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FIXED POINTS UNDER THE ACTION OF UNIPOTENT ELEMENTS OF SL_n IN THE FLAG VARIETY*

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1. Introduction and notation

All varieties and algebraic groups are taken over an algebraically closed field k of arbitrary characteristic. G is a connected semisimple group, V its variety of unipotent elements and $\mathscr{B}$ the variety of Borel subgroups of G.

Let $X = \{(B, x) \mid x \in B\} \subset \mathscr{B} \times V$. Springer basically proved that if G has a separable universal covering, then the projection $f: X \to V$ is a desingularization [8]. We study the fibers of it when $G = SL_n$ and $\mathscr{B} = \mathfrak{F}$, the variety of full flags on an n dimensional space, we denote these fibers by $\mathscr{B}_u = \mathfrak{F}_u$, identifying the latter with the flags fixed by u.

For an arbitrary semisimple group there exist characterizations of the regular and subregular elements of V in terms of these fibers [7].

In our case, let $u \in SL_n$, we write $u \sim \lambda = (\lambda_1, \cdots, \lambda_a)$ if the Jordan blocks of u are of sizes $\lambda_1 \geq \cdots \geq \lambda_a$. We also write $\lambda' = \mu = (\mu_1, \cdots, \mu_b)$ for the partition of n dual to λ, (u is regular when $a = 1$, subregular when $a = 2$ and $\lambda_2 = 1$).

We associate a Young diagram with λ_i blocks in the ith column to the element u, so that there are dim $\ker(u - 1)^j$ blocks in the first j rows of the diagram. To such a diagram there correspond standard tableaux obtained by placing the numbers $1, \cdots, n$ one in each block so that they increase up and to the left.

THEOREM 1.1. *The number of irreducible components of $\mathfrak{F}_u$ is the number of tableaux for the Young diagram of u. These components are equidimensional.*

This is proved in [8]. We give a proof of it in section 2 from which a description of the components of $\mathfrak{F}_u$ is also given.

In section 3, the components of $\mathfrak{F}_u$ of the form P/B with P parabolic are determined and it is found that they exhaust the components exactly when u is regular, subregular or the identity.

In section 4, the elements of one hook type are introduced and it is proved that their components are rational, smooth and finite unions of orbits under the centralizer. Their intersections turn out to have the same properties, their dimensions are also calculated.

Finally, section 5 presents an example in SL_6 of a component with singularities.

N. Spaltenstein [5], [6] has independently obtained results overlapping with the present work and with [8], in particular Theorem 1.1 and the example in section 5.

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* This is essentially the author's Ph.D. thesis written at UCLA under R. Steinberg in 1976.

2. A description of the components of $\mathfrak{F}_u$

In this section, V is an n dimensional vector space over k, $G = SL(V)$, $\tilde{G} = GL(V)$ and $u \in \operatorname{End}(V)$ is nilpotent, $1 + u$ being unipotent in G.

We assume $u \sim \lambda = (\lambda_1, \cdots, \lambda_a)$. One can choose spaces $V_1, \cdots, V_b$ such that $V = V_1 \oplus \cdots \oplus V_b$ with $u^b = 0$ and $u^{b-1} \neq 0$ which satisfy the conditions $\ker u^i = V_1 \oplus \cdots \oplus V_i$ for all i, with a sequence of linear maps each given by $u: V_b \to V_{b-1} \to \cdots \to V_2 \to V_1 \to 0$, all injective except for the last one. Thus, $\operatorname{im} u^i = u^i(V_{i+1} \oplus \cdots \oplus V_b)$. Here $\dim V_i = \mu_i$, $\dim \ker u^i = \mu_1 + \cdots + \mu_i$, $\dim \operatorname{im} u^i = \mu_{i+1} + \cdots + \mu_b$, with $\mu = \lambda'$.

Let $m(\lambda)$ be the number of components of $\mathfrak{F}_u$, $Z(u)$ the centralizer of u in G and $\tilde{Z}(u)$ the centralizer in $\tilde{G}$.

$\tilde{Z}(u)$ then fixes each $\ker u^i$. We define $H_u = \tilde{Z}(u) \cap \tilde{G}^{V_1} \cap \cdots \cap \tilde{G}^{V_b}$, where $\tilde{G}^{V_i}$ is the subset of $\tilde{G}$ fixing V_i. Following Steinberg we prove

THEOREM 2.1. (a) *H_u is connected. If we identify V_{i+1} with a subspace of V_i via u and if $x \in \tilde{G}^{V_1} \cap \cdots \cap \tilde{G}^{V_b}$, then $x \in \tilde{Z}(u)$ iff the action of x on V_i extends the action of x on V_{i+1}, in that case x stabilizes the flag $u^{b-1}V_b \subset \cdots \subset u^2V_3 \subset uV_2 \subset V_1$ of V_1 of type $(\mu_b, \cdots, \mu_1)$ and x acts on V_i via the pull back by $(u^{i-1})^{-1}$ on $u^{i-1}V_i$.*

(b) *If $\mu_i > \mu_{i+1}$ and the line $L \subset (\ker u \cap \operatorname{im} u^{i-1}) = u^{i-1}V_i$, but $L \not\subseteq \operatorname{im} u^i$, i.e., $L \not\subseteq u^iV_{i+1}$, then u induces u' on V/L and $u' \sim {}_i\mu$, where ${}_i\mu$ is obtained from μ replacing μ_i by $\mu_i - 1$.*

(c) *The orbits of H_u on V_1 are the set theoretic differences $u^{i-1}V_i - u^iV_{i+1}$.*

(d) *The irreducible components of $\mathfrak{F}_u$ have the same dimension. If $\mathfrak{F}_i$ is the subset of $\mathfrak{F}_u$ projecting into $u^{i-1}V_i - u^iV_{i+1}$, then $\mathfrak{F}_i$ is locally closed and has $m({}_i\mu')$ irreducible components. Hence $\mathfrak{F}_u$ has $\Sigma m({}_i\mu')$ irreducible components.*

$$\dim \mathfrak{F}_u = \Sigma_{i<j} \min\{\lambda_i, \lambda_j\} = \tfrac{1}{2} \Sigma_i(\mu_i^2 - \mu_i).$$

Proof. (a) H_u is connected being the group of units of an algebra. On V_i, $ux = xu$ iff x and uxu^{-1} have the same effect on uV_i.

