{\mathcal{U}}_\lambda\}_\lambda$ is a uniformity on $\hat{X}$. For each $x \in X$, the neighbourhood filter η_x of x is a minimal Cauchy filter, and the function defined by $x \mapsto \eta_x$ is a dense uniform embedding of X in $\hat{X}$. It is easily shown that $\hat{X}$ is Hausdorff. In order to show that $\hat{X}$ is complete, one shows that every minimal Cauchy filter in $\hat{X}$ converges.

In the previous paragraph $\hat{X}$ is described as the completion of X. However since the uniformity $\{\hat{\mathcal{U}}_\lambda\}_\lambda$ consists of finite coverings, $\hat{X}$ is totally bounded and, being complete, is therefore compact. Furthermore, one can show that any continuous mapping $f : X \to Y$, where Y is a compact Hausdorff space, is uniformly continuous. Since X is dense in $\hat{X}$, it therefore has an extension $\hat{f} : \hat{X} \to Y$. Thus $\hat{X}$ is in fact the Stone-Čech compactification of X.

The following characterization of minimal Cauchy filters, which is used in proving that $\hat{X}$ is complete, is often needed in the sequel.

PROPOSITION 4.9. *A Cauchy filter* η *is a minimal Cauchy filter if and only if* $\{\mathrm{St}(B, \mathscr{U}_\lambda): B \in \eta, \mathscr{U}_\lambda$ *arbitrary*$\}$ *is a base for* η.
If η is a minimal Cauchy filter and $A \in \eta$, it follows from (4.9) that $A \supset \mathrm{St}(B, \mathscr{U}_\lambda)$, for some $B \in \eta$ and some $\mathscr{U}_\lambda$. Hence $B \cap \mathrm{St}(X - A, \mathscr{U}_\lambda) = \varnothing$. It follows that $C = X - \mathrm{St}(X - A, \mathscr{U}_\lambda)$ is a zero-set having the property that $B \subset C \subset A$, and consequently $C \in \eta$. This shows that the collection of zero-sets in η is a base for η. Since this collection is necessarily a z-filter (for the definition, see [3]), we call it the *z-filter base* of η. The following proposition is an immediate consequence of this and (4.9):

PROPOSITION 4.10. *A Cauchy filter* η *is a minimal Cauchy filter if and only if* $\{\mathrm{St}(B, \mathscr{U}_\lambda): B$ *is a zero-set in* η, $\mathscr{U}_\lambda$ *arbitrary*$\}$ *is a base for* η.

Finally we prove the following proposition, which is often overlooked.

PROPOSITION 4.11. $\{\hat{U}: U$ *is a cozero-set in* $X\}$ *is a basis for the uniform topology on* $\hat{X}$.

Proof: We can indeed define $\hat{A} = \{\eta \in \hat{X}: A \in \eta\}$ for any subset A of X. We will prove that $\hat{A}$ is always open in X. Let $\eta \in \hat{A}$. Since $A \in \eta$ and η is a minimal Cauchy filter, by (4.9) there is some $B \in \eta$ and some $\mathscr{U}_\lambda$ such that $\mathrm{St}(B, \mathscr{U}_\lambda) \subset A$. We claim that $\mathrm{St}(\eta, \hat{\mathscr{U}}_\lambda) \subset \hat{A}$. In order to see this, let $\eta \in \hat{U}$, where $\hat{U} \in \hat{\mathscr{U}}_\lambda$. Then $U \in \eta$ and so $U \cap B \neq \varnothing$; i.e., $U \subset \mathrm{St}(B, \mathscr{U}_\lambda)$. Thus $U \subset A$, which implies that $\hat{U} \subset \hat{A}$. This shows that $\mathrm{St}(\eta, \hat{\mathscr{U}}_\lambda) \subset \hat{A}$; i.e., η is an interior point of $\hat{A}$.
From this it follows that $\bigcup\{\{\mathrm{St}(\eta, \hat{\mathscr{U}}_\lambda)\}_\lambda: \eta \in \hat{X}\}$ is a base for the uniform topology on $\hat{X}$, and hence $\{\hat{U}: U$ is a cozero-set in $X\}$ is a base for the uniform topology on X. q.e.d.

