Boletín de la Sociedad Matemática Mexicana Vol. 24 No. 1, 1979

COMPUTATION OF A BOUNDARY POINT OF TEICHMÜLLER SPACE

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1. Statement of Results

In this paper we compute a boundary point of the Teichmüller space of the Fuchsian group G_0 generated by the two hyperbolic fractional linear transformations

$$A_0(z) = (\sqrt{2}\, z + R)/(z/R + \sqrt{2}),$$

$$B_0(z) = (\sqrt{2}\, z + iR)/(-iz/R + \sqrt{2}).$$

G_0 is a free group on these two generators acting on the disk $\Delta = \{\,|z| < R\}$, and the quotient $\Delta/G_0 = S_0$ is a simply punctured surface of genus 1.

A *Kleinian group* is a group of fractional linear transformations which acts discontinuously on some open set in the extended complex plane $\hat{\mathbf{C}}$. A Kleinian group is *Fuchsian* if it has an invariant disk, *quasifuchsian* if it has an invariant Jordan domain. A function ϕ is a *quadratic differential* for a general Fuchsian group G_0 if $\phi(\gamma(z))\gamma'(z)^2 = \phi(z)$ for $\gamma \in G_0$, $z \in \Delta$. Let ϕ be holomorphic. If the Schwarzian differential equation

$$[F] = \phi,$$

where $[F] = (F''/F')' - (\tfrac{1}{2})(F''/F')^2$ is the classical Schwarzian derivative, has a solution F which is schlicht in Δ and which can be extended quasiconformally to all of $\hat{C}$, then we say that $\phi \in T(G_0)$. When this holds, the norm

$$\|\phi\| = \sup\{\phi(z)\,|\,(R^2 - |z|^2)/4R^2,\ z \in \Delta\}$$

is finite. Let $B_2(G_0, \Delta)$ denote the complex vector space of holomorphic quadratic differentials for G_0 in Δ which have finite norm. Then the subset $T(G_0) \subseteq B_2(G_0, \Delta)$ is called the *Bers embedding* [3] of the Teichmüller space of G_0 (or of S_0); it is a simply-connected open set whose boundary $\partial T(G_0)$ lies between the spheres of radii $\frac{1}{2}$ and $\frac{3}{2}$.

One thinks of an element $\phi \in B_2(G_0, \Delta)$ as parameterizing a deformation G of the base group G_0. The following is a brief description of the bifurcation which occurs as ϕ crosses from the interior to the exterior of $T(G_0)$. When $\phi \in T(G_0)$, the mapping F conjugates $\gamma \in G_0$ into $\chi(\gamma) = F\gamma F^{-1}$, which is fractional linear transformation in its action on $F(\Delta)$. The image group $G = \chi(G_0)$ is quasifuchsian and $\chi: G \rightarrow G_0$ is an isomorphism. When $\phi \in \partial T(G_0)$, G

* Included are the results of the author's doctoral dissertation written at Northwestern University. The author wishes to express his gratitude for the guidance provided by his thesis advisor, William Abikoff, at the University of Illinois, Champaign-Urbana.

is no longer quasifuchsian (though F is still schlicht in Δ); it is called a b-*group* because there is a simply-connected component, $F(\Delta)$, of its set of points of discontinuity. When ϕ lies in the exterior of $T(G_0)$, G and χ are still well-defined, but G may not be Kleinian, and we may only assert that χ is a homomorphism.

Certain qualitative properties of $\partial T(G_0)$ were discovered by Bers [3], such as the existence of *cusps* (boundary groups in which the image $\chi(\gamma)$ of some hyperbolic element $\gamma \in G_0$ is parabolic) and *totally degenerate* groups (where $F(\Delta)$ is the entire set of ordinary points of the action of G on $\hat{\mathbf{C}}$). Abikoff [1] showed that if G_1 is a *regular* b-*group* (the precise definition will not be given here), then G_1 appears on the boundary of $T(G_0)$ for some Fuchsian group G_0.

