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SOME DYNAMICAL PROPERTIES OF CERTAIN DIFFERENTIABLE MAPPINGS OF AN INTERVAL

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1. Introduction

The aim of this paper is to present some qualitative and quantitative results concerning the behavior of the derivatives of iterates of some smooth mappings of an interval. The estimates obtained allow one to give some sufficient conditions for existence of an invariant measure absolutely continuous with respect to the Lebesgue measure for these mappings. The technique used in this paper is partially similar to the technique used by Bunimovič in [1] and is partially similar to the methods used by Sinaj on other occasions. In particular Lemma 4 was proved jointly by Sinaj and the author of this article (in 1968, the result was not published). The other version of this lemma has also been proved by Jakobson [3]. One result (Section 6, Example 1) has been proved just recently by Misiurewicz [5] in a stronger version.

Section 2 contains the definitions, notation and the statements of some results which are used in this article.

In Section 3 we present some basic estimates (Lemma 2a, 2b) and certain general results on some dynamical properties of the studied mappings (Theorems 1 and 2).

In Section 4 we study the behavior of the derivatives of the iterates at single points (Lemma 6, Corollary 2) and on some intervals (Theorem 3).

In Section 5 a sufficient condition is given for the existence of an invariant measure absolutely continuous with respect to the Lebesgue measure (Theorem 4).

In Section 6 three groups of mappings are presented, which admit an invariant measure absolutely continuous with respect to the Lebesgue measure.

2. Notation and definitions

Through this paper we study some differentiable mappings of class C^2 of the interval $\langle 0, 1\rangle$ in itself: $f: \langle 0, 1\rangle \to \langle 0, 1\rangle$. The composition $\underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ is denoted by f^n; we denote $x_n = f^n(x)$, $x_0 = x$. The same convention is used in the case of sets: $A_n = f^n(A)$. We denote by

$$C_n = \{x: (f^n)'(x) = 0\} \cup \{0, 1\}, \qquad n = 1, 2, \cdots,$$
$$C_n^0 = \{x: (f^n)'(x) = 0\}.$$

Assuming C_n to be finite we put in increasing order the points of C_n: $0 = c_{n,0} < c_{n,1} < \cdots < c_{n,r_n} = 1$, where $r_n = \text{Card } C_n - 1$. Since $(f^n)'(x) = f'(x_{n-1})(f^{n-1})'(x)$, we have

(1) $$C_n = C_{n-1} \cup f^{-(n-1)}(C_1).$$

We set $R = \bigcup_{n=1}^{\infty} f^n(C_1)$, $R^0 = \bigcup_{n=1}^{\infty} f^n(C_1^0)$.

Let $\Delta_{n,j} = (c_{n,j-1}, c_{n,j})$, $j = 1, \cdots, r_n$. The family of intervals $\{\Delta_{n,j}\}_{j=1}^{r_n}$ is denoted by Δ_n, and we set diam $\Delta_n = \max_{1\le j\le r_n} |\Delta_{n,j}| = \max_{1\le j\le r_n} |c_{n,j} - c_{n,j-1}|$. In virtue of (1) Δ_{n+1} is a refinement of Δ_n. Let $(\Delta_{n,j_n})_{n=1}$ be a decreasing sequence of intervals: $\Delta_{n,j_n} \supset \Delta_{n+1,j_{n+1}}$. Let

$$K = \bigcap_{n=1}^{\infty} \bar{\Delta}_{n,j_n}. \tag{2}$$

The set K is either a point, or a closed interval: $K = \langle \alpha, \beta \rangle$, $\beta > \alpha$.

Finally, we set $\lambda = \max_{x\in\langle 0,1\rangle} |f'(x)|$, $\vartheta = \max_{x\in\langle 0,1\rangle} |f''(x)|$. By λ_i, ϑ_i we denote some other estimates of $|f'(x)|$ and $|f''(x)|$ restricted to some sets.

By A' we denote the completion of a set $A \subset \langle 0, 1\rangle$ to the whole interval: $A' = \langle 0, 1\rangle - A$.

We say that a point $x_0 \in \langle 0, 1\rangle$ of a mapping f is a periodic half-attracting point if x_0 is periodic: $f^p(x_0) = x_0$ for some p, and there exists a $\delta > 0$ such that for every $x \in (x_0 - \delta, x_0)$ or $x \in (x_0, x_0 + \delta)$, $f^{kp}(x) \to x_0$ as $k \to +\infty$.

Definition 1. Let $f \in C^3$. The Schwartzian derivative Sf of the function f is defined as follows:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

The idea of using this notion is due to Singer [6].

PROPOSITION *1. Let $f, g: \langle 0, 1\rangle \to \langle 0, 1\rangle$. $Sf \le 0$, $Sg \le 0$. Then $S(f\circ g) \le 0$.*

PROPOSITION *2. If $Sf \le 0$, then $|f'|$ has no positive local minima.*

Definition 2. Let $g \in C^0$. We say that the function g is *strongly decreasing* if there exists a constant number $w < 0$ such that for any $0 \le y < x \le 1$ we have $g(x) - g(y) \le w(x - y)$.