At the beginning of this section we indicated that there is no account of the Stone-Čech compactification as a completion in which precisely our point of view is taken. The presentations in the following sources partially coincide with ours. In [5] uniformities are treated as families of coverings, but the completion is formed from the equivalences classes of Cauchy filters. In [1] the completion is formed from the minimal Cauchy filters and proposition (4.9) is proved, the uniformity being considered as a family of relations. The most readily available source in which the Stone-Čech compactification is considered as a completion is [2]. Here the completion is formed from the equivalence classes of Cauchy filters and uniformities, or uniform structures, are considered as families of relations.

5. The Correspondence between $\mathscr{M}_e$ and $\hat{X}$

The purpose of this section is to explicitly define the correspondence between $\mathscr{M}_e$ and $\hat{X}$ which commutes with the embeddings of X and to show that it is a homeomorphism. The correspondence and its inverse are given by propositions (5.2) and (5.3), respectively. These propositions also show that the traces of the e-ultrafilters involved are the z-filter bases of the respective minimal Cauchy filters. In proposition (5.4) it is shown that the correspondence has the required properties.

Throughout this section X is a completely regular space furnished with the uniformity $\{\mathscr{U}_\lambda\}_\lambda$ of all finite coverings of cozero-sets. The notation is the same as in previous section.

We need the following preliminary result.

PROPOSITION 5.1. *If η is a minimal Cauchy filter and A is a z-set in η, then there is a continuous function $f:X \to [0, 1]$ such that $A = Z_f(\frac{1}{2})$ and* $\mathrm{Im}\, Z_f \subset \eta$. *Furthermore $f^{-1}(0) \in \eta$.*

Proof: Let A be a zero-set in a minimal Cauchy filter η. By (4.10) there is a z-set B in η and a finite covering $\mathscr{U}_\lambda$ of X by cozero-sets such that $A \supset \mathrm{St}(B, \mathscr{U}_\lambda)$. Since B, $X - \mathrm{St}(B, \mathscr{U}_\lambda)$ are disjoint z-sets, by (4.3) there is a continuous function $g:X \to [0, \frac{1}{2}]$ such that $g^{-1}(0) = B$, $g^{-1}(\frac{1}{2}) = X - \mathrm{St}(B, \mathscr{U}_\lambda)$. Supposing that $h:X \to [\frac{1}{2}, 1]$ is a continuous function such that $A = Z_h(\frac{1}{2})$, we define

$$f(x) = \begin{cases} g(x), & x \in A, \\ h(x), & x \in X - \mathrm{St}(B, \mathscr{U}_\lambda). \end{cases}$$

Since $g(x) = \frac{1}{2} = h(x)$ for $x \in A - \mathrm{St}(B, \mathscr{U}_\lambda)$, $f:X \to [0, 1]$ is a continuous function. Also $A = Z_f(\frac{1}{2})$ and, since $B = f^{-1}(0)$, $\mathrm{Im}\, Z_f \subset \eta$ and $f^{-1}(0) \in \eta$.

PROPOSITION 5.2. *If ξ is an e-ultrafilter, then* $(\mathrm{Tr}\, \xi)^+$ *is a minimal Cauchy filter, and* $\mathrm{Tr}\, \xi$ *is the z-filter base of* $(\mathrm{Tr}\, \xi)^+$.

Proof: It is shown just before proposition (3.18) that $\mathrm{Tr}\, \xi$ is a filter-base. Thus $(\mathrm{Tr}\, \xi)^+$ is a filter. In order to prove that $(\mathrm{Tr}\, \xi)^+$ is a Cauchy filter, let U_1, $U_2, \cdots, U_n$ be a finite covering of X by cozero-sets. It must be shown that $(\mathrm{Tr}\, \xi)^+$ contains some U_i. By (4.4) there is a finite covering $A_1, A_2, \cdots, A_n$ of X by zero-sets such that $A_i \subset U_i$ for each i. By (4.3) there is a continuous function $f_i:X \to [0, 1]$ such that $f_i^{-1}(0) = A_i$, $f_i^{-1}(1) = X - U_i$ for each i. Observe that $Z_{f_1} \wedge Z_{f_2} \wedge \cdots \wedge Z_{f_n} = Z_0$, where 0 is the zero function on X. Since $Z_0 \in \xi$ and since ξ, being an e-ultrafilter, is prime by (3.5 (a)), it follows that some $Z_{f_i} \in \xi$. Since, for example $Z_{f_i}(\frac{1}{2}) \subset U_i$, this proves that $U_i \in (\mathrm{Tr}\, \xi)^+$.