For the group G_0 defined above, $B_2(G_0, \Delta)$ is one dimensional. Writing $\phi = \sigma\phi_0$, $\sigma \in \mathbf{C}$, where ϕ_0 is fixed, we identify $T(G_0)$ with a bounded open domain in $\mathbf{C}$. The algorithm described in Section 5 converges to the intersection σ_0 of $\partial T(G_0)$ with the positive real axis, providing the first known example of an explicit boundary group of a given Teichmüller space. The proof of convergence is based on two theorems:

THEOREM 1. $\sigma_0/2$ *is the least eigenvalue of the differential operator*

$$Lu = -u'' - (Q/4)u$$

on the interval $[0, \frac{1}{2}]$ *with the boundary conditions*

$$u'(0) = 0 = u'(\tfrac{1}{2}),$$

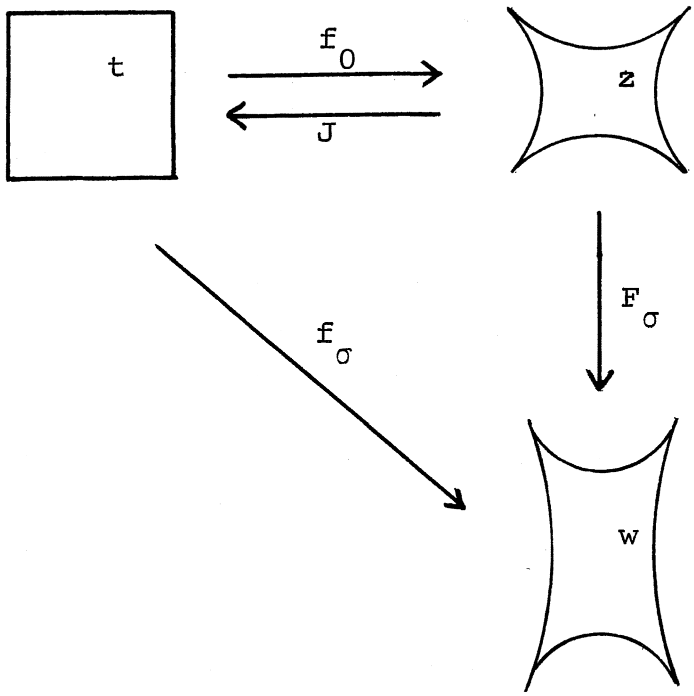
and $y_1(t, \sigma_0)$ *is the corresponding eigenfunction.*

THEOREM 2. σ_0 *is the only boundary point of* $T(G_0)$ *on the ray* $\sigma > 0$.

The normalizations and definitions which give the preceding statements a precise meaning will be found in the following section. We mention here that the coefficient Q in the definition of the operator L is a translate of the classical Weierstrass $\wp$-function. Theorem 1 follows from a study of the geometry of a fundamental domain in the image group. The proof of Theorem 2 is based on the observation of Nehari [10] that the map F_σ determined by σ is schlicht on Δ if and only if no solution of the differential equation (2b) vanishes twice in Δ.

2. The Lattice Model

We define our basis element ϕ_0 for $B_2(G_0, \Delta)$ as follows: Let D_0 be the fundamental domain for G_0 in Δ bounded by arcs of the four circles $|z - i^k R\sqrt{2}| = 1$, $k = 0, 1, 2, 3$, and consider the conformal map J from D_0 to the unit square $\Pi = \{t: |Re\ t| < \frac{1}{2}, |\mathrm{Im}\ t| < \frac{1}{2}\}$. We assume $J(0) = 0$ and that J takes the corners of D_0 to corners of Π. For convenience in calculation we fix the radius R of Δ so that $J'(0) = 1$. Observe that J extends analytically to all of Δ and that the image $J(\Delta)$ is C minus a lattice $\mathscr{L}$. Defining $\phi_0 = J'^2$, one verifies that $\phi_0 \in B_2(G_0, \Delta)$. For a complex number σ we will denote by F_σ the solution of $[F] = \sigma\phi_0$ normalized by $F_\sigma(0) = 0$, $F_\sigma'(0) = 1$. We will write G_σ, χ_σ



for the corresponding group and isomorphism. It will also be useful to consider the inverse function $f_0 = J^{-1}$ and the composition $f_\sigma = F_\sigma \circ f_0$. By comparing the singularities of f_σ at the points of $\mathscr{L}$ with those of the Weierstrass $\wp$-function (of periods 1 and i) one can verify (cf. [8]) that

$$[f_0] = \tfrac{1}{2}Q$$

where $Q(t) = \wp(t + (1 + i)/2)$. It then follows from the Schwarzian "chain rule" that