PROPOSITION *3. Let $f: \langle 0, 1\rangle \to \langle 0, 1\rangle$ be of class C^2 such that $h \stackrel{df}{=} \frac{f''}{f'}$ is strongly decreasing on the intervals where it is continuous. Then for any $n = 1, 2, \cdots$ the function $|(f^n)'|$ has no local positive minima.*

Proof: It is enough to approximate f by a function $g \in C^3$ in C^2-topology and apply Proposition 1 and 2.

Definition 3. We say that a set $A \subset \langle 0, 1\rangle$ is *totally wandering* if for any $n \ne m$ $A_n \cap A_m = \varnothing$. We say that a set $A \subset \langle 0, 1\rangle$ is *trivially totally wandering* if there exists a periodic point $x_0 \in \langle 0, 1\rangle$ such that the limit set $\omega(A)$ is equal to the orbit of x_0:

$$\omega(A) = \{x_0, f(x_0), \cdots, f^{p-1}(x_0)\},$$

where $f^p(x_0) = x_0$.

Section 3.

THEOREM 1. *Let* $f: \langle 0, 1\rangle \to \langle 0, 1\rangle$. *We assume*

A.1. $f \in C^2$

A.2. Card $C_1 < +\infty$

A.3. $f'' |_{C_1^0} \neq 0$

A.4. *There exists a number* $\lambda_0 > 1$ *such that setting* $V_0 = \{x: | f'(x) | > \lambda_0\}$ *the following inclusion holds*: $R \subset V_0$.

Then the dynamical system $(\langle 0, 1\rangle, f)$ *has no non-trivial totally wandering intervals.*

For the proof we need some lemmas.

Let V_1 be an open set such that $V_1 \supset \bar{V}_0$, and let $\inf_{x\in V_1} | f'(x) | \overset{df}{=} \lambda_1 > 1$.

LEMMA *1. Assume that for every* $j = 0, 1, \cdots, k$, $f^j(x)$, $f^j(y)$ *belong to the same component of* V_1. *Then the following inequality holds:*

$$\left|\frac{(f^j)'(x)}{(f^j)'(y)}\right| \leq \delta \qquad \textit{for every } j = 0, 1, \cdots, k,$$

where $\delta = \exp\left\{\dfrac{\vartheta_1}{\lambda_1(\lambda_1 - 1)}\right\}$ *and* $\vartheta_1 = \max_{x\in V_1} | f''(x) |$.

Proof. By assumption $| f'(x) | \geq \lambda_1 > 1$ for $x \in V_1$. Using the Taylor formula and the inequality $1 + u \leq e^u$ we have

$$(1) \qquad \left|\frac{(f^j)'(x)}{(f^j)'(y)}\right| = \prod_{s=0}^{j-1} \left|\frac{f'(x_s)}{f'(y_s)}\right| \leq \prod_{s=0}^{j-1}\left[1 + \frac{| f'(y_s) - f'(x_s) |}{| f'(x_s) |}\right]$$

$$\leq \prod_{s=0}^{j-1}\left[1 + \frac{| f''(\xi) | \cdot | y_s - x_s |}{\lambda_1}\right] \leq \exp\left\{\frac{\vartheta_1}{\lambda_1}\sum_{s=0}^{j-1} | y_s - x_s |\right\}.$$

Since $| y_{s+1} - x_{s+1} | = | f'(\eta_s) | \cdot | y_s - x_s | \geq \lambda_1 | y_s - x_s |$, $s = 0, \cdots, j - 1$, we have $\sum_{s=0}^{j-1} y_s - x_s | \leq | y_j - x_j | \sum_{s=0}^{j-1} \dfrac{1}{\lambda_1^{j-s}} \leq \dfrac{1}{\lambda_1 - 1}$.

Therefore

$$(2) \qquad \left|\frac{(f^j)'(x)}{(f^j)'(y)}\right| \leq \exp\left\{\frac{\vartheta_1}{\lambda_1} \cdot \frac{1}{\lambda_1 - 1}\right\} \overset{df}{=} \delta.$$

LEMMA *2a. Assume that all the assumptions of Theorem* 1 *are satisfied. Then there exist two numbers* $\delta > 0$ *and* $d_0 > 0$ *such that if* $U \overset{df}{=} \{x: | f'(x) | < \delta\}$, $x \in U_\delta - C_1^0$, *then there exists an integer* $k = k(x)$ *such that*

$$(a) \qquad | (f^{k+1})'(x) | \geq \frac{d_0}{| f'(x) |}$$

$$(b) \qquad f^j(x) \in V_1 \qquad \text{for every } j = 1, \cdots k.$$

Proof. Let δ be a small number such that the following conditions hold:

1) $\inf_{x\in U_\delta} | f''(x) | \overset{df}{=} \vartheta_2 > 0$.