We use (4.9) to show that $(\mathrm{Tr}\, \xi)^+$ is a minimal Cauchy filter. Let $Z_f(\alpha)$ be an element in the base $\mathrm{Tr}\, \xi$ of $(\mathrm{Tr}\, \xi)^+$, where f is a non-negative function such that $Z_f \in \xi$ and $\alpha > 0$. Let $\mathscr{U}_\lambda = \{U, V\}$ be the finite covering of X by cozero-sets defined by $U = f^{-1}([0, 3\alpha/4))$, $V = f^{-1}((\alpha/2, \infty)$. Then $\mathrm{St}(Z_f(\alpha/4), \mathscr{U}_\lambda) = U \subset Z_f(\alpha)$. Since $Z_f(\alpha/4)$ also belongs to $\mathrm{Tr}\, \xi$, it follows from (4.9) that $(\mathrm{Tr}\, \xi)^+$ is a minimal Cauchy filter.

Finally we show that $\mathrm{Tr}\, \xi$ is the z-filter base of $(\mathrm{Tr}\, \xi)^+$. Since $\mathrm{Tr}\, \xi$ consists of zero sets, it is contained in the z-filter base of $(\mathrm{Tr}\, \xi)^+$. In order to prove the converse inclusion, let A be a zero set in $(\mathrm{Tr}\, \xi)^+$. Since $(\mathrm{Tr}\, \xi)^+$ is a minimal Cauchy filter, by (5.1) there is a continuous function $f:X \to [0, 1]$ such that $A = Z_f(\frac{1}{2})$ and $\mathrm{Im}\, Z_f \subset (\mathrm{Tr}\, \xi)^+$. We use (3.15 (b)) to show that $Z_f \in \xi$. Let $Z_g \in \xi$. Since $\mathrm{Im}\, Z_f$, $\mathrm{Im}\, Z_g \subset (\mathrm{Tr}\, \xi)^+$ and $(\mathrm{Tr}\, \xi)^+$ is a filter, $Z_f(\alpha) \cap Z_g(\alpha) \neq \varnothing$ for each $\alpha > 0$. This means that $\varnothing \notin \mathrm{Im}(Z_f \vee Z_g)$. Since Z_g is an arbitrary element of ξ and ξ is an e-ultrafilter, it follows from (3.15 (b)) that $Z_f \in \xi$. Hence $A = Z_f(\frac{1}{2}) \in \mathrm{Tr}\, \xi$. That is, the z-filter base of $(\mathrm{Tr}\, \xi)^+$ is contained in, and hence equal to, $\mathrm{Tr}\, \xi$. q.e.d.

PROPOSITION 5.3. *If η is a minimal Cauchy filter, then $\xi = \{Z_f : \operatorname{Im} Z_f \subset \eta\}$ is an e-ultrafilter, and* Tr ξ *is the z-filter base of* η.