(b) and (c) are clear.

(d) It is easy to see that $\dim \mathfrak{F}_u$ is as stated.

Let $g: \mathfrak{F}_i \to \mathscr{P}(V)$ be the projection, then $g(\mathfrak{F}_i) =$ lines in $u^{i-1}V_i - u^iV_{i+1}$, this is an orbit of H_u. Let $K = H_u{}^L$, the stabilizer of a fixed line L (as in (b)) in H_u.

We identify $g(\mathfrak{F}_i)$ with H_u/K.

The fibers of $\mathfrak{F}_i$ are all of the form $\mathfrak{F}_{u'}$ with $u' \sim {}_i\mu'$. H_u acts on $\mathfrak{F}_u$.

Let A be the set of flags $(W_1, \cdots, W_n)$ in $\mathfrak{F}_i$ with $W_1 = L$.

Let Y be a component of A. Here, $\mathfrak{F}_i = H_u(A)$. The set $H_u(Y)$ is clearly irreducible.

We claim that $H_u(Y)$ is an irreducible component of $\mathfrak{F}_i$. For this it suffices to check that $\dim H_u(Y) = \dim \mathfrak{F}_u$ and that $H_u(Y)$ is closed in $\mathfrak{F}_i$, which will also prove the equidimensionality of the components.

By induction and the dimension of the fibers of g we have dim $H_u(Y)$ = dim $Y + (\mu_i - 1)$ = dim $\mathfrak{F}_u - (\mu_i - 1) + (\mu_i - 1)$ = dim $\mathfrak{F}_u$.

The set of flags $H_u(Y)$ consists of the flags F with $g(F)^{-1}(F) \in Y$, a closed condition, where we used the previous identification.

Since K is connected, it acts trivially on the components of A. Hence the bijection $H_u(Y) \leftrightarrow Y$, and $m(\mu) = \Sigma m({}_i\mu')$. Q.E.D.

This also proves Theorem 1.1 because standard tableaux satisfy the recursion in (d), [1].

The completeness of Grassmannians guarantees that $g: \mathfrak{F} \to \mathcal{P}(V)$ is closed.

Let X be a fixed component of $\mathfrak{F}_u$, $g(X)$ is then closed and irreducible. In fact, $g(X) = \{L \in \mathcal{P}(V) \mid L \subseteq (\ker u \cap \operatorname{im} u^s)\}$ for some s. We write the number 1 in the last block of the $(s + 1)$th row of the diagram of u, so that the action induced by u on V/L with a line $L \subseteq (\ker u \cap \operatorname{im} u^s)$, $L \not\subseteq \operatorname{im} u^{s+1}$, is that of a nilpotent u' whose diagram is obtained from that of u deleting the block where 1 was put. If L as above is fixed, the elements of X with $W_1 = L$ correspond to a component Z of $\mathfrak{F}_{u'}$. By induction, there is a standard tableau for Z which is completed to one for X adding 1 to each number there and the new block with the number 1.

PROPOSITION 2.2. *Given u and a tableau for it, the component X of $\mathfrak{F}_u$ corresponding to it is the closure in $\mathfrak{F}$ of the set A of flags $W_1 \subset W_2 \subset \cdots \subset W_n$ such that*

(a) $u(W_{i+1}) \subset W_i$ *for* $i = 0, 1, \cdots, n-1$ *and*

(b) $W_{i+1} \subset W_i + \operatorname{im} u^{j-1}$, *where* $i + 1$ *occurs in the* j^{th} *row of the diagram. Here* $W_0 = (0)$ *and* $u^0 = 1$.

Proof: The flags satisfying (a) clearly belong to $\mathfrak{F}_u$. Conversely, if $u(W_i) \subset W_i$ for all i and assuming $W_{i+1} = W_i + \langle v \rangle$, we have $u(v) = cv + w$, the nilpotence of u giving $c = 0$, i.e., $u(W_{i+1}) \subset W_i$.

If 1 occurs in the $(s + 1)$th row of the diagram, then the set of flags in X with $W_1 \subset (\ker u \cap \operatorname{im} u^s)$, $W_1 \not\subseteq \operatorname{im} u^{s+1}$ is dense in X by 2.1.

The flags with $W_1 = L$ fixed as above satisfying (a) and (b) correspond to the flags in V/L satisfying the analogous version of (a) and (b) for the induced u'. The closure $\bar{A}$ contains the closure of this set which is the subset of X with $W_1 = L$ inductively; and as L covers $(\ker u \cap \operatorname{im} u^s) - \operatorname{im} u^{s+1}$ this describes a dense part of X. Hence $\bar{A} = X$. Q.E.D.

3. P/B

We fix a Borel subgroup B_0, which we assume consisting of all upper triangular matrices of G, and call standard the parabolic subgroups of G containing B_0.

If U_P is the unipotent radical of P, then $u \in U_P$ iff all Borel subgroups of P are in $\mathcal{B}_u$.

The conjugacy classes of parabolic subgroups are given by unordered partitions $\lambda = (\lambda_1, \cdots, \lambda_a)$ and each has exactly one representative which is standard consisting of all matrices in G of the form

$$\begin{bmatrix} \boxed{*} & & & * \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot \\ & & & \boxed{*} \end{bmatrix}$$

where the diagonal blocks are of sizes $\lambda_1, \cdots, \lambda_a$. Here U_P is the collection of matrices as above with identity matrices as diagonal blocks. We write the Levi decomposition $P = MU_P$, where $M \cong G \cap \Pi_i GL_{\lambda_i}$.