2) Every component of U_δ contains exactly one point of C_1^0.

3) $f(\bar{U}_\delta) \subset V_1$.

In view of assumptions A.1–A.4 such δ exists. Denote $E = \{x: | f'(x) | < 1\}$, $E_m = \left\{ x: \frac{1}{2^m} \leq | f'(x) | < \frac{1}{2^{m-1}} \right\}$, $m = 1, 2, \cdots$. Without loss of generality we may assume that $U_\delta - C_1^0 = \bigcup_{m-m_0}^{\infty} E_m$ for some m_0. Let $a \in C_1^0$ and $x \in E_m \subset U_\delta$ belong to the same component of U_δ. Then

$$| x_1 - a_1 | = | f(x) - f(a) | = | f'(\xi) | \cdot | x - a | \leq \frac{1}{2^{m-1}} | x - a | ,$$

and (we omit some details)

$$| x_1 - a_1 | \geq \frac{1}{2^{m+1}} | x - a | .$$

On the other hand

$$f'(x) = f'(x) - f'(a) = f''(\eta)(x - a),$$

which implies

$$\frac{1}{2^m} \frac{1}{\vartheta} \leq | x - a | = \frac{| f'(x) |}{| f''(\eta) |} \leq \frac{1}{2^{m-1}} \frac{1}{\vartheta_2} .$$

Hence

$$\frac{1}{2^{2m}} \frac{1}{\vartheta} \leq | x_1 - a_1 | \leq \frac{1}{2^{2(m-1)}} \frac{1}{\vartheta_2} . \tag{3}$$

Let $b = \operatorname{dist}(R, V_1')$. Since $| f'(x) | \geq \lambda_1 > 1$ for $x \in V_1$, there exists an integer $k = k(x)$ such that

$$| f^{k-1}(x_1) - f^{k-1}(a_1) | \leq b \quad \text{and} \quad | f^k(x_1) - f^k(a_1) | > b. \tag{4}$$

In view of Lemma 1 we get

$$\begin{aligned} | f^k(x_1) - f^k(a_1) | &= | (f^k)'(\xi) | \cdot | x_1 - a_1 | \\ &= \textstyle\prod_{i=1}^{k} | f'(\xi_i) | \cdot | x_1 - a_1 | \\ &= \textstyle\prod_{i=1}^{k} \frac{| f'(\xi_i) |}{| f'(a_i) |} | x_1 - a_1 | \cdot \prod_{i=1}^{k} | f'(a_i) | \\ &\leq | x_1 - a_1 | \ \delta \textstyle\prod_{i=1}^{k} | f'(a_i) | , \end{aligned} \tag{5}$$

where $a_i < \xi_i < x_i$. The inequalities (3), (4) and (5) imply

$$\textstyle\prod_{i=1}^{k} | f'(a_i) | \geq \frac{b}{\delta} \frac{1}{| x_1 - a_1 |} \geq \frac{b}{\delta} \vartheta_2 2^{2(m-1)}. \tag{6}$$

Finally, once again using Lemma 1 we obtain

$$\begin{aligned} |(f^{k+1})'(x)| &= |f'(x)| \prod_{i=1}^{k} |f'(x_i)| \\ &\geq |f'(x)| \frac{\prod_{i=1}^{k} |f'(x_i)|}{\prod_{i=1}^{k} |f'(a_i)|} \prod_{i=1}^{k} |f'(a_i)| \\ &\geq |f'(x)| \frac{1}{\delta}\frac{b}{\delta} \vartheta_2 2^{2(m-1)} \geq \frac{b\vartheta_2}{\delta^2}\frac{1}{2^m} \cdot 2^{2(m-1)} \\ &= \frac{b\vartheta_2}{\delta^2} 2^m \geq \frac{d_0}{|f'(x)|} \end{aligned}$$

where

$$d_0 = \frac{b\vartheta_2}{\delta^2}. \tag{7}$$

Corollary 1. *If $x \in U_\delta$, then there exists a number k such that $f^j(x) \in V_1$, for $j = 1, \cdots, k$ and*

$$|(f^{k+1})'(x)| \geq \frac{d_0}{\lambda}.$$

It follows immediately from Lemma (2a).

Remark 1. Let $x \in E_m \subset U_j$, i.e. $\frac{1}{2^m} \leq |f'(x)| < \frac{1}{2^{m-1}}$. Then the number $k = k(x)$ in Lemma (2a) has the following property: if x and $a \in C_1^0$ belong to the same component of U_δ, then the points a_j, x_j, also belong to the same components of V_1 for $j = 1, \cdots, k$. Moreover

$$m \geq \frac{1}{2} k \frac{\log \lambda_1}{\log 2} - \frac{1}{2}\frac{\log b\vartheta}{\log 2}.$$

The last inequality follows easily from (3) and (4).