Proof: We first show that ξ is an e-filter. Since $\operatorname{Im} Z_0 = \{X\}$, where 0 is the zero function on X, $Z_0 \in \xi$. Thus ξ is non-empty. Now take $Z_f, Z_g \in \xi$. Then $Z_f(\alpha), Z_g(\alpha) \in \eta$ for each $\alpha > 0$, and so $(Z_f \vee Z_g)(\alpha) = Z_f(\alpha) \cap Z_g(\alpha) \in \eta$ for each $\alpha > 0$. Thus $Z_f \vee Z_g \in \xi$. Now take $Z_f \leq Z_g$ and $Z_g \in \xi$. Then $Z_f(\alpha) \supset Z_g(\alpha)$ and $Z_g(\alpha) \in \eta$ for each $\alpha > 0$. Hence $Z_f(\alpha) \in \eta$ for each $\alpha > 0$. That is, $\operatorname{Im} Z_f \subset \eta$ and so $Z_f \in \xi$. Now take $Z_f \in \xi$ and $\beta > 0$. Then $Z_{\beta f}(\alpha) = Z_f(\alpha/\beta) \in \eta$ for each $\alpha > 0$. That is, $\operatorname{Im} Z_{\beta f} \subset \eta$, which implies that $Z_{\beta f} \in \xi$. Finally, if $Z_f \in \xi$ then $\varnothing \notin \operatorname{Im} Z_f$, because $\varnothing \notin \eta$. This proves that ξ is an e-filter, by (3.9)–(3.12).

We use (3.15 (b)) to show that ξ is an e-ultrafilter. Let Z_f have the property that $\varnothing \notin \operatorname{Im}(Z_f \vee Z_g)$ for each $Z_g \in \xi$. We have to show that $Z_f \in \xi$.

We define $\eta' = \bigcup \{\operatorname{Im}(Z_f \vee Z_g) : Z_g \in \xi\}$. We shall show that η' is a filter base which generates η. From this it will easily follow that $Z_f \in \xi$.

In order to see that η' is a filter base, observe first that $\varnothing \notin \eta'$ by definition. Now let $(Z_f \vee Z_{g_1})(\alpha_1)$, $(Z_f \vee Z_{g_2})(\alpha_2)$ be arbitrary elements of η', where $Z_{g_1}, Z_{g_2} \in \xi$ and $\alpha_1, \alpha_2 > 0$. By (3.2), the intersection of these elements contains $(Z_f \vee Z_{g_1} \vee Z_{g_2})(\alpha_3)$, where $\alpha_3 = \min(\alpha_1, \alpha_2)$. Since ξ is an e-filter, $Z_{g_1} \vee Z_{g_2} \in \xi$ and this implies that $(Z_f \vee Z_{g_1} \vee Z_{g_2})(\alpha) \in \eta'$. Thus η' is a filter base.

Now let η'' denote the filter generated by η. We show that $\eta = \eta''$. Firstly, if A is a z-set in η, then by (5.1) there is a continuous function $g: X \to [0, 1]$ such that $A = Z_g(\frac{1}{2})$ and $\operatorname{Im} Z_g \subset \eta$. Thus $Z_g \in \xi$ and so $\operatorname{Im}(Z_f \vee Z_g) \subset \eta'$. Thus $Z_f(\frac{1}{2}) \cap Z_g(\frac{1}{2}) \in \eta'$ by (3.2), from which it follows that $A \in \eta''$. This proves that the z-filter base of η is contained in η'', and so $\eta \subset \eta''$. From this it follows that η'' is a Cauchy filter, and so to prove that $\eta = \eta''$ it suffices to prove that η'' is a minimal Cauchy filter. For this purpose we use (4.9). Let $(Z_f \vee Z_g)(\alpha)$ be an element in the base η' of η'', where g is a non-negative function such that $Z_g \in \xi$ and $\alpha > 0$. We may suppose that f is also a non-negative function, in which case $Z_f \vee Z_g = Z_h$, where $h = f \vee g$, by (3.5). Let $\mathscr{U}_\lambda = \{U, V\}$ be the finite covering of X by cozero sets defined by $U = h^{-1}([0, 3\alpha/4))$, $V = h^{-1}((\alpha/2, \infty))$. Then $\operatorname{St}(Z_h(\alpha/4), \mathscr{U}_\lambda) = U \subset Z_h(\alpha) = (Z_f \vee Z_g)(\alpha)$. Since $Z_h(\alpha/4)$ also belongs to η', it follows from (4.9) that η'' is a minimal Cauchy filter. Thus $\eta = \eta''$.