(1a) $$[f_\sigma] = \tfrac{1}{2}Q + \sigma.$$

Our aim is to take advantage of the additive dependence on σ rather than the multiplicative dependence expressed in

(1b) $$[F_\sigma] = \sigma J'^2$$

It is classical that the solutions of these Schwarzian equations are of the form $f_\sigma = y_2/y_1$, $F_\sigma = Y_2/Y_1$ where $y_i(t, \sigma)$ are solutions of the *associated linear equations*

(2a) $$y'' + ((Q + 2\sigma)/4)y = 0,$$

(2b) $$Y'' + (\sigma/2)J'^2Y = 0,$$

with the initial conditions

(3a) $$y_1(0) = 1 = y_2'(0), \qquad y_1'(0) = 0 = y_2(0);$$

(3b) $$Y_1(0) = 1 = Y_2'(0), \qquad Y_1'(0) = 0 = Y_2(0).$$

In the next section we will refer to the following elementary properties of $Q(t) = q_1t^2 + q_2t^6 + q_3t^{10} + \cdots$.

PROPOSITION. *$Q(t)$ is real on the boundary of Π. It is real and nonpositive on the real axis. Further,*

$$Q(t) = Q(-t) = Q(t+1) = Q(t+i) = -Q(it).$$

3. Proofs of Theorems 1 and 2

Let $D_\sigma = F_\sigma(D_0)$. For $\sigma \in T(G_0)$, D_σ is a fundamental domain for G_σ in $F_\sigma(\Delta)$.

LEMMA 1. *For real σ, the four sides bounding D_σ are arcs of circles (or straight lines).*

Proof. For a smooth path $w(r)$, $r \in \mathbb{R}$ in the plane, the curvature $\kappa(r)$ satisfies

$$d\kappa/dr = (1/|w'|)\mathrm{Im}((w''/w')' - (\tfrac{1}{2})(w''/w')^2)$$

(see [2], Chapter 1, Exercise 3). Now let $t(r)$ trace a side of Π in a linear manner. Then $wt(r) = f_\sigma(t(r))$ traces a side of D_σ. Since $[w] = (\frac{1}{2})Q + \sigma$ is real when σ is real, we see that $d\kappa/\mathrm{d}r = 0$, $\kappa =$ constant.

The boundary group G_{σ_0} is a cusp; we may describe σ_0 as the first positive value for which the element $A_\sigma = \chi_\sigma(A_0)$ is a parabolic transformation (see the remarks in the following section). Because of the symmetry, A_{σ_0} must fix the point at infinity and thus be a translation $z + 2a$, $a \in \mathbb{R}$. Since A_{σ_0} pairs the left and right sides of D_{σ_0}, these sides must be straight vertical segments, by Lemma 1. This is the key geometrical idea in what follows. For $|\sigma| < \sigma_0$ the vertical sides (in fact, all four sides) of D_σ are concave outward; it is easily verified that in this case D_σ is the Ford fundamental domain [6] for G_σ in $F_\sigma(\Delta)$. We mention here that because of the fourfold symmetry of G_0 it can be shown that $T(G_0)$ is symmetric in the real and imaginary axes in the σ-plane (we disbelieve, however, that it has fourfold symmetry). The behavior of $\chi_\sigma(B_0)$ for $\sigma < 0$ mirrors that of $\chi_\sigma(A_0)$ for $\sigma > 0$; thus $B_{(-\sigma_0)}$ is also parabolic, a vertical translation.

We now make some observations about the functions $y_1(t, \sigma)$. For $|\sigma| \le |\sigma_0|$ the conformal mapping $f_\sigma = y_2/y_1$ has no poles on $[-\frac{1}{2}, \frac{1}{2}]$, and since the Wronskian $y_1y_2' - y_1'y_2$ is identically 1, y_1 and y_2 never vanish simultaneously. Thus $y_1(t, \sigma) \neq 0$ for $|t| \le \frac{1}{2}$, $|\sigma| \le \sigma_0$. Now fix $\sigma = \sigma_0$. From (2a) we may write