The parabolic subgroups with conjugate Levi factors are called associates, they are those with the same ordered partition.

We state a theorem of Richardson [4] which holds for an arbitrary connected semisimple group, in (c) below he assumes that the number of unipotent conjugacy classes is finite, which is no longer necessary in view of [3], where that finiteness is proved for reductive groups.

THEOREM 3.1. *With the above notation.*

(a) *There exists a dense open subset of U_P such that each of its elements is contained in finitely many G-conjugates of U_P.*

(b) *The union of all G-conjugates of U_P is a closed and irreducible subset A of G of dimension* $\dim G - \dim P/U_P$.

(c) *A contains a unique dense class C of the same dimension as its own. Here $C \cap U_P$ is dense in U_P and forms a single class under P.*

(d) *If $u \in C$ as above, then* $\dim \mathscr{B}_u = \dim P/B$.

(e) *$Z(u)$ acts transitively on the finite set of conjugates of U_P containing u.*

Let $f: \mathscr{B} \to G/B_0$ be the isomorphism of varieties given by $f(xB_0x^{-1}) = xB_0$.

LEMMA 3.2. *The following are equivalent:*

(a) *u is regular in U_P*

(b) *$u \in U_P$ and* $\dim P/B = \dim \mathscr{B}_u$, *where B is Borel in P.*

(c) *$g\ P_0/B_0$ is an irreducible component of $\mathscr{B}_u$, where $B = g\ B_0\ g^{-1}$, $P = g\ P_0 g^{-1}$ and P_0 is standard.*

Proof: (a) $\Rightarrow$ (b) is 3.1 (d). (b) $\Rightarrow$ (c) because $f(P/B) = \{gpg^{-1}g\ B_0 \mid p \in P_0\} = g\ P_0/B_0$.

As P/B is a component of $\mathscr{B}_u$, $u \in U_P$. Also $\dim P/B = \dim \mathscr{B}_u$ by the equidimensionality of the components of $\mathscr{B}_u$. Thus, $r + 2 \dim P/B = r + 2 \dim \mathscr{B}_u = \dim Z_G(u) \geq \dim Z_P(u)$ and if $C_P(u)$ is the conjugacy class of u in P, then $\dim C_P(u) = \dim P - \dim Z_P(u) \geq \dim P - r - 2 \dim P/B = \dim U_P$, i.e., u is regular in U_P. Hence (c) $\Rightarrow$ (a). Q.E.D.

The equality $r + 2 \dim \mathscr{B}_u = \dim Z_G(u)$ used above holds in SL_n; in general, it also holds when the characteristic is "very good" [8].

LEMMA 3.3. (a) *If u is regular in U_P, then each element of the conjugacy class of u in G is contained in a constant finite number m of conjugates of U_P.*

(b) *If P is standard parabolic, then there is a bijection between the distinct conjugates $g_1 U_P g_1^{-1}, \cdots, g_m U_P g_m^{-1}$ of U_P containing u and the distinct components $g_1 P/B_0, \cdots, g_m P/B_0$ of $\mathscr{B}_u$ which are translates of P/B_0.*

Proof: (a) is clear and (b) follows from it and the fact that P is the normalizer of U_P in G.

LEMMA 3.4. *Let the standard parabolic subgroup P be associated with the unordered partition $s = (s_1, \cdots, s_t)$. Let $u \sim \lambda$ be regular in U_P. Then $\lambda = \mu'$, where μ is the ordered partition of s.*

Proof: We choose a basis so that the situation is as at the beginning of this section, *e.g.* $M = G \cap \Pi\, GL(V_i)$ with $V = V_1 \oplus \cdots \oplus V_t$, our basis being the union of bases $\{v_{ij}\}$ for each V_i.

We rearrange the basis as follows: take the first vector of the basis of V_i as $i = 1, \cdots, t$, then take the second vector when it exists as i increases, etc.

Let u be the unipotent element of G which is in upper triangular Jordan canonical form with respect to the new basis whose blocks correspond to the span of the v_{ij} with j fixed so that $u \sim \mu'$. Clearly $u \in U_P$. To prove that u is regular in U_P it is enough to verify the equality $\dim P/B_0 = \dim \mathscr{B}_u$. Since $u \sim \mu' = (\mu_1', \cdots, \mu_q')$, it follows that

(1) $$\dim \mathscr{B}_u = \Sigma_{i<j} \min\{\mu_i', \mu_j'\} = \mu_2' + 2\mu_3' + \cdots + (q-1)\mu_q'.$$

Also $\dim P/B_0$ is the number of positive roots of a Levi subgroup of P. Thus,

(2) $$\dim P/B_0 = [1 + 2 + \cdots + (\mu_1 - 1)] + \cdots + [1 + 2 + \cdots + (\mu_t - 1)]$$

But μ_2' is the number of parts of μ of size > 1, hence μ_2' equals the sum of all one's in (2), μ_3' is the number of parts of μ of size > 2, hence $2\mu_3'$ equals the sum of all two's in (2), etc. Hence $\dim P/B_0 = \dim \mathscr{B}_u$. Q.E.D.

3.5. *Proposition: If $u \sim \lambda$, then the components of $\mathscr{B}_u$ of the form gP/B_0 with $g \in G$ and P standard parabolic are in bijection with the standard parabolic associates of partition λ'.*

Proof: $\tilde{Z}(u)$ is connected and acts transitively on the finite set of conjugates of U_P where u is regular, hence $m = 1$ in 3.3. The conclusion follows from the lemmas. Q.E.D.

It is known [7], that all components of $\mathscr{B}_u$ are of the form in 3.5 when u is regular, subregular or the identity. An easy combinational argument now proves the converse:

COROLLARY 3.6. *All irreducible components of $\mathscr{B}_u$ are of the form gP/B_0 with P standard parabolic iff u is regular, subregular or the identity.*

4. One hook case

In this section, u is a nilpotent element of End (V). We say that it is of *one hook type* when $u \sim \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_a)$ with $\lambda_1 > 1$ and $\lambda_2 = 1$, i.e., when its Jordan decomposition has exactly one block of size bigger than 1.

