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ORIENTABILITY OF BUNDLES WITH RESPECT TO CERTAIN SPECTRA

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§1. Introduction

Let BO_n be the classifying space for real n-plane bundles, and $BO_n[t]$ be the space obtained from BO_n by killing the homotopy groups in dimensions less than t. Let $MO[t]$ denote the associated Thom spectrum, *localized at the prime* 2.

Given a ring spectrum R with unit, then for any space X, the map $X \to X \wedge R$ given by the unit in R is called the *Hurewicz map.*

We say that the ring spectrum R with unit orients $BO[t]$-bundles if there exists a map of ring spectra $MO[t] \to R$. Recall that if α is an n-plane bundle over X, then "R orients α" means that we have a map $X^\alpha \to R \wedge S^n$ of the Thom space X^α so that $S^n \to X^\alpha \to R \wedge S^n$ is the Hurewicz map for S^n. Thus if R orients $BO[t]$-bundles, it orients n-plane bundles trivial over the $(t-1)$-skeleton.

Let $\alpha_k = (E_k, p_k, X)$ be the associated bundle of k-frames, i.e. with fibre $V_{n,k}$, the Stiefel manifold of k-frames in n-space. Recall from [12], that α_k has an R-orientation through dimension t if there exists a mapping

$$X/E_k \to R \wedge (\Sigma V_{n,k})^{(t)}$$

so that the composite

$$(\Sigma V_{n,k})^{(t)} \to X/E_k \to R \wedge (\Sigma V_{n,k})^{(t)}$$

is the Hurewicz map for the homotopy t-skeleton of $\Sigma V_{n,k}$.

Let $\rho(t)$ be the vector field number, i.e., if $t = 4a + b$ where $0 \le b \le 3$, then $\rho(t) = 8a + 2^b$.

Then our main result is:

THEOREM 1.1. *Let α be an n-plane bundle over X. Suppose α is trivial over the $(\rho(t)$-$1)$-skeleton of X. Then α_k is R-orientable through dimension $2(n-k)$ if R orients $BO[\rho(t)]$-bundles and $n \equiv 0 \pmod{2^t}$.*

In the course of establishing (1.1), we obtain

THEOREM 1.2. *The spectra $RP_n^{\,n+k} \wedge R$ and $RP_m^{\,m+k} \wedge R$ are stably homotopy equivalent if $n \equiv m$ mod 2^t) and R orients $BO[\rho(t)]$-bundles.*

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This result may be helpful in understanding the stable homotopy type of stunted real projective spaces, (cf. [9]).

As above, for complex bundles we may consider $BU_n[t]$ and the associated Thom spectrum $MU[\rho(t)]$ localized at the prime 2. Again we say R orients $BU[\rho(t)]$ bundles if there is a map of ring spectra $MU[\rho(t)] \to R$. If α is a complex n-plane bundle over X, we denote by $\tilde{\alpha}_k$ the associated bundle of *real* k-frames, i.e. with fiber $V_{2n,k}$. Then similarly to (1.1), we obtain:

THEOREM 1.3. *Let α be a complex n-plane bundle over X. Suppose α is trivial over the $(\rho(t)$-1)-skeleton of X. Then $\tilde{\alpha}_k$ is R-orientable through dimension $2(2n - k)$ if R orients $BU[\rho(t)]$-bundles and $n \equiv 0 \pmod{2^t}$.*

The orientation maps M Spin $\to bo$ of [3] and $MU \to BP$ of [13] give from (1.1) and (1.3),

COROLLARY 1.4. *The associated bundles of real k-frames for spin-bundles are bo-orientable and those associated to complex bundles are BP-orientable through the stable range.*

The result (1.4) provides a natural construction of the orientation maps for *Sp*-bundles produced in a very unnatural way in [7].

We hope that (1.1) and (1.3) will enable us to further exploit the approach to obstruction theory initiated in [6], [7] and [12].

Using (1.4) and calculation of $[RP^n, \Sigma RP^k \wedge MU]$ affords an obstruction theoretic proof of the strong geometric dimension results of Astey in [1] which imply the non-immersion results of [2].