(4) $$f_{\sigma_0}(t) = \int_0^t y_1(t)^{-2}\, dt.$$

Since $f_{\sigma_0}(t+1) = f_{\sigma_0}(t) + 2a$, we see upon differentiating that $y_1(t)^2$ is periodic: $y_1(t+1, \sigma_0) = \pm y_1(t, \sigma_0)$. If the minus sign were to hold, then $y_1(1, \sigma_0) = -y_1(0, \sigma_0) = -1$ by (3a); then y_1 must vanish for some $0 < t < 1$. This is absurd since f_{σ_0} has no pole on $[-\frac{1}{2}, \frac{1}{2}]$, hence nowhere on the real axis. Hence the plus sign holds, and $y_1(t, \sigma_0)$ extends throughout the horizontal strip $|\operatorname{Im} t| < \frac{1}{2}$ as an even function of period 1. We have established

LEMMA 2. *$y_1(t, \sigma_0)$ is a bounded nonvanishing function for real values of t.*

Proof of Theorem 1. The eigenvalues $\lambda_0 < \lambda_1 < \cdots$ of the Sturm-Liouville operator L form a strictly increasing sequence tending to infinity, and each eigenspace is one-dimensional [7]. The statement $Lu = (\sigma/2)u$ is equivalent to $y = u$ satisfying the associated linear equation (2a). By the normalization (3a), $y_1'(t, \sigma)$ vanishes at $t = 0$ for all σ. We must now look at the remaining boundary condition. As already remarked, the right and left sides of D_σ are straight for $\sigma = \sigma_0$ but not for $|\sigma| < \sigma_0$. Since the right side is the image of the side $t = \frac{1}{2} + ir$, $-\frac{1}{2} < r < \frac{1}{2}$ of Π under $w = f_\sigma(\frac{1}{2} + ir)$, the condition on σ_0 is that the curvature

$$\kappa = \operatorname{Im} \bar{w}'w'' / |w'|^3$$

vanishes (we already know it is constant). But the symmetry $w(-\bar{r}) = \overline{w(r)}$ shows that w' is imaginary, w'' real at $r = 0$ for any real σ, and $w' \neq 0$ since f_σ is conformal. Thus the straightness is equivalent to

$$f_\sigma''(\tfrac{1}{2}) = 0.$$

Differentiating (4) gives

$$f_\sigma'' = -2y_1^{-3}y_1'.$$

Since by Lemma 2, $y_1(\frac{1}{2}, \sigma_0) \neq 0$, the condition on σ_0 becomes $y_1'(\frac{1}{2}, \sigma_0) = 0$. This is the boundary condition at $t = \frac{1}{2}$, and shows at once that $\sigma_0/2$ is the least positive eigenvalue of L. But there are no negative eigenvalues: for any eigenvalue λ with eigenvector u,

$$\begin{aligned} \lambda \textstyle\int u^2\,dt &= \textstyle\int(-u''u - (Q/4)u^2)\,dt \\ &= \textstyle\int(u'^2 - (Q/4)u^2)\,dt \\ &> 0, \end{aligned}$$

since $Q \leq 0$ on $\mathbb{R}$; therefore $\lambda > 0$. This concludes the proof.

LEMMA 3. *Let $f: D_1 \to D_2$, $F: D_2 \to D_3$ be conformal. Write $[F] = \Phi$, $[F \circ f] = \phi$. Let Y be any solution of $2Y'' + \Phi Y = 0$ in D_2. Then*

$$y = (Y \circ f) f'^{-1/2}$$

is a solution of $2y'' + \phi y = 0$ in D_1.

The proof is a direct calculation; the result is that the solutions of the linear equation for $F \circ f$ are the solutions for F "pulled back" to D_1 as differentials of type $(-\frac{1}{2}, 0)$. Applying the lemma to F_σ and f_0 (choosing the positive square root of $f_0'(t)$ at $t = 0$) we obtain an isomorphism between the solution spaces of (2a) and (2b) such that $y_1 \leftrightarrow Y_1$ and $Y_2 \leftrightarrow Y_2$. This enables us to study the growth rate of $Y_1(t, \sigma_0)$ by looking at $y_1(t, \sigma_0)$ instead.

LEMMA. 4. $f_0'(t) \to 0$ as $t \to \pm\infty$.