Given the Young diagram of u one has a basis of V such that each block corresponds to a basis vector and u acts sending it to the block above it or to zero if no such block exists.

Let b be the number of rows in the diagram, then $u^b = 0$ and $u^{b-1} \neq 0$. For u nilpotent we have $u(\ker u^{i+1}) \subset \operatorname{im} u \cap \ker u^i$ for $i = 0, 1, \cdots, b - 1$. In our case we also have $\operatorname{im} u^{b-i} = \operatorname{im} u \cap \mathit{ker}\, u^i$ and then

(3) $$u\,(\ker u^{i+1}) \subset \operatorname{im} u^{b-i} \quad \text{for} \quad i = 0, 1, \cdots, b-1$$

We now fix a basis $\{w_1, \cdots, w_n\}$ for V as above ordered so that the subindices of the w_i are given by

1	$b+1$	$\cdots$	n
2			
$\vdots$			
b			

It is easy to see that the centralizer $\tilde{Z}(u)$ in $GL(V)$ consists of the invertible matrices of the form

$$\left[\begin{array}{cccc|ccc} a_1 & a_2 & \cdots & a_b & * & \cdots & * \\ & \ddots & \ddots & \vdots & & & \\ & & \ddots & a_2 & & 0 & \\ & 0 & & \ddots & & & \\ & & & a_1 & & & \\ \hline & & & * & & & \\ & 0 & & \vdots & & * & \\ & & & * & & & \end{array}\right]$$

We now invoke some results of Ehresmann [2], we also use a notation which is close to his: $\mathcal{G}_{d,n}$ is the Grassmannian of all d dimensional subspaces of V; for a sequence $1 \leq i_1 < i_2 < \cdots < i_m < n$, we define the variety of flags of type $(i_1, \cdots, i_m)$ as the closed subvariety of $\mathcal{G}_{i_1,n} \times \cdots \times \mathcal{G}_{i_m,n}$ of all elements

$(W_{i_1}, \cdots, W_{i_m})$ with $W_{i_1} \subset \cdots \subset W_{i_m}$. This is a rational, irreducible and homogeneous projective variety of dimension $i_1(i_2 - i_1) + i_2(i_3 - i_2) + \cdots + i_m(n - i_m)$.

We fix a flag $F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset V = F_n$. For a given i with $1 \leq i \leq n - 1$ and a given sequence $1 \leq a_1 < \cdots < a_i \leq n$ we define the Schubert variety $[a_1, \cdots, a_i] \subset \mathscr{G}_{i,n}$ of all W_i with $\dim (W_i \cap F_{a_j}) \geq j$. It is irreducible of dimension $(a_1 - 1) + (a_2 - 2) + \cdots + (a_i - i)$.

$$\text{Let } S = \begin{bmatrix} a_1, \cdots, a_\alpha & & \\ b_1, b_2, \cdots, b_\beta & & \\ \cdots\cdots\cdots & & \\ c_1, c_2, c_3, \cdots, c_\gamma & & \end{bmatrix} \text{ be the Schubert}$$

variety of all flags $W_\alpha \subset W_\beta \subset \cdots \subset W_\gamma$ of type $(\alpha, \beta, \cdots, \gamma)$ such that for each $i, j, \cdots, k$ we have $\dim(W_\alpha \cap F_{a_i}) \geq i$, $\dim(W_\beta \cap F_{b_j}) \geq j, \cdots, \dim(W_\gamma \cap F_{c_k}) \geq k$. Then S is irreducible if each integer appearing in a row occurs in the row below it either in the same column or else in a column to the right of the previous occurrence, in that case

$$\dim S = \Sigma_i(a_i - i) + \Sigma_j(b_j - j) + \cdots + \Sigma_k(c_k - k), \tag{4}$$

where $i = 1, 2, \cdots, \alpha$ and $j, \cdots, k$ take the values of those indices which appear for the first time.

If the flags are of type $(1, 2, \cdots, n)$, then each irreducible Schubert variety has one new integer in each row and thus can also be described by a permutation on n letters.

In SL_n, the Weyl group is the symmetric group on n letters and the Schubert variety S corresponding to the permutation w is the image in $G/B \cong \mathscr{B} \cong \mathfrak{F}$ of $\overline{Bn_wB}$, the closure of the double coset of n_w, a representative of w, where B is the stabilizer of $F_1 \subset F_2 \subset \cdots \subset F_{n-1}$.

THEOREM 4.1 *Let u be a one hook type nilpotent element of* End(V). *Then the centralizer of u has finitely many orbits in each component of $\mathfrak{F}_u$. In the correspondence of tableaux and components we associate.*

$$\begin{array}{|c|c|c|} \hline n & \cdots & \cdots \\ \hline i_{b-1} & \multicolumn{2}{c}{} \\ \cline{1-1} \vdots & \multicolumn{2}{c}{} \\ \cline{1-1} i_2 & \multicolumn{2}{c}{} \\ \cline{1-1} i_1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

to the collection of flags $W_1 \subset W_2 \subset \cdots \subset W_{n-1}$ *satisfying*

$$\text{(5)}\qquad \begin{aligned} \operatorname{im} u^{b-1} &\subset W_{i_1} \subset \ker u \\ \operatorname{im} u^{b-2} &\subset W_{i_2} \subset \ker u^2 \\ &\vdots \\ \operatorname{im} u &\subset W_{i_{b-1}} \subset \ker u^{b-1} \end{aligned}$$

Proof: Let A be the set of flags (5). It is a closed subvariety of $\mathfrak{F}$, (3) above guarantees that $A \subset \mathfrak{F}_u$.