§2. Proof of the Results

Let γ be the canonical bundle over BO_N. Let ξ be the Hopf line bundle over RP^{k-1} and consider the tensor product bundle $\gamma \otimes \xi$ over $BO_N \times RP^{k-1}$. With Z_2-coefficients

$$H^*(BO_N \times RP^{k-1}) = Z_2[W_1, \cdots, W_N] \otimes Z_2[x]/(x^k)$$

and we have

LEMMA 2.1. *The Stiefel-Whitney class $W_N(\gamma \otimes \xi)$ is given by*

$$W_N(\gamma \otimes \xi) = \textstyle\sum_{i=0}^{N} W_i \otimes x^{N-i}$$

Proof. By the splitting principle, $W_N(\gamma \otimes \xi)$ is the class $(x + x_1) \cdots (x + x_N)$. We may then write

$$W_N(\gamma \otimes \xi) = \textstyle\sum_{i=0}^{N} \sigma_i(x_1, \cdots, x_N) \otimes x^{N-i}$$

which is (2.1).

We will denote $BO_N[\rho(t)]$ by Y_N. Assume $N \equiv 0 \pmod{2^t}$. Let $\bar{\gamma}$ be the induced bundle from γ. Consider the tensor product bundle $\bar{\gamma} \otimes \xi$ over $Y_N \times RP^{k-1}$. Since the $(\rho(t)\text{-}1)$-skeleton of $Y_N \times RP^{k-1}$ is $* \times RP^m$, where $m = \min(\rho(t)\text{-}1, k-1)$ and $\bar{\gamma} \otimes \xi$ restricted to RP^m is $N\xi$ and $N \equiv 0 \pmod{2^t}$, it follows that $\bar{\gamma} \otimes \xi$ is trivial over the $(\rho(t)\text{-}1)$-skeleton of $Y_N \times RP^{k-1}$ and hence $\bar{\gamma} \otimes \xi$ gives rise to a unique mapping:

$$Y_N \times RP^{k-1} \xrightarrow{f} Y_N$$

which makes the following diagram

(2.2)
$$\begin{array}{ccc} Y_N \times RP^{k-1} & \xrightarrow{f} & Y_N \\ \downarrow & & \downarrow \\ BO_N \times RP^{k-1} & \xrightarrow[g]{} & BO_N \end{array}$$

commutative, where g classifies $\gamma \otimes \xi$.

We have the following result due to J. C. Becker [4, (3.11)] and I. M. James [11].

PROPOSITION 2.3. *Let α be a vector bundle over X. Then α has k-linearly independent sections implies that $\alpha \otimes \xi$ over $X \times RP^{k-1}$ has a never-zero section.*

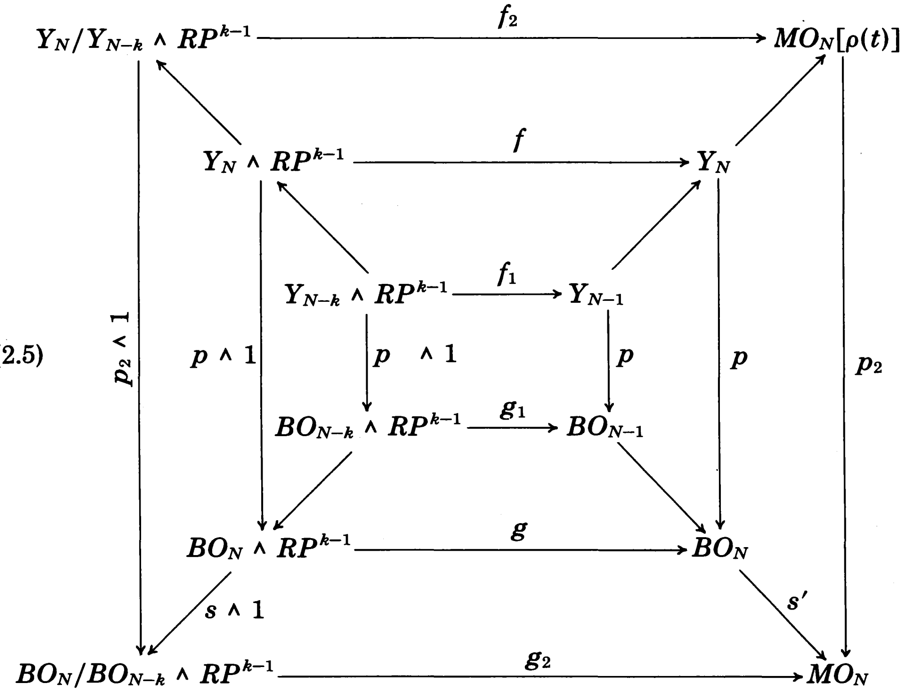
We use (2.3) to enlarge (2.2) as follows:

(2.4)
$$\begin{array}{ccccccc} Y_N \times RP^{k-1} & & & \xrightarrow{f} & & & Y_N \\ \big\downarrow & \nwarrow & & & & \nearrow & \big\downarrow \\ \big\downarrow & & Y_{N-k} \times RP^{k-1} & \xrightarrow{f_1} & Y_{N-1} & & \big\downarrow \\ \big\downarrow & & \downarrow & & \downarrow & & \big\downarrow \\ \big\downarrow & & BO_{N-k} \times RP^{k-1} & \xrightarrow{g_1} & BO_{N-1} & & \big\downarrow \\ \big\downarrow & \swarrow & & & & \searrow & \big\downarrow \\ BO_N \times RP^{k-1} & & & \xrightarrow{g} & & & BO_N \end{array}$$

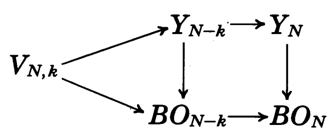
The existence of f_1 and g_1 follows since $\bar{\gamma}$ and γ restricted to Y_{N-k} and BO_{N-k} respectively have k-sections, so the restricted associated bundles $\bar{\gamma} \otimes \xi$ and $\gamma \otimes \xi$ have a section by (2.3).

Passing to stable maps, we may enlarge (2.4) to

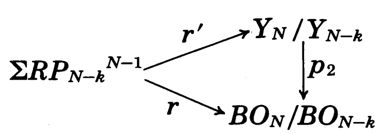
(2.5)



where the diagonal maps are of cofibration sequences and the ones on the left are obtained from those on the right by smashing with RP^{k-1} the corresponding projections.



and $RP_{N-k}{}^{N-1} \subset\!\!\rightarrow V_{N,k}$, so that we obtain a commutative diagram:



Now, it is well known that there are unique classes $v_i \in H^i(BO_N/BO_{N-k})$ with $N - k + 1 \leq i \leq N$ so that

$$r^*v_i = \sigma^*y^{i-1}$$

$$s^*v_i = W_i$$

where $y \in H^1(RP^{N-1})$ is the generator and σ^* is the suspension isomorphism in cohomology.

Let U and $\bar{U}$ be the Thom classes of MO_N and $MO_N[\rho(t)]$ respectively.

LEMMA 2.6. *We have*

$$f_2^* \bar{U} = \textstyle\sum_{i=n-k+1}^{N} p_2^* v_i \otimes x^{N-i}.$$

Proof. Since $p\,(\min)_1{}^* U = \bar{U}$, we have to consider $f_2{}^* p_1{}^* U = (p_2 \wedge 1)_* g_2{}^* U$. So it suffices to study $g_2{}^* U$. Now

$$(s \wedge 1)_* g_2{}^* U = g_* s'_* U = g_* W_N = \Sigma W_i \otimes x^{N-i}$$

also $(s \wedge 1)_*(\Sigma v_i \otimes x^{N-i}) = \Sigma W_i \otimes x^{N-i}$ but $(s \wedge 1)_*$ is a monomorphism, hence $g_2{}^* U = \Sigma v_i \otimes x^{N-i}$ and (2.6) follows.

If we now consider the map $\alpha = f_2 \circ (r' \wedge 1)$

$$(2.7) \qquad \Sigma RP_{N-k}{}^{N-1} \wedge RP_1{}^{k-1} \xrightarrow{\ \alpha\ } MO_N[\rho(t)]$$

then

$$\alpha_* \bar{U} = \textstyle\sum_{i=N-k+1}^{N-1} \sigma_* y^{i-1} \otimes x^{N-i}.$$