Proof. Since $A_0'(x) = R^2/(x + R\sqrt{2})^2 \le \frac{1}{2}$ for $x \ge 0$, we have in particular for $t \in [0, 1]$ that $f_0(t + 1) = A_0 f_0(t)$, $f_0'(t + 1) \le (\frac{1}{2}) f_0'(t)$, and inductively

$$f_0'(t + n) \le 2^{-n} f_0'(t), \qquad n = 1, 2, \cdots.$$

Now let $n \to \infty$. The proof for $t \to -\infty$ follows similarly.

By Lemmas 3 and 4, $Y_1(x) = y_1(t) f_0'(t)^{1/2}$ is the product a nonvanishing function and one which tends to zero.

LEMMA 5. *$Y_1(x, \sigma_0) \to 0$ as $x \to \pm R$ along real values; Y_1 does not vanish in $-R < x < R$.*

Proof of Theorem 2. Introduce the Prüfer transform [7] of equation (3b),

$$\tan \Phi = Y_1/Y_1',$$
$$\rho^2 = Y_1'^2 + Y_1^2.$$

The first equation with the added condition

$$\Phi(0) = \pi/2 \tag{5}$$

defines Φ continuously on $(-R, R)$. We take ρ to be everywhere positive. Differentiating gives

$$\Phi' = (Y_1'/\rho)^2 + (\sigma J'^2/2)(Y_1/\rho)^2$$

so

$$\Phi' = \cos^2 \Phi + (\sigma J'^2/2) \sin^2 \Phi. \tag{6}$$

We observe that $\Phi(x)$ can be an integral multiple of π only where $Y_1(x)$ vanishes, and $\Phi'(x) = 1$ at such a point. Therefore Φ "counts" the zeroes of Y_1 in the sense that if $x_1 < x_2$ and

$$m\pi < \Phi(x_1) < (m + 1)\pi$$
$$n\pi < \Phi(x_2) < (n + 1)\pi$$

then Y_1 vanishes precisely $n - m$ times between x_1 and x_2.

Write $\Phi_\sigma(x)$ to indicate the dependence on σ. By Lemma 5 we can extend Φ_{σ_0} continuously to $[-R, R]$ with $\Phi_{\sigma_0}(-R) = 0$, $\Phi_{\sigma_0}(R) = \pi$. Now regard (6) as a differential equation of the form

$$\Phi_\sigma'(x) = h_\sigma(\Phi(x), x).$$

Since $h_\sigma(\Phi, x)$ is a strictly increasing function of σ for fixed (Φ, x), by standard theory [7] the solutions $\Phi_\sigma(x)$ with the common initial condition (5) increase strictly with σ for fixed x. Therefore $\Phi_\sigma(R) > \pi$ when $\sigma > \sigma_0$. A similar argument shows that $\Phi_\sigma(-R) < 0$.

We conclude that $Y_1(x, \sigma)$ vanishes at least twice in the open interval $(-R, R)$. By the result of Nehari mentioned earlier, F_σ is not schlicht and σ is not in the closure of $T(G_0)$ when $\sigma > \sigma_0$.

4. Remarks

(i) *The accessory parameter problem.* The determination of σ_0, which will be presented numerically in the following section, can be rephrased in terms of the accessory parameter problem for conformal mapping [11]. In general if h: $\{|s|<1\} \to D$ is a conformal mapping from the unit disk to a domain bounded by a finite number of circular arcs and straight line segments, then $[h]$ is a rational function with double poles at the points $s_1, \cdots, s_n$ on $|s|=1$ mapping to vertices of D. Some of the coefficients of this function are given by the angles at the vertices $h(s_i)$, but it is difficult in general to describe how the remaining ones, and indeed, the values s_i, depend on the geometry of D.