Let $\mathcal{O} = \mathcal{O}(\alpha_1, \cdots, \alpha_{b-1}; \beta_1, \cdots, \beta_{b-1})$ be the subset of A of all flags with im $u^{b-j} \subseteq W_{\alpha_j}$, im $u^{b-j} \not\subseteq W_{\alpha_j-1}$, $W_{\beta_j} \subseteq \ker u^j$ and $W_{\beta_j+1} \not\subseteq \ker u^j$. It is stable under $\tilde{Z}(u)$. If $\mathcal{O} \neq \varnothing$ then

$$\text{(6)}\qquad 1 \leq \alpha_1 \leq i_1 \leq \beta_1 < \alpha_2 \leq i_2 \leq \beta_2 < \cdots < \alpha_{b-1} \leq i_{b-1} \leq \beta_{b-1} < n$$

We now show that $\mathcal{O}$ is an orbit of the centralizer. For this purpose, we fix a flag $W_1 \subset W_2 \subset \cdots \subset W_n$ in $\mathcal{O}$ and describe it by a matrix M as follows:

$$M = \left[\begin{array}{c|c} I_b & P \\ \hline Q & R \end{array}\right] \quad \text{with} \quad Q = \begin{bmatrix} 0 & \begin{matrix} * \\ \vdots \\ * \end{matrix} \end{bmatrix} \quad \text{and}$$

I_b *a* $b \times b$ identity matrix. Here we require that W_1 be the span of the $(b + 1)$th column, $\cdots$, that W_{α_1-1} be the span of the $(b + 1)$th through $(b + \alpha_1 - 1)$th columns, that W_{α_1} be spanned by W_{α_1-1} and the first column, etc. so that each of the first b columns of M is used as soon as possible. In this ordering of the columns of M we would have obtained the same order for another $\mathcal{O}$ with the same α-sequence.

For a fixed flag, the matrix M can be simplified so that we can assume that the $(b - 1) \times (n - b)$ matrix N of the last rows of P has at most one nonzero entry in each row, that this entry is 1 and that for the lth row of N it does not occur if $\beta_l + 1 = \alpha_{l+1}$ and that when it occurs it does so in the $(\beta_l + 1 - l)$th column, this for $l = 1, 2, \cdots, b - 2$. Also that the last row of N is zero.

The computation

$$\left[\begin{array}{c|c} I_b & S \\ \hline 0 & I_{n-b} \end{array}\right] \left[\begin{array}{c|c} I_b & 0 \\ \hline 0 & (R-QP)^{-1} \end{array}\right] \left[\begin{array}{c|c} I_b & 0 \\ \hline -Q & I_{n-b} \end{array}\right] \left[\begin{array}{c|c} I_b & P \\ \hline Q & R \end{array}\right] = \left[\begin{array}{c|c} I_b & \begin{matrix} 0 \cdots 0 \\ N \end{matrix} \\ \hline 0 & I_{n-b} \end{array}\right]$$

where S is so that

$$P + S = \left[\begin{array}{c} 0 \cdots 0 \\ \hline N \end{array}\right]$$

shows that M can be transformed into a canonical matrix of $\mathcal{O}$ by the action of $\tilde{Z}(u)$ proving that $\mathcal{O}$ is indeed an orbit of the centralizer. If the matrices in the left member of (7) are multiplied from right to left, one sees at the first step that $R - QP$ has an inverse.

Here $\dim \mathfrak{F}_u = 1 + 2 + \cdots + ((\dim \ker u) - 1)$. We next compute the dimension of $\mathscr{C} = \mathcal{O}(i_1, \cdots, i_{b-1}; i_1, \cdots, i_{b-1})$.

Let $i_0 = 0$, $i_n = n$ and $W_1 \subset W_2 \subset \cdots \subset W_n$ a flag in $\mathscr{C}$. Here $(\operatorname{im} u^{b-j} + W_{i_{j-1}})/W_{i_{j-1}}$ is a line in $\ker u^j / W_{i_{j-1}}$.

Let $\mathfrak{F}_j$ be the variety of flags of type $(1, 2, \cdots, i_j - i_{j-1})$ in a vector space of dimension $\dim \ker u^j - i_{j-1}$.

Let $p_1 : \mathscr{C} \to \mathfrak{F}_1$ be the natural projection. Then $p_1(\mathscr{C})$ is the subvariety S_1 of $\mathfrak{F}_1$ of all flags where W_{i_1} contains a line, all fibers of P_1 are of the same form, viewing them as flags in V/W_{i_1} and projecting any of them in $\mathfrak{F}_2$ we obtain S_2: The subvariety of flags $W_1 \subset \cdots \subset W_{i_2 - i_1}$ containing a line. In this way, we have subvarieties S_i of $\mathfrak{F}_i$ with $S_b = \mathfrak{F}_b$, $\dim \mathscr{C} = \sum_{j=1}^{b} \dim S_j$ and with S_j of type

$$\begin{bmatrix} l \\ l-1, l \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ l-r, l-r+1, \cdots, l \\ 1, l-r, l-r+1, \cdots, l \end{bmatrix} \quad \text{for } j < b, \text{ where } r = i_j - i_{j-1} - 2 \text{ and}$$

$l = \dim \ker u^j - i_{j-1}$. Thus, $\dim S_j = (l-1) + (l-2) + \cdots + (l-r-1)$, by (4), this is a sum of decreasing consecutive integers. For S_1 it begins at $\dim \ker u - 1$ and ends at $\dim \ker u - i_1 + 1$, for S_2 it begins at $\dim \ker u^2 - i_1 - 1 = \dim \ker u - i_1$ and ends at $\dim \ker u^2 - i_2 + 1, \cdots$, for S_{b-1} it ends at $\dim \ker u^{b-1} - i_{b-1} + 1$, $\dim S_b = \dim \mathfrak{F}_b = 1 + 2 + \cdots + (n - i_{b-1} - 1)$ and $n - i_{b-1} - 1 = \dim \ker u^{b-1} - i_{b-1}$. Hence $\dim \mathfrak{F}_u = \dim \mathscr{C}$.