Lemma 2b. *There exists a number $d_1 > 0$ such that if I is an interval contained in U_δ, then there exists a number k such that*

$$|I_k| \geq d_1 \frac{|I|}{\max_{x\in I} |f'(x)|};$$

the constant d_1 is equal to $\frac{b\vartheta_2^2}{\delta^2\lambda\vartheta^2}$.

Proof. Let $U_\delta = (\alpha, \beta)$, let $a \in C_1^0$ and I belong to the same component of U_δ. Assume $a < \alpha < \beta$. We set $N = (a, \beta)$, $M = (a, \alpha)$. Let k be such that

$$|N_k| \leq b, \ |N_{k+1}| > b.$$

(we replace x by β in (4)). Thus $N_j \subset V_1$ for $j = 1, \cdots, k$, what implies

$(f^j)'(x) \neq 0$ for every $j = 1, \cdots, k-1$, $x \in N_1$. Hence

$$b < |N_{k+1}| = \int_{N_k} |f'(x)|\, dx \leq \lambda |N_k|$$
$$= \int_{N_1} |(f^{k-1})'(x)|\, dx \leq \lambda \max_{x\in N_1} |(f^{k-1})'(x)| \cdot |N_1|$$

and

$$\max_{x\in N_1} |(f^{k-1})'(x)| \geq \frac{b}{\lambda |N_1|}.$$

By Lemma 1 and by the last inequality we get (note that $I_k \subset N_k$)

$$\begin{aligned} |I_k| &= \int_{I_1} |(f^{k-1})'(x)|\, dx \\ &\geq \min_{x\in I_1} |(f^{k-1})'(x)| \cdot |I_1| \\ &\geq |I_1| \min_{x\in N_1} |(f^{k-1})'(x)| \geq |I_1| \frac{1}{\delta} \max_{x\in N_1} |(f^{k-1})'(x)| \\ &\geq \frac{b}{\delta\lambda} \frac{I_1}{N_1}. \end{aligned} \tag{8}$$

Now we shall estimate $\frac{|I_1|}{|N_1|}$ from below. By the property 1) of U_δ from Lemma 2a the function $|f'(x)|$ is increasing on the interval (α, β). Thus by Taylor's formula we have

$$\begin{aligned} |I_1| &= |f(\beta) - f(\alpha)| = |f'(\eta)| \cdot |\beta - \alpha| \geq |f'(\alpha)| \cdot |\beta - \alpha| \\ &= |f'(\alpha)| \cdot |I|. \end{aligned} \tag{9a}$$

$$|M_1| = |f(\alpha) - f(a)| = |f'(\xi)| \cdot |\alpha - a|, \quad \text{where } \xi \in (a, \alpha), \tag{9b}$$

$$f'(\xi) = f'(a) + f''(\zeta)(\xi - a) = f''(\zeta)(\xi - a), \quad \text{where } \zeta \in (a, \xi) \tag{9c}$$

$$f'(\alpha) = f'(a) + f''(\omega_1)(\alpha - a) = f''(\omega_1)(\alpha - a),\ \omega_1 \in (a, \alpha), \tag{9d}$$

$$f'(\beta) = f'(a) + f''(\omega_2)(\beta - \alpha) = f''(\omega_2)(\beta - \alpha),\ \omega_2 \in (a, \beta). \tag{9e}$$

By (9b), (9c), (9d) we have

$$\begin{aligned} |M_1| &= f'(\xi)| \cdot |\alpha - a| \leq |f''(\zeta)| \cdot |\alpha - a|^2 \\ &\leq \theta |\alpha - a|^2 \leq \theta \frac{|f'(\alpha)|^2}{|f''(\omega_1)|^2} \leq \frac{\theta}{\theta_2^2} |f'(\alpha)|^2. \end{aligned} \tag{10}$$

By Lemma 1, (9d) and (9e) we obtain

$$
\begin{aligned}
\frac{|M_1|}{|N_1|} &= \frac{\int_M |f'(x)|\,dx}{\int_N |f'(x)|\,dx} \ge \frac{\min_{x\in M} |f'(x)|}{\max_{x\in N} |f'(x)|} \cdot \frac{|M|}{|N|} \\
&\ge \frac{\min_{x\in N} |f'(x)|}{\max_{x\in N} |f'(x)|} \frac{|M|}{|N|} \ge \frac{1}{\delta} \frac{|M|}{|N|} \\
&= \frac{1}{\delta} \frac{|\alpha - a|}{|\beta - a|} = \frac{1}{\delta} \frac{|f'(\alpha)|}{|f'(\beta)|} \frac{|f''(\omega_2)|}{|f''(\omega_1)|} \\
&\ge \frac{1}{\delta} \frac{\theta_2}{\theta} \frac{|f'(\alpha)|}{|f'(\beta)|}.
\end{aligned}
\tag{11}
$$