Because of the symmetry, our problem has essentially one accessory parameter. We may factor $h = f_\sigma \circ g$ where

$$t = g(s) = k \int_0^s (1+s^4)^{-1/2}\, ds$$

is the elliptic integral mapping $|s|<1$ onto Π (choose the positive square root of $1+s^4$ at $s=0$; then choose the constant k appropriately). Then

$$[h] = ((Q \circ g)/2 + \sigma)g'^2 + [g].$$

A simple computation gives

$$[g](s) = -6s^2(1+s^4)^{-2}.$$

From the power series expansion

$$g(s) = k\int_0^s (1 - s^4/2 + \cdots)\, ds$$

$$= ks - ks^5/10 + \cdots$$

we get

$$Q(g(s)) = k^2 q_1 s^2 + 0(|s|^6).$$

Since $Q(t)$, regarded as a conformal mapping, sends Π twice over $\hat{\mathbf{C}}$ taking real values on the boundary of Π, $Q \circ g$ extends by reflection in $|s|=1$ and must be a rational function of order 4. One finds that

$$Q(g(s)) = (k^2 q_1 s^2)/(1+s^4)$$

which gives

$$[h] = \frac{(q_1 k^4/2 - 6)s^2 + k^2\sigma(1+s^4)}{(1+s^4)^2} \tag{7}$$

Expanding this in partial fractions exhibits the singularities at the fourth roots of -1. The problem, restated, is to determine $\sigma = \sigma_0$ so that in the mapping of the disk determined by (7) the images of the two arcs $|\arg s| < \pi/4$, $|\arg s - \pi| < \pi/4$ of the circle $|s|=1$ are straight (see also [4], [5]).

(ii) *The quasiconformal extension of* F_σ. For $\sigma \in T(G_0)$ the homeomorphic extension of F_σ to $\hat{\mathbf{C}}$ can be taken to be quasiconformal (see [3]) and can even be chosen so that its *Beltrami differential*

$$\mu = (\partial F/\partial \bar{z})/(\partial F/\partial z)$$

is of the special type known as a *Teichmüller differential*:

$$\mu = k\,|\,\psi\,|/\psi$$

where $\psi \in B_2(G_0, E)$ is a quadratic differential in the exterior E of Δ, and $0 \leq k \leq 1$. In our case we can take $\psi_0(z) = J'(1/z)^2$, $z \in E$. Then all Teichmüller differentials are

$$\mu_\tau = \tau\,|\,\psi_0\,|/\psi_0, \qquad |\,\tau\,| < |.$$

This determines a 1-to-1 correspondence $\tau \leftrightarrow \sigma$ between $|\,\tau\,| < 1$ and $T(G_0)$. By the results of [3] it is holomorphic (this does not hold in higher-dimensional Teichmüller spaces).

The correspondence is completely unknown, except that the real and imaginary values of τ correspond, respectively, to real and imaginary values of σ, that $\tau = 1$ corresponds to $\sigma = \sigma_0$ determined in this paper, and that if $\tan(\theta/2)$ is rational then $\tau = re^{i\theta}$ tends to a definite value $\sigma(\theta)$ which is therefore an accessible boundary point of the region $T(G_0)$. These rational rays correspond to those quadratic differentials on E whose horizontal trajectories project to closed curves on $\bar{S}_0 = E/G_0$ (these are known as *Strebel differentials*). As $r \rightarrow 1$ the change in conformal structure on $F_\sigma(E)/G_0$ corresponds to "pinching" one of these closed curves to a point. This is how it is shown [9] that the limit $\sigma(\theta)$ as $r \rightarrow 1$ exists, that $G_{\sigma(\theta)}$ is a regular b-group and a cusp, and that $A_{\sigma(\theta)}$ is parabolic.

5. Calculation of the Boundary Point

The first step is the computation of the Taylor coefficients q_j of $Q(t)$. These are all real numbers. Then the differential equation (2a) is solved by the standard term-by-term method, giving a power series $y_1(t)$. This series is differentiated termwise, and lastly the value $t = \frac{1}{2}$ is substituted, giving a value for $y_1'(\frac{1}{2}, \sigma)$. It is shown below that this vanishes for precisely one value of σ in the range $0 < \sigma < 2$, and this value σ_0 can therefore be approximated by successively subdividing the interval and seeing where $y_1'(\frac{1}{2})$ is positive or negative.

The Fortran programs used for the following calculations are presented in [13] with a detailed error analysis. The author wishes to thank Northwestern University for the use of its CDC 6600 Computer.