Each element of $\bar{\mathscr{C}}$ satisfies (a) and (b) in Proposition 2.2 for the given tableau. We now claim that $A = \bar{\mathscr{C}}$ and hence that A is irreducible. More precisely, the closure of the orbit $\mathcal{O} = \mathcal{O}(\alpha_1, \cdots, \alpha_{b-1}; \beta_1, \cdots, \beta_{b-1})$, which is a union of orbits, consists of those orbits whose α-sequence is obtained from that of $\mathcal{O}$ by decreasing its entries and whose β-sequence is obtained by increasing its entries.

Having fixed an α-sequence, at the beginning of this proof we associated matrices to flags in orbits with that α-sequence. We also found a canonical matrix for each orbit. The matrices obtained from a canonical one allowing arbitrary elements of k after a leading one in the submatrix N and replacing those leading one's by nonzero elements of k are seen to be in the same orbit, whose closure then contains the canonical matrices of the orbits with the same α-sequence and with entries in its β-sequence bigger than or equal to the corresponding ones of $\mathcal{O}$, and hence the complete orbits. The proof interchanging the roles of α and β is similar.

This completes the proof. Q.E.D.

REMARK 4.2. *We call* $(i_1, i_2, \cdots, i_{b-1})$ *the component of* $\mathfrak{F}_u$ *as above. The components of the form* P/B *of section* 3 *are those where the sequence complementary to* $i_1, i_2, \cdots, i_{b-1}, n$ *consists of consecutive integers.*

COROLLARY 4.3. *The number of orbits under $Z(u)$ in $\mathfrak{F}_u$ is finite. Each orbit is of the form $\mathcal{O}(\alpha_1, \cdots, \alpha_{b-1}; \beta_1, \cdots, \beta_{b-1}) = \mathcal{O}$ with $1 \le \alpha_1 \le \beta_1 < \alpha_2 \le \beta_2 < \cdots < \alpha_{b-1} \le \beta_{b-1} < n$. The closure of $\mathcal{O}$ consists of the orbits whose* α-sequence (*resp.* β-sequence) *is obtained from that of $\mathcal{O}$ by increasing* (*resp.* decreasing) *some of its entries. $\mathcal{O}$ is contained in the components $(i_1, \cdots, i_{b-1})$ with $\alpha_j \le i_j \le \beta_j$ for all j. Any intersection of components is the intersection of two components, it is irreducible and a union of orbits under the centralizer. Any nonempty orbit is dense in some intersection of components.*

Proof: All this is clear, e.g., $\mathcal{O}(\alpha_1, \cdots, \alpha_{b-1}; \beta_1, \cdots, \beta_{b-1})$ is dense in $(\alpha_1, \cdots, \alpha_{b-1}) \cap (\beta_1, \cdots, \beta_{b-1})$

THEOREM 4.4. *The components $(\alpha_1, \cdots, \alpha_{b-1})$ and $(\beta_1, \cdots, \beta_{b-1})$ of $\mathfrak{F}_u$ have nonempty intersection J when $A_1 \le B_1 < A_2 \le B_2 < \cdots < A_{b-1} \le B_{b-1}$, where $A_i = \min\{\alpha_i, \beta_i\}$ and $B_i = \max\{\alpha_i, \beta_i\}$.* Then

$$\dim J = \dim \mathcal{O}(A_1, \cdots, A_{b-1}; B_1, \cdots, B_{b-1}) = \dim \mathfrak{F}_u - \textstyle\sum_{j=1}^{b-1} (B_j - A_j).$$

Proof: We write $B_0 = 0$ and $B_b = n$. Let $\mathfrak{F}_j$ be the variety of flags of type $(1, 2, \cdots, B_j - B_{j-1})$ in a vector space of dimension $\dim \ker u^j - B_{j-1}$, this for $1 \le j \le b$. Let S_1 be the projection of $\mathcal{O}(B_1, \cdots, B_{b-1}; B_1, \cdots, B_{b-1})$ and T_1 that of $\mathcal{O}(A_1, \cdots, A_{b-1}; B_1, \cdots, B_{b-1})$ in $\mathfrak{F}_1$. For each orbit, the fibers are of the same form and define projections S_2, T_2 respectively in $\mathfrak{F}_2$, etc. until finally $S_b \cong T_b \cong \mathfrak{F}_b$.

Since $\dim \mathfrak{F}_u = \dim \mathcal{O}(B_1, \cdots, B_{b-1}; B_1, \cdots B_{b-1}) = \sum_{j=1}^{b} \dim S_j$ and

$$\dim \mathcal{O}(A_1, \cdots, A_{b-1}; B_1, \cdots B_{b-1}) = \textstyle\sum_{j=1}^{b} \dim T_j,$$

it suffices to show that $\dim T_j = \dim S_j - (B_j - A_j)$ for all $1 \le j \le b - 1$.

The Schubert varieties S_j and T_j are respectively of types

$$\begin{bmatrix} l \\ l-1, l \\ \cdots\cdots\cdots \\ l-r, l-r+1, \cdots, l \\ 1, l-r, l-r+1, \cdots, l \end{bmatrix} \text{ and } \begin{bmatrix} l \\ l-1, l \\ \cdots\cdots\cdots \\ l-q, l-q+1, \cdots, l \\ 1, l-q, l-q+1, \cdots, l \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1, l-r, l-r+1, l-r+2, \cdots, l \end{bmatrix}$$

with the same l and r, and q is so that in the second symbol 1 leads $1 + (B_j = A_j)$ rows. Thus, by (4)

$$\dim S_j = (l-1) + (l-2) + \cdots + (l-r-1) \tag{8}$$

$$\dim T_j = (l-1) + (l-2) + \cdots + (l-q-1) + (l-q-3) + (l-q-4) + \cdots + (l-r-2) \tag{9}$$

In these sums, the number of summands is the same, the first few terms are equal, and each of the last $B_j - A_j$ terms of (9) is a unit less than the corresponding term in (8). The conclusion follows. Q.E.D.