Let

$$\hat{\alpha}: \Sigma RP_{N-k}{}^{N-2} \to MO_N[\rho(t)] \wedge RP_{M-k}{}^{M-2}$$

be the dual map to α. Then it is easy to see that

$$\hat{\alpha}^*(\bar{U} \otimes z^{M-k+i}) = \sigma^* y^{N-k+i}$$

and $\hat{\alpha}^*$ is epimorphic in mod 2 cohomology. Thus if we consider

$$\begin{array}{ccc} MO[\rho(t)] \wedge \Sigma RP_{N-k}{}^{N-2} & \xrightarrow{\ 1 \wedge \tilde{\alpha}\ } & MO[\rho(t)] \wedge MO[\rho(t)] \wedge RP_{M-k}{}^{M-2} \\ & \searrow^{\beta} & \downarrow{\scriptstyle \mu \wedge 1} \\ & & MO[\rho(t)] \wedge RP_{M-k}{}^{M-2} \end{array}$$

then β^* in cohomology is an epimorphism. But now

$$\dim H^q(MO[\rho(t)] \wedge \Sigma RP_{N-k}{}^{N-2}) = \dim H^{q+M-N-1}(MO[\rho(t)] \wedge RP_{M-k}{}^{M-2})$$

and hence β^* is an isomorphism, so that β is a homotopy equivalence. If now $\rho: MO[\rho(t)] \to R$ is a map of ring spectra with unit, then $\rho^* 1_R = \underline{U}$ and the same arguments prove that $R \wedge \Sigma RP_{N-k}{}^{N-2}$ and $R \wedge RP_{M-k}{}^{M-2}$ are stably homotopically equivalent.

Proof of (1.2). We use naturality of the constructions as follows. Given that $n \equiv m \pmod{2^t}$ and given k, set $h = 2^L - m$, $N = h + n$, then $N \equiv o \pmod{2^t}$.

Up to homotopy, we have a commutative diagram:

(2.8)
$$\begin{array}{ccc}
Y_N/Y_{n-h} \wedge RP^{h-k-2} & \xrightarrow{\rho \wedge 1} & Y_N/Y_{N-h+k+1} \wedge RP^{h-k-2} \\
\downarrow{\scriptstyle 1 \wedge \underline{i}} & & \downarrow{\scriptstyle f_2'} \\
Y_N/Y_{N-h} \wedge RP^{h-1} & \xrightarrow[f_2]{} & MO[\rho(t)]
\end{array}$$

and hence a commutative diagram up to homotopy

$$\begin{array}{ccc}
RP_{N-h}{}^{N-1} \wedge RP^{h-k-2} & \xrightarrow{c \wedge 1} & RP_{N-h+k+1}{}^{N-1} \wedge RP^{h-k-2} \\
\downarrow{\scriptstyle 1 \wedge i} & & \downarrow{\scriptstyle \alpha'} \\
RP_{N-h}{}^{N-1} \wedge RP^{h-1} & \xrightarrow{\alpha} & MO[\rho(t)]
\end{array}$$

Let $D(i): DRP^{h-1} \to DRP^{h-k-2}$ be the dual of i

$D(\alpha): RP_{N-h}{}^{N-1} \to MO[\rho(t)] \wedge DRP^{h-1}$ the dual of α

and $D(\alpha'): RP_{N-h+k+1}{}^{N-1} \to MO[\rho(t)] \wedge DRP^{h-k-2}$ the dual of α'.

Then the following diagram is homotopy commutative

(2.10)
$$\begin{array}{ccc}
RP_{N-h}{}^{N-1} & \xrightarrow{c} & RP_{N-h+k+1}{}^{N-1} \\
\downarrow{\scriptstyle D(\alpha)} & & \downarrow{\scriptstyle D(\alpha')} \\
MO[\rho(t)] \wedge DRP^{h-1} & \xrightarrow[1 \wedge D(i)]{} & MO[\rho(t)] \wedge DRP^{h-k-2}
\end{array}$$

This follows from the definition of duals [14, Theorem 5.9] and (2.9). Hence (2.10) extends to a homotopy commutative diagram:

$$\begin{array}{ccc}
MO[\rho(t)] \wedge RP_{N-h}{}^{N-1} & \xrightarrow{1 \wedge c} & MO[\rho(t)] \wedge RP_{N-h+k+1}{}^{N-1} \\
\downarrow & & \downarrow \\
MO[\rho(t)] \wedge DRP^{h-1} & \xrightarrow{1 \wedge D(i)} & MO[\rho(t)] \wedge DRP^{h-k-2}
\end{array}$$

Now the vertical maps are stable homotopy equivalences, so the cofibers of the horizontal maps are stable homotopy equivalences, i.e.

$$MO[\rho(t)] \wedge RP_{N-h}{}^{N-h+k} \cong MO[\rho(t)] \wedge DRP_{h-k-1}{}^{h-1}$$

and (1.2) follows, by passing to R as above.

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