The function $Q(t)$. In the classical Weierstrass notation,

$$\begin{aligned}\wp'(t)^2 &= 4\wp(t)^3 - g_2\wp(t) - g_3\\ &= 4(\wp(t) - e_1)(\wp(t) - e_2)(\wp(t) - e_3).\end{aligned}$$

The calculation of the modular invariants g_i, e_i is easily accomplished with the rapidly-converging series for the Jacobian "theta constants" [12]. We find in particular that $e_3 = g_3 = 0$ and

$$Q(\tfrac{1}{2}) = e_2 = -6.87518581802037 \cdots$$
$$Q\left(\frac{i}{2}\right) = e_1 = -e_2$$
$$g_2 = 189.07272012923385229 \cdots$$

Further, from (8) we have $Q'' = 6Q^2 - g_2/2$ which gives the recursion relation

(9) $$q_1 = -g_2/4$$

$$q_{j+1} = \frac{4}{(2j+1)(4j+1)} \sum_{k=1}^{j} q_k q_{j+1-k}$$

Starting with the above value of g_2, one obtains the coefficients successively. Unfortunately these numbers grow quite rapidly, and for $j > 7$ most of the accuracy is lost. In order to compute a few more terms, a different formula was used: begin with the Weierstrass ζ-function

$$\zeta(z) = \frac{1}{t} + \sum{}' \left(\frac{1}{t-\omega} + \frac{t}{\omega^2} + \frac{1}{\omega}\right)$$

(the Σ' means $\omega = m + ni \neq 0$), which satisfies $\zeta' = -\wp$. Substitute $t + (1+i)/2$ for t and rewrite each of the fractions $1/(t\text{-constant})$ as a geometric series. By uniform and absolute convergence the resulting double series can be rearranged, gathering like powers of t. Differentiating gives $-Q(t)$, and one arrives at

$$q_j = (4j-1) \sum c_{mn}^{-4j},$$

where $c_{mn} = (m - \tfrac{1}{2}) + i(n - \tfrac{1}{2})$. Observing that

$$c_{mn} = \overline{c_{m,1-n}} = \overline{-c_{1-m,n}} = -c_{1-m,1-n},$$

the terms can be paired off, giving finally

(10) $$q_j = 4(4j-1)\left(\sum_{m=1}^{\infty} (-4)^{-j}(m - \tfrac{1}{2})^{-4j} + 2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1} \operatorname{Re} c_{mn}^{-4}\right).$$

It was found that it was sufficient to use only the first term in the first summation:

$$q_j^* = (-1)^j(4j-1)4^{j+1};$$

the sum of the remaining terms decreases rapidly as j increases. The bound on the error $|q_j^* - q_j|$ worked out in [13] gives a bound on the growth of $|q_j|$, and thus a limit on the error when $Q(t)$ is approximated by a polynomial $\sum_{j=1}^{N} q_j^* t^{4j-2}$. The calculations were performed with $N = 10$. Since tabulated values of the $\wp$-function for the periods 1 and i are not easily found in the literature, a brief table of the coefficients and values of $Q(t)$ is appended.

Computation of $y_1'(\tfrac{1}{2}, \sigma)$. Substituting the expression

(11) $$y(t) = \sum_1^\infty a_n t^{n-1}$$

into (2a) and applying the initial conditions $a_1 = y_1(0) = 1$, $a_2 = y_1'(0) = 0$ gives

$$a_{2n} = 0,$$

$$a_{2n+1} = \frac{-1}{4(2n-3)(2n-1)} \left(2\sigma a_{2n-3} + \sum_{j=1}^{[(n-1)/2]} a_{2n-1-4j} q_j\right)$$

It is easily verified that $|a_{2n-1}| \le 2^{n-1}$ for $1 \le n \le 5$ when σ is in the range $|\sigma| < 2$. Then inductively it follows that $10|a_{2n-1}| \le 2^n$ for $n \ge 6$. This makes it possible to compute a bound for the error in discarding a tail of the series (11). Values of $y_1'(\frac{1}{2}, \sigma)$ for various values of σ are given in Table 3.

Approximation of σ_0. We see that $y_1'(\frac{1}{2}, 0) \approx .401$, $y_1'(\frac{1}{2}, 2) \approx -.111$ so there is at least one eigenvalue $\sigma/2$, that is, at least one zero of $y_1'(\frac{1}{2}, \sigma)$, in the range $0 < \sigma < 2$. If $\sigma_0/2 < \sigma_1/2 < \cdots$ are the eigenvalues of L, then (again by Sturm-Liouville theory) the eigenfunction $y_1(t, \sigma_n)$ vanishes n times in the interval $(0, \frac{1}{2}]$. Since y_1 is an even function, $y_1(t, \sigma_1)$ vanishes twice in $|t| \le \frac{1}{2}$.