THEOREM 4.5. *The closure A of any orbit $\mathcal{O} = \mathcal{O}(\alpha_1, \cdots, \alpha_{b-1}; \beta_1, \cdots, \beta_{b-1})$ is rational and smooth,* i.e., *every intersection of irreducible components of $\mathfrak{F}_u$ is rational and smooth.*

Proof: For the nonsingularity, it is enough to see that one point in each orbit under $Z(u)$ in A is simple in A.

For this purpose, let $\{w_1, \cdots, w_n\}$ be a basis of V such that the action of u is given by

$$w_b \to w_{b-1} \to \cdots \to w_2 \to w_1 \to 0,\ w_{b+1} \to 0,\ w_{b+2} \to 0,\ \cdots,\ w_n \to 0.$$

Let $\mathcal{O}' = \mathcal{O}(\gamma_1, \cdots, \gamma_{b-1}; \delta_1, \cdots, \delta_{b-1})$ be an orbit in A. As in the proof of Theorem 4.1, it is possible to modify $\{w_1, \cdots, w_n\}$ replacing each w_{b+i} with $i > 0$ by itself or by $w_{b+i} + w_j$ with $j < b$ and then permuting the resulting vectors to obtain a basis $\{v_1, \cdots, v_n\}$ of V such that the flag $F_0: V_1 \subset V_2 \subset \cdots \subset V_n$ with each $V_i = \langle v_1, \cdots, v_i \rangle$ belongs to $\mathcal{O}'$.

Let Ω be the big cell of $\mathfrak{F}$ adapted to F_0: Ω consists of all flags $W_1 \subset W_2 \subset \cdots \subset W_n$ with each W_i spanned by the first i columns of the matrix M.

$$M = \begin{bmatrix} 1 & & & & 0 & \\ a_{21} & 1 & & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & 1 & \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn-1} & 1 \end{bmatrix}$$

where the coordinates are with respect to $\{v_1, \cdots, v_n\}$. In this way, $k[\Omega] = k[a_{ij}]$, a polynomial algebra; and F_0 is the origin in this affine space.

Let I be the ideal of $\Omega \cap A$ in $k[\Omega]$. The conditions $W_{\beta_j} \subset \ker u^j$ and $\operatorname{im} u^{b-j} \subset W_{\alpha_j}$ for $1 \leq j \leq b-1$ give some elements of I as follows.

The space $\ker u^i$ has a basis consisting of a subset of $\{v_1, \cdots, v_n\}$ together with some other vectors of the form $v_c - v_d$, while $\operatorname{im} u^{b-1} = \langle v_{\gamma_1}, \cdots, v_{\gamma_i} \rangle$.

Assuming $\varnothing \neq \mathcal{O}' \subset A$ we have

$$1 \leq \gamma_1 \leq \alpha_1 \leq \beta_1 \leq \delta_1 < \gamma_2 \leq \alpha_2 \leq \beta_2 \leq \delta_2 < \cdots < \gamma_{b-1} \leq \alpha_{b-1} \leq \beta_{b-1} \leq \delta_{b-1}.$$

In particular,

$$\beta_{j-1} < \gamma_j \leq \beta_j \qquad \text{for} \qquad 2 \leq j \leq b-1 \tag{10}$$

Let $T(\Omega)$ and $T(\Omega \cap A)$ be the tangent spaces at F_0 to Ω and $\Omega \cap A$ respectively.

The inclusion $W_{\beta_j} \subset \ker u^j$ implies that if $l \leq \beta_j$, then for the entries in the lth columns of M we have $b - j$ independent linear equations of the form $a_{rl} = 0$ or $a_{ql} = a_{sl}$ with $q \neq s$, which give linear polynomials of I whose differentials set equal to zero decrease $\dim T(\Omega)$ by $(b-1)\beta_1 + (b-2)(\beta_2 - \beta_1) + \cdots + (\beta_{b-1} - \beta_{b-2}) = \beta_1 + \beta_2 + \cdots + \beta_{b-1}$ units.

The condition im $u^{b-j} \subset W_{\alpha_j}$ implies that v_{γ_j} is a linear combination with certain coefficients of the first α_j columns of M.

Each of these linear equations is equivalent to n scalar equations of which the first α_j serve to find the coefficients just mentioned, which turn out to be a collection of zeroes, a 1 and several polynomials in the a_{xy} with no constant term. The remaining $n - \alpha_j$ scalar equations give elements of I, they are polynomials whose leading forms are linear, as they are all $a_{t\gamma_j}$ with $t > \alpha_j$.

Hence setting the differentials of these elements of I equal to zero further reduces dim $T(\Omega)$ by at least $(n - \alpha_j) - (b - j)$ units for $1 \leq j \leq b - 1$ because at most $b - j$ equations from before are redundant by (10).

Thus, [codim $T(\Omega \cap A)$ in $T(\Omega)$] $\geq (\beta_1 + \beta_2 + \cdots + \beta_{b-1}) + [(n - \alpha_1) - (b - 1)] + [(n - \alpha_2) - (b - 2)] + \cdots + [(n - \alpha_{b-1}) - 1] = \sum_{j=1}^{b-1} (\beta_j - \alpha_j) + [n - (b - 1)] + [n - (b - 2)] + \cdots + [n - 1]$. But dim $\mathfrak{F}_u = 1 + 2 + \cdots + (n - b)$ and dim $\Omega = 1 + 2 + \cdots + (n - 1)$. Hence [codim $T(\Omega \cap A)$ in $T(\Omega)$] $\geq$ dim Ω − dim $\mathfrak{F}_u + \sum_{j=1}^{b-1} (\beta_j - \alpha_j)$ = dim Ω − dim A by Theorem 4.4, which proves the nonsingularity of A.