We can now show that ξ is an e-ultrafilter. Since $Z_0 \in \xi$ (where 0 is the zero-function on X), $\operatorname{Im}(Z_f \vee Z_0) \subset \eta' \subset \eta$. However $\operatorname{Im} Z_f = \operatorname{Im}(Z_f \vee Z_0)$ and $\operatorname{Im} Z_f \subset \eta$ implies that $Z_f \in \xi$. This proves that ξ is an e-ultrafilter by (3.15 (b)).

Finally we show that Tr ξ is the z-filter base of η. It is clear that Tr ξ is contained in the z-filter base of η. In order to prove the converse inclusion, let A be a zero-set in η. By (5.1) there is a continuous function $g: X \to [0, 1]$ such that $A = Z_g(\frac{1}{2})$ and $\operatorname{Im} Z_g \subset \eta$. Since $Z_g \in \xi$ it follows that $A \in \operatorname{Tr} \xi$. Thus the z-filter base of η is contained in, and hence is equal to, Tr ξ. q.e.d.

PROPOSITION 5.4. *The function defined by*

$$\xi \mapsto (\operatorname{Tr} \xi)^+,$$

where ξ is an e-ultrafilter, is a homeomorphism between $\mathscr{M}_e$ and $\hat{X}$ which commutes with the embeddings of X.

Proof: By (5.2) the range of the function described is contained in $\hat{X}$. We shall first show that this function commutes with the embeddings of X in $\mathcal{M}_e$ and in $\hat{X}$.

The embeddings of X in $\mathcal{M}_e$ and in $\hat{X}$ are given by

$$x \mapsto \xi_x = \{Z_f : f(x) = 0\},$$

$$x \mapsto \eta_x = \text{the } ngbd \text{ filter of } x,$$

respectively, according to (3.17) and Section 4. We have to prove that $\eta_x = (\text{Tr } \xi_x)^+$. Let A be a zero set in η_x. By (5.1) there is a continuous function $f: X \to [0, 1]$ such that $A = Z_f(\frac{1}{2})$, Im $Z_f \subset \eta_x$ and $f^{-1}(0) \in \eta_x$. The last condition implies that $f(x) = 0$. Hence $Z_f \in \xi_x$ and $A \in \text{Tr } \xi_x$. Since the zero sets in η_x form a base for η_x, it follows that $\eta_x \subset (\text{Tr } \xi_x)^+$. The converse inclusion can also be proved directly. However, the equality $\eta_x = (\text{Tr } \xi_x)^+$ follows immediately from the inclusion $\eta_x \subset (\text{Tr } \xi_x)^+$, because $(\text{Tr } \xi_x)^+$ is a minimal Cauchy filter by (5.2).

In order to show that the function from $\mathcal{M}_e$ to $\hat{X}$ is a homeomorphism, it is enough to show that it is one-one, onto and continuous, because $\mathcal{M}_e$ is compact and X is Hausdorff.

In order to show that the function is one-one, let ξ, ξ' be distinct e-ultrafilters. Since ξ is an e-ultrafilter and $\xi \neq \xi'$, there is an element $Z_f \in (\xi - \xi')$. Since ξ' is an e-ultrafilter, by (3.15) (b) there is an element $Z_g \in \xi$ such that $\phi \in \text{Im}(Z_f \vee Z_g)$. This means by (3.2) that $Z_f(\alpha) \cap Z_g(\alpha) = \emptyset$ for some $\alpha > 0$. Since Tr ξ, Tr ξ' are filter bases (this is established before (3.18)), it follows that Tr $\xi \neq$ Tr ξ', and hence that $(\text{Tr } \xi)^+ \neq (\text{Tr } \xi')^+$.

In order to show that the function is onto, let η be a minimal Cauchy filter. By (5.3) $\xi = \{Z_f : \text{Im } Z_f \subset \eta\}$ is an e-ultrafilter having the property that Tr ξ is the z-filter base of η. However, by (5.2) Tr ξ is also the z-filter base of $(\text{Tr } \xi)^+$. It follows that $\eta = (\text{Tr } \xi)^+$.