Now compare the equations

$$y_1'' + ((Q + 2\sigma_1)/4)y_1 = 0,$$
$$y'' + k^2 y = 0,$$

where $k < 2\pi$. The solution $\cos kt$ of the second equation does not vanish in $|t| \le \frac{1}{2}$, and the second equation is a Sturm majorant of the first if $\sigma_1 < 2k^2$. In that case, y_1 does not vanish in $|t| < \frac{1}{2}$, a contradiction. Hence $\sigma_1 \ge 8\pi^2$. This is well out of the range $0 < \sigma < 2$ and the process of successive subdivisions is guaranteed to converge to the desired value σ_0. This value is $\sigma_0 \approx 1.552$ approximately.

The boundary group G_{σ_0}. Once the Taylor coefficients of $y_1(t, \sigma_0)$ are known, by (4) we may integrate the square of the reciprocal of the power series term by term to obtain f_{σ_0}. From the geometry of D_{σ_0}, we may write the element B_{σ_0} as

$$B_{\sigma_0} = \begin{pmatrix} b/a & i(b^2 - a^2)/a \\ 1/ia & b/a \end{pmatrix}$$

(its isometric circle is centered at $-ib$, with radius a). Numerically we find that $f_{\sigma_0}(\frac{1}{2}) \approx .5218$, $f_{\sigma_0}(i/2) \approx .4606i$, giving $a \approx .5218$, $b \approx .9214$. Thus the generators of the boundary group G_{σ_0} are

$$A_{\sigma_0} \approx \begin{pmatrix} 1 & 1.044 \\ 0 & 1 \end{pmatrix}$$

$$B_{\sigma_0} \approx \begin{pmatrix} 1.883 & 1.328i \\ -1.917i & 1.882 \end{pmatrix}$$

TABLE 1. Taylor coefficients of $Q(t) = \wp(t + (1 + i)/2) = \sum_1^\infty q_j t^{4j-2}$

j	q_j	j	q_j
1	−47.2681800323085	6	376832.000408
2	446.856168713345	7	−1769472.00
3	−2816.27704417200	8	8126464.00
4	15360.0335013207	9	−36700160.000
5	−77823.9842681082	10	163577856.0000

TABLE 2. Selected values of $Q(t) = \wp(t + (1 + i)/2)$

T	$Q(T)$	T	$Q(T)$
0	0	.30	−3.9443
.01	−.0047	.31	−4.1679
.02	−.0189	.32	−4.3904
.03	−.0425	.33	−4.6109
.04	−.0756	.34	−4.8281
.05	−.1182	.35	−5.0407
.06	−.1701	.36	−5.2475
.07	−.2316	.37	−5.4473
.08	−.3024	.38	−5.6387
.09	−.3826	.39	−5.8205
.10	−.4722	.40	−5.9914
.11	−.5712	.41	−6.1503
.12	−.6793	.42	−6.2959
.13	−.7967	.43	−6.4272
.14	−.9231	.44	−6.5431
.15	−1.0585	.45	−6.6429
.16	−1.2026	.46	−6.7256
.17	−1.3553	.47	−6.7906
.18	−1.5164	.48	−6.8375
.19	−1.6855	.49	−6.8657
.20	−1.8624	.50	−6.8751
.21	−2.0467		
.22	−2.2379		
.23	−2.4355		
.24	−2.6390		
.25	−2.8478		
.26	−3.0612		
.27	−3.2784		
.28	−3.4986		
.29	−3.7209		

TABLE 3. Values of $y_1'(\frac{1}{2}, \sigma)$

σ	$y_1'(\frac{1}{2}, \sigma)$	σ	$y_1'(\frac{1}{2}, \sigma)$
0	.4013		
.1	.3747	1.1	.1143
.2	.3482	1.2	.0889
.3	.3218	1.3	.0635
.4	.2955	1.4	.0383
.5	.2693	1.5	.0131
.6	.2432	1.6	−.0119
.7	.2172	1.7	−.0369
.8	.1913	1.8	−.0617
.9	.1656	1.9	−.0864
1.0	.1399	2.0	−.1111

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