The formulas (9a) and (10) give us

$$
\begin{aligned}
\frac{|I_1|}{|N_1|} &= \frac{|I_1|}{|M_1|} \frac{|M_1|}{|N_1|} \ge \frac{\theta_2^2}{\theta} \frac{|I|}{|f'(\alpha)|} \frac{1}{\delta} \frac{\theta_2}{\theta} \frac{|f'(\alpha)|}{|f'(\beta)} \\
&= \frac{\theta_2^3}{\delta\theta^2} \frac{|I|}{|f'(\beta)|}.
\end{aligned}
$$

The last inequality and (8) imply

$$
|I_k| \ge \frac{b}{\delta\lambda} \frac{|I_1|}{|N_1|} \ge \frac{b\theta_2^3}{\delta^2\lambda\theta^2} \frac{|I|}{|f'(\beta)|} = d_1 \frac{|I|}{\max_{x\in I} |f'(x)|},
\tag{12}
$$

where $d_1 = \dfrac{b\theta_2^3}{\delta^2\lambda\theta^2}$.

Remark 2. In fact, Lemma (2a) can be deduced from Lemma (2b). But in this way the constant d_0 given by (7) would be replaced by d_1. For some applications it is important that the constant d_0 in Lemma (2a) is exactly of the form (7) (see Section 6, Example (3)).

LEMMA 3. *Let $I \subset \langle 0, 1\rangle$ be an interval such that $\sum_{n=0}^{\infty} |I_n| < +\infty$, $I_n = f^n(I)$. We suppose that the assumptions of Theorem 1 are fulfilled. Then*

$$
l \stackrel{\text{df}}{=} \sum_{n=0}^{\infty} \frac{|I_n|}{\max_{x\in I_n} |f'(x)|} < +\infty.
$$

Proof. Let $I = \langle \alpha, \beta\rangle$. We divide all the intervals I_n, $n = 0, 1, \cdots$, in two groups G' and G'': 1) $I_n \in G'$ if I_n is not contained in U_δ (U_δ from Lemma 2a); 2) $I_n \in G''$ if $I_n \subset U_\delta$.

By assumption, we have

$$
\sum_{n=0}^{\infty} |I_n| < +\infty.
\tag{13}
$$

Thus if n_0 is big enough then for $n \geq n_0$ $|I_n| < \frac{\delta}{2}$. Suppose $n \geq n_0$ and $I_n \in G'$: then dist $(I_n, C_1^0) \geq \frac{\delta}{2}$ and $\min_{x \in I_n} |f'(x)| \geq \delta_1 > 0$, where δ_1 is a constant number. The intervals $I_n \in G''$ form a sequence $(I_{n_i})_{i=1}^{\infty}$ such that $I_{n_i} \in G''$ and some consecutive I_n, $n = n_i + 1, \cdots, n + k(n_i)$, do not belong to G''; the number $k(n_i)$ is the number k from Lemma 2b, $k(n_i)$ satisfies (12) for $I = I_{n_i}$. Of course $n_i + k(n_i) < n_{i+1}$. In virtue of Lemma 2b and (13) we have

$$\begin{aligned} l &= \sum_{n=0}^{\infty} \frac{|I_n|}{\max_{x \in I_n} |f'(x)|} \\ &= \sum_{n=0}^{n_0-1} \frac{|I_n|}{\max_{x \in I_n} |f'(x)|} \\ &\quad + \sum_{I_n \in G',\, n \geq n_0} \frac{|I_n|}{\max_{x \in I_n} |f'(x)|} \\ &\quad + \sum_{I_n \in G'',\, n \geq n_0} \frac{|I_n|}{\max_{x \in I_n} |f'(x)|} \\ &\leq \sum_{n=0}^{n_0-1} \frac{|I_n|}{\max_{x \in I_n} |f'(x)|} + \frac{1}{\delta_1} \sum_{I_n \in G',\, n \geq n_0} |I_n| \\ &\quad + \frac{1}{d_0} \sum_{I_n \in G'',\, n \geq n_0} |I_{n+k(n)}| < +\infty. \end{aligned}$$

Remark 3. We notice that if for every n, $|I_n| < \frac{\delta}{2}$ then

$$\sum_{n=0}^{\infty} \frac{|I_n|}{\max_{x \in I_n} |f'(x)|} \leq \left(\frac{1}{\delta_1} + \frac{1}{d_0}\right) \sum_{n=0}^{\infty} |I_n|.$$

Remark 4. Let $u_n = \max_{x \in I_n} |f'(x)|$, $v_n = \min_{x \in I_n} |f'(x)|$, and assume $v_n > 0$ for every n. Then if $\sum_{n=0}^{\infty} \frac{|I_n|}{u_n} < +\infty$, then also $\sum_{n=0}^{\infty} \frac{|I_n|}{v_n} < +\infty$. Moveover, if $\frac{|I_n|}{u_n} \leq \epsilon$ for every n, where $\epsilon\theta < 1$, then

$$\sum_{n=0}^{m} \frac{|I_n|}{v_n} \leq \frac{1}{1 - \epsilon\theta} \sum_{n=0}^{m} \frac{|I_n|}{u_n} \quad \text{for every } m = 1, 2, \cdots.$$

Proof. Let $y_n, z_n \in I_n$ be two points such that $u_n = |f'(y_n)|$, $v_n = |f'(z_n)|$. By Taylor's formula $\pm u_n = f'(y_n) = f'(z_n) + f''(\eta)(y_n - z_n)$, where $\eta \in (z_n, y_n)$.

Hence

$$u_n \leq v_n + \theta |I_n| \tag{14}$$

and

$$\frac{|I_n|}{u_n} \geq \frac{|I_n|}{v_n + \theta|I_n|} = \frac{\dfrac{|I_n|}{v_n}}{1 + \theta\dfrac{|I_n|}{v_n}}.$$

This implies easily the convergence of $\sum_{n=0}^{\infty} \frac{|I_n|}{v_n}$. If $\frac{|I_n|}{u_n} < \epsilon$ for every n, then $\frac{|I_n|}{v_n} < \frac{\epsilon}{1-\epsilon\theta}$ and $1 + \theta\frac{|I_n|}{v_n} \leq \frac{1}{1-\epsilon\theta}$.

Hence

$$\sum_{n=0}^{m} \frac{|I_n|}{v_n} \leq \frac{1}{1-\epsilon\theta} \sum_{n=0}^{m} \frac{|I_n|}{u_n}.$$

LEMMA 4. *Assume f to be of class C^2. Let $I = \langle\alpha, \beta\rangle \subset \langle 0, 1\rangle$ be an arbitrary interval such that $(f^n)'(x) \neq 0$ for every $x \in I$ and for every $n = 1, 2, \cdots$. If*

$$\sum_{n=0}^{\infty} \frac{|I_n|}{\max_{x\in I_n} |f'(x)|} < +\infty,$$

then there exist two intervals $U_\alpha = (\alpha', \alpha)$ and $U_\beta = (\beta, \beta')$, $\alpha' < \alpha$, $\beta < \beta'$, such that

$$\sum_{n=0}^{\infty} |f^n(U_\alpha)| < +\infty, \qquad \sum_{n=0}^{\infty} |f^n(U_\beta)| < +\infty.$$

Proof. We set $u_n = \max_{x\in I_n} |f'(x)|$, $v_n = \min_{x\in I_n} |f'(x)|$.

Since $(f^n)'(x) \neq 0$ for every $x \in I$ and for every n, we have $v_n > 0$ for $n = 1, 2, \cdots$. By Remark 4 we have

$$l = \sum_{n=0}^{\infty} \frac{|I_n|}{v_n} < +\infty. \tag{15}$$

By elementary arguments we get

$$|I_n| = |f(I_{n-1})| = \int_{I_{n-1}} |f'(x)|\, dx \geq v_{n-1}|I_{n-1}| \geq \cdots \geq |I| \prod_{j=0}^{n-1} v_j,$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{u_n} \prod_{j=0}^{n-1} v_j \leq \frac{1}{|I|} \sum_{n=0}^{\infty} \frac{|I_n|}{u_n} = \frac{1}{|I|} \sum_{n=0}^{\infty} \frac{|I_n|}{\max_{x\in I_n} |f'(x)|} < +\infty. \tag{16}$$

The inequalities (14) and (15) imply

$$\begin{aligned} \prod_{j=0}^{n-1} u_j &\leq \prod_{j=0}^{n-1} (v_j + \theta|I_j|) = \prod_{j=0}^{n-1} v_j\left(1 + \frac{\theta}{v_j}|I_j|\right) \\ &\leq \prod_{j=0}^{n-1} v_j \exp\left\{\theta \sum_{j=0}^{n-1} \frac{|I_j|}{v_j}\right\} \leq \prod_{j=0}^{n-1} v_j e^{\theta l}. \end{aligned} \tag{17}$$

The formulas (16) and (17) give us

$$\sum_{n=1}^{\infty} \frac{1}{u_n} \prod_{j=0}^{n-1} u_j \leq e^{\theta l} \sum_{n=1}^{\infty} \frac{1}{u_n} \prod_{j=0}^{n-1} v_j < +\infty. \tag{18}$$

Now we shall construct U_α, i.e. we have to define α'. Let $U_\alpha = (\alpha', \alpha)$, $f^n(U_\alpha) = (\alpha_n', \alpha_n)$, $n = 0, 1, \cdots$ (it is not necessary that for every n $\alpha_n' < \alpha_n$). By the Taylor's formula we have

$$\begin{aligned} |\alpha_{n+1}' - \alpha_{n+1}| = |f(\alpha_n') - f(\alpha_n)| &\leq |f'(\xi)| \cdot |\alpha_n' - \alpha_n| \\ &\leq |\alpha_n' - \alpha_n| (|f'(\alpha_n)| + |f''(\eta)| \cdot |\alpha_n' - \alpha_n|) \\ &\leq |\alpha_n' - \alpha_n| (|f'(\alpha_n)| + \theta |\alpha_n' - \alpha_n|). \end{aligned}$$

In other words we have

$$|f^{n+1}(U_\alpha)| \leq |f^n(U_\alpha)| (|f'(\alpha_n)| + \theta |f^n(U_\alpha)|).$$

We are looking for an α' such that $|f^n(U_\alpha)| \to 0$ as $n \mapsto +\infty$. Let

$$\eta_{n+1} = \eta_n(u_n + \theta\eta_n), \qquad n = 0, 1, \cdots. \tag{19}$$

We note that $u_n \geq |f'(\alpha_n)|$. If we will find an η_0 such that $\sum_{n=0}^{\infty} \eta_n < +\infty$, then α': $\alpha' = \alpha - \eta_0$ will satisfy our assertion (because $|f^n(U_\alpha)| \leq \eta_n$, $n = 0, 1, \cdots$).

We set

$$\eta_n = \delta_n \prod_{j=0}^{n-1} u_j, \qquad n = 1, 2, \cdots. \tag{20}$$

Then (19) takes the form

$$\delta_{n+1} \prod_{j=0}^{n-1} u_j = \delta_n \prod_{j=0}^{n-1} u_j (u_n + \theta\sigma_n \prod_{j=0}^{n-1} u_j),$$

$$\delta_{n+1} = \delta_n \left(1 + \frac{\theta}{u_n} \prod_{j=0}^{n-1} u_j \, \delta_n \right), \qquad n = 1, 2, \ldots, \tag{21}$$

$$\delta_1 = \delta_0 \left(1 + \frac{\theta}{u_0} \delta_o \right).$$

Let $w_n = \dfrac{\theta}{u_n} \prod_{j=0}^{n-1} u_j$, $n = 1, 2, \cdots$, $w_o = \dfrac{\theta}{u_0}$. By (18) we have

$$\sum_{n=1}^{\infty} w_n < +\infty.$$

In the terms of w_n the formula (21) takes form

$$\delta_{n+1} = \delta_n(1 + w_n\delta_n), \qquad n = 0, 1, 2, \cdots, \tag{22}$$

Now we set $\delta_0 = (1 + \sum_{n=1}^{\infty} w_n)^{-2}$. We shall show that

$$\delta_n \leq \delta_0(1 + w_1 + \cdots + w_{n-1}), \qquad n = 1, 2, \cdots. \tag{23}$$

Indeed, for $n = 1$ we have $\delta_1 = \delta_0(1 + w_0\delta_0) < \delta_0(1 + w_0)$. Suppose that (23) holds for some n; we shall estimate δ_{n+1}:

$$\begin{aligned}\delta_{n+1} &= \delta_n(1 + w_n\delta_n) = \delta_n + w_n\delta_n^2 \\ &\leq \delta_0(1 + \cdots + w_{n-1}) + w_n\delta_0^2(1 + \cdots + w_{n-1})^2 \\ &= \delta_0\left[1 + \cdots + w_{n-1} + w_n\frac{(1 + \cdots + w_{n-1})^2}{(1 + \sum_{n=1}^{\infty} w_n)^2}\right] \\ &\leq \delta_0[1 + w_1 + \cdots + w_n]\end{aligned}$$

what finishes the proof of (23).

The inequality (23) shows that the sequence $(\delta_n)_0^\infty$ is bounded:

$$\begin{aligned}\delta_n &\leq \delta_0(1 + w_1 + \cdots + w_{n-1}) \\ &\leq \delta_0(1 + \textstyle\sum_{n=1}^{\infty} w_n) = (1 + \sum_{n=1}^{\infty} w_n)^{-1} = \sqrt{\delta_0}, \qquad n = 0, 1, \cdots.\end{aligned}$$

In view of (20) and (18) we obtain (note that $\dfrac{\lambda}{u_n} \geq 1$):

$$\begin{aligned}\textstyle\sum_{n=1}^{\infty} \eta_n &= \textstyle\sum_{n=1}^{\infty} \delta_n \prod_{j=0}^{n-1} u_j \\ &\leq \sqrt{\delta_0}\sum_{n=1}^{\infty}\frac{\lambda}{u_n}\prod_{j=0}^{n-1} u_j = \lambda\sqrt{\delta_0}\sum_{n=1}^{\infty}\frac{1}{u_n}\prod_{j=0}^{n-1} u_j < +\infty,\end{aligned}$$

In the same way we construct the interval U_β.

Proof of Theorem 1. Let $I = \langle\alpha, \beta\rangle$ be a totally wandering interval, i.e. $I_n \cap I_m = \varnothing$ for $n \neq m$. Then

$$\textstyle\sum_{n=0}^{\infty} |I_n| < +\infty.$$

By Lemma 3 we get

$$\sum_{n=0}^{\infty}\frac{|I_n|}{\max_{x\in I_n}|f'(x)|} < +\infty.$$

By Lemma 4 there exist some intervals $U_\alpha = (\alpha', \alpha)$ and $U_\beta = (\beta, \beta')$ such that $|f^n(U_\alpha)|, |f^n(U_\beta)| \to 0$ as $n \to +\infty$. We set $V_\alpha = \bigcup U_\alpha$, $V_\beta = \bigcup U_\beta$, where the unions are taken over all possible U_α, U_β, which have the above property. Let $\Im = V_\alpha \cup I \cup V_\beta \overset{df}{=} (\bar\alpha, \bar\beta)$. We have two possibilities:

1) $\Im_n \cap \Im_m = \varnothing$ for every $n \neq m$. But then applying once again Lemma 4, it turns out that there exist two intervals $U_{\bar\alpha}$, $U_{\bar\beta}$ such that $|f^n(U_{\bar\alpha})|$, $|f^n(U_{\bar\beta})| \to 0$ as $n \to +\infty$, which contradicts the definition of V_α and V_β.

2) Let $\Im_n \cap \Im_m \neq \varnothing$ for some $m > n$. If $\bar\alpha_n$ (or $\bar\beta_n$) belongs to $\Im_m$, then we can find a neighborhood U of $\bar\alpha$, $f^m(\bar U) \subset \Im_m$, and because of that $|f^n(U)| \to 0$.

Thus it contradicts the definition of V_α (or V_β). Therefore $\Im_m = \Im_n$, i.e. $f^{(m-n)}(\Im_n) = \Im_m$. We set $f^{2(m-n)} = g$, $\Im_n = L = (\bar\alpha_n, \bar\beta_n)$. Then $g(L) = L$, and

$g(\bar{\alpha}_n) = \bar{\alpha}_n$, $g(\bar{\beta}_n) = \bar{\beta}_n$. For each closed interval $N \subset L$ we have $|g^k(N)| \to 0$ as $k \to +\infty$. Hence we conclude that g has exactly one fixed half-attracting point $p \in L$ and for any $x \in L$ we have $g^k(x) \to p_0$ as $k \to +\infty$. Since $I_n \subset L$, this establishes our assertion.

THEOREM 2. *Suppose that all the assumptions* A.1–A.4 *of Theorem* 1 *are fulfilled. Moreover, we assume* A.5:*f has no periodic, half-attracting points. Then* $\lim_n \operatorname{diam} \Delta_n = 0$.

Proof. Suppose that our assertion is false, i.e. that there exists an $\epsilon_1 > 0$ such that $\operatorname{diam} \Delta_n \geq \epsilon_1$ for every n. Then there exists an interval $K = \langle \alpha, \beta \rangle$ given by (2) of the section 2, such that $\beta > \alpha$. We shall prove that at least one end-point of K is an accumulation point of the set $\bigcup_{n=1}^{\infty} C_n$. Indeed, for every interval $\Delta_{n,j} = (c_{n,j-1}, c_{n,j})$ there exists an integer m such that $f^m(\Delta_{n,j})$ contains at least one point of C_1^0. If not, it must exist a periodic, half attracting point (we skip the details). Suppose β is an accumulation point of the set $\bigcup_{n=1}^{\infty} C_n$. Now, either α is also an accumulation point of this set, or there exists an n_0 such that $\alpha \in C_{n_0}$. If the second possibility holds, then there exists another point $\bar{\beta} < \alpha$ such that the interval $K' = \langle \bar{\beta}, \alpha \rangle$ is also of the type (2) of the section 2. The point $\bar{\beta}$ has to be an accumulation point of the set $\bigcup_{n=1}^{\infty} C_n$, and $\beta, \bar{\beta} \in \bigcup_{n=1}^{\infty} C_n$. Now we set $L = f^{n_0+1}(\langle \bar{\beta}, \beta \rangle) \overset{df}{=} \langle \omega_1, \omega_2 \rangle$. The interval L has the following properties: for any $n = 1, 2, \cdots$. $(f^n)'(x) \neq 0$ for every $x \in L$, and the end-points ω_1, ω_2 are two accumulation points of the set $\bigcup_{n=1}^{\infty} C_n$. The intervals $L_n = f^n(L)$, $n = 0, 1, \cdots$, cannot be pairwise disjoint: if they were, the mapping f would have a periodic, half-attracting point (see the proof of Theorem 1) which contradicts A.5. If for some $m > n$ $L_n \cap L_m \neq \varnothing$