In order to prove that the function is continuous at $\xi \in \mathcal{M}_e$, let U be a basic open neighbourhood of $(\text{Tr } \xi)^+$ in $\hat{X}_e$, where U is a cozero-set in X. Since $(\text{Tr } \xi)^+ \in U$, it follows that $U \in (\text{Tr } \xi)^+$. Thus $U \supset Z_g(\alpha)$, for some $Z_g \in \xi$ and some $\alpha > 0$. Since $X - U$, $Z_g(\alpha)$ are disjoint z-sets, by (4.3) there is a continuous function $f: X \to [0, 2\alpha]$ such that $f^{-1}(0) = X - U$, $f^{-1}(2\alpha) = Z_g(\alpha)$. Since $Z_f(\alpha) \cap Z_g(\alpha) = \emptyset$, it follows from (3.9) and (3.12) that $Z_f \notin \xi$. Thus $\mathcal{M}_e - {}^\circ(Z_f)$ is a neighbourhood of ξ in $\mathcal{M}_e$, and we show that it is mapped into $\hat{U}$. Let $\xi' \in (\mathcal{M}_e - {}^\circ(Z_f)$. Since $Z_f \notin \xi'$ and ξ' is an e-ultrafilter, it follows from (3.15 (b)) that $\emptyset \in \text{Im}(Z_f \vee Z_h)$ for some $Z_h \in \xi'$. Consequently $Z_f(\beta) \cap Z_h(\beta) = \emptyset$ for some $\beta > 0$, by (3.2). Since $X - U \subset Z_f(\beta)$, it follows that $U \supset Z_h(\beta)$. Hence $U \in (\text{Tr } \xi')^+$. That is $(\text{Tr } \xi')^+ \in U$. q.e.d.

6. The Connection Between $\mathcal{M}_e$ and $\mathcal{M}_z$

In Section 5 we have established the natural correspondence between $\mathcal{M}_e$ and $\hat{X}$. In the present section we indicate the correspondence between $\hat{X}$ and the space $\mathcal{M}_z$ of all z-ultrafilters on X (X is always a completely regular space). In this way we want to point out that the correspondence between $\mathcal{M}_e$ and $\mathcal{M}_z$ relies essentially on the uniformity on X described in Section 4.

We refer to [3] for the theory of z-filters. We preserve the notation of the previous sections. The proofs are not included, because they are not difficult. However, this is precisely due to the definition of $\hat{X}$ as stated in Section 4. Let us point out that the proofs of results of this section become more complicated if one uses the definition of uniformities in terms of relations or the definition of the completion in terms of equivalence classes of Cauchy filters.

PROPOSITION 6.1. *If* $\eta \in \hat{X}$ *then* $\zeta = \{A \subset X$: *A is a z-set which meets every* $B \in \eta\}$ *is a z-ultrafilter.*

Observe that it is enough to prove only the intersection property of ζ.

PROPOSITION 6.2. *If* $\zeta \in \mathcal{M}_z$ *then* $\eta = \{\mathrm{St}(A, \mathcal{U}_\lambda)$: $A \in \zeta$, $\mathcal{U}_\lambda$ *is arbitrary*$\}^+$ *belongs to* $\hat{X}$.

PROPOSITION 6.3. *The mapping of* $\hat{X}$ *in* $\mathcal{M}_z$ *given by* $\eta \mapsto \zeta$, *where* ζ *is defined as in* (6.1), *is a homeomorphism of* $\hat{X}$ *onto* $\mathcal{M}_z$ *which commutes with the embeddings of X into* $\hat{X}$ *and* $\mathcal{M}_z$.

Composing the homeomorphism $\mathcal{M}_e \to \hat{X}$ (Section 5) and the above homeomorphism $\hat{X} \to \mathcal{M}_z$ we get a natural correspondence between $\mathcal{M}_e$ and $\mathcal{M}_z$. One can show, however, that the basic closed sets in $\mathcal{M}_e$ are z-sets, but a basic closed set $A \subset \mathcal{M}_z$ is a z-set if and only if $A = \bigcap_{n=1}^{\infty} U_n$, where U_n is a cozero-set in $\mathcal{M}_z$. Therefore, the above correspondence, although natural, does not carry over basic closed sets. (See also a remark on the bottom of p. 105 of [3]).