The leading forms of those polynomials in I whose differentials we have used give some of the a_{ij} in terms of dim $(\Omega \cap A)$ remaining coordinates. Thus $\Omega \cap A$ is isomorphic to affine space and A is rational. Q.E.D.

The two opposite extreme cases covered by the one hook case are the subregular elements and the elements given by one root, i.e. those of type (2, 1, 1, $\cdots$, 1). We write in the latter case.

$$x = 1 + u = \begin{bmatrix} 1 & & & & 1 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{bmatrix}, \quad \text{so that } x \in \text{center of } U,$$

where U is the unipotent radical of B. Using the imbedding $x_\gamma : \mathbf{G}_a \to U$ from the additive group into U corresponding to the root $\gamma = (1, n)$ we can write $x = x_\gamma(1)$.

Each component of $\mathfrak{F}_x = \mathscr{B}_x$ is a generalized Schubert variety given by the condition im $u \subset W_i \subset$ ker u. There are $n - 1$ of them as $1 \leq i < n$. As seen before, such a component is described by a permutation σ such that 1 and n appear as neighbors in that order in the sequence $\sigma(1), \sigma(2), \cdots, \sigma(n)$, and the other integers appear in totally reversed order.

If for such σ, $\sigma(i) = 1$, then σ makes negative all positive roots except for $(1, i + 1), (2, i + 1), \cdots, (i - 1, i + 1); (i, i + 1); (i, i + 2), \cdots, (i, n)$, hence we see directly that $l(\sigma) = 1 + 2 + \cdots + (n - 2) = \dim \mathfrak{F}_x$. Also, σ^{-1} is a permutation of maximal length among those keeping $(1, n)$ positive.

This can be interpreted observing that $x_\gamma(1)y\, n_w B = y\, x_\gamma(1)\, n_w B$ for all $y \in U$, where n_w is a representative of w in the Weyl group of G, so that $x_\gamma(1)y\, n_w B = y\, n_w B$ *iff* $x_\gamma(1) n_w B = n_w B$, i.e., $x_\gamma(1)$ fixes all elements of the Bruhat cell

$U\, n_w B$ *iff* $x_\gamma(1)$ fixes one of them, like $n_w B$, and this occurs when $n_w^{-1} x_\gamma(1) n_w \in B$, that is when $w^{-1}(\gamma) > 0$. Thus, $\mathscr{B}_x$ consists of a union of Bruhat cells and the components are the closures of those of maximal dimension.

The closure of an orbit under the centralizer of u in $\mathscr{B}_x$ is the generalized Schubert variety $\overline{B n_w B}$ with w a permutation such that in the sequence $w(1), w(2), \cdots, w(n)$ the number 1 occurs before n (not necessarily as neighbors) and the other intergers occur in totally reversed order, so we obtain.

COROLLARY 4.6. *If w is as above, then $\overline{B n_w B}$ is smooth.*

5. A component with singularities

The earliest case of a component of $\mathfrak{F}_u$ with singular points occurs in SL_6 with $u \sim (2, 2, 1, 1)$ so that dim $\mathfrak{F}_u = 7$, it has tableau

6	5	3	1
4	2		

To see this, let $\{v_1, \cdots, v_6\}$ be a basis for V such that the action of u is given by $v_1 \to 0$, $v_3 \to 0$, $v_5 \to v_2 \to 0$ and $v_6 \to v_4 \to 0$.

Let Ω be the big cell of $\mathfrak{F}$ adapted to this basis: it consists of the flags $W_1 \subset W_2 \subset \cdots \subset W_6$ with W_i supplementary to span of $\{v_{i+1}, \cdots, v_6\}$. The elements of Ω can be described by strictly lower triangular matrices requiring that each W_i be the span of the first i columns. We take the appropriate coordinate functions as affine coordinates for Ω.

Let Z be the subvariety of Ω given by

$$\begin{bmatrix} 1 & & & & & \\ a & 1 & & & 0 & \\ b & 0 & 1 & & & \\ c & x & \alpha & 1 & & \\ 0 & 0 & \beta & 0 & 1 & \\ 0 & 0 & \beta x & 0 & y & 1 \end{bmatrix}$$

where the letters can take arbitrary values in k. This is the trace of our component in Ω.

Let Ω' be the big cell adapted to the new basis given by $w_1 = v_2$, $w_2 = v_4$, $w_3 = v_1$, $w_4 = v_3$, $w_5 = v_5$, $w_6 = v_6$. Then our component $\bar{Z}$ has trace in Ω' contained in the affine subspace of Ω' given by

$$\begin{bmatrix} 1 & & & & & \\ X_3 & 1 & & & 0 & \\ X_1 & X_4 & 1 & & & \\ X_2 & X_5 & X_6 & 1 & & \\ 0 & 0 & X_8 & X_7 & 1 & \\ 0 & 0 & X_9 & X_{10} & X_{11} & 1 \end{bmatrix}$$

including the origin there, the ideal $I(\bar{Z} \cap \Omega')$ in $k[\Omega]$ is $\cup_\rho$ $(X_1X_5 - X_2X_4, X_1X_8$

$+ X_2X_7 - X_1X_6X_7$, $X_5X_7 + X_4X_8 - X_4X_6X_7$, $X_4X_9 + X_1X_8 - X_3X_4X_8$, $X_1X_7 + X_4X_{10} - X_3X_4X_7):X_4{}^\rho$. Call it I.

It is not difficult to see that any polynomial in I having nonzero linear form has that form equal to cX_9 with $c \in k$ so that if T is the tangent space to $\bar{Z}$ at the origin of Ω', then dim $T \geq 10$ and the point is singular. N. Spaltenstein [5] independently proved that the singular locus of the component in question is isomorphic to $\mathscr{P}^1 \times \mathscr{P}^1 \times \mathscr{P}^1$, (and that it is homogeneous in a sense made precise there), all this over the complex field.

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