Finally let us observe that if ξ is an e-filter, then Tr ξ is an e-filter as defined in [3], p. 33. Conversely, if $\mathfrak{F}$ is an e-filter in the sense of [3], p. 33, (i.e. $\mathfrak{F}$ is a z-filter such that $\mathfrak{F} = \bigcup \{\mathrm{Im}\, Z_f : \mathrm{Im}\, Z_f \subset \mathfrak{F}\}$), then $\xi = \{Z_f : \mathrm{Im}\, Z_f \subset \mathfrak{F}\}$ is an e-filter. In [3], 2L15 it is indicated that Tr ξ, for $\xi \in \mathcal{M}_e$, is contained in a unique z-ultrafilter and, conversely, that every z-ultrafilter contains a unique Tr ξ, $\xi \in \mathcal{M}_e$. By the previous results we see that the bridge between Tr ξ ($\xi \in \mathcal{M}_e$) and the unique z-ultrafilter which contains Tr ξ is provided by the minimal Cauchy filter $(\mathrm{Tr}\, \xi)^+$ in X. (Compare (5.2), (6.1)).

7. The Connection between e-Ultrafilters and Maximal Ideals of $C^*(X)$

Let X be a completely regular space. Define $C_+^*(X) = \{f \in C^*(X) : f \geq 0\}$. $C_+^*(X)$ with the natural partial order ($f, g \in C_+^*(X)$; $f \leq g$ if $g - f \geq 0$) and with the operations $\vee$, $\wedge$ defined at the beginning of Section 3, is a distributive lattice. It follows from (3.4), (3.5), (4.6), (3.7) that the lattices $C_+^*(X)$ and $\mathscr{C}$ are lattice-isomorphic. Denote by $\mathcal{M}_+$ the maximal lattice ideal space of $C_+^*(X)$. With the natural $h - k$ topology, $\mathcal{M}_+$ is a compact space (2.8). By (2.9), $\mathcal{M}_+$ is naturally homeomorphic with $\mathcal{M}_e$. One can show that $\mathcal{M}_+$ is the Stone-Čech compactification of X. The constructions of Stone-Čech compactifications by using Banach lattices can be found in [6], [7].

Consider now the real Banach algebra $C^*(X)$ with unit – the constant function 1. Let $\mathcal{M}$ be the maximal ideal space of $C^*(X)$ equipped with the usual hull-kernel ($h - k$) topology; namely, the base of closed sets of $\mathcal{M}$ is given by the collection of $f_* = \{J \in \mathcal{M} : f \in J\}$, $f \in C^*(X)$. ($\mathcal{M}$, $h - k$) is a compact space (see [8] 11.6) and, since $C^*(X)$ is a regular Banach algebra, $(\mathscr{M}, h - k)$ is Hausdorff (see [8], 22 for the definitions and proofs).

One can show that if $J \in \mathscr{M}$, then

(7.1) $\quad f \in J$ if and only if $|f| \in J$; for each $f \in C^*(X)$,

(7.2) $\quad$ if $f \in J, g \in C^*(X), \quad 0 \leq g \leq f,$ then $g \in J$.

By (7.1) and (7.2) one can prove that the mapping $\mathscr{M} \to \mathscr{M}_+$ defined by $J \mapsto J_+ = J \cap C_+{}^*(X)$ is a $1 - 1$ mapping of $\mathscr{M}$ onto $\mathscr{M}_+$. Its inverse $\mathscr{M}_+ \to \mathscr{M}$ is defined by $A \mapsto^+ A = \{f \in C^*(X) : |f| \in A\}$. This mapping is a homeomorphism, because it maps the basic closed set $f_* = |f|_*$ (use (7.1) on $\mathscr{M}$ precisely onto the basic closed set $°|f|$ on $\mathscr{M}_+$, for each $f \in C^*(X)$. The natural embedding of X into $\mathscr{M}$ is given by $x \mapsto M_x = \{f \in C^*(X) : f(x) = 0\}$.

Hence we have indicated the correspondence between $\mathscr{M}$ and $\mathscr{M}_e$, which is a homeomorphism, preserving natural embeddings of X into $\mathscr{M}$ and $\mathscr{M}_e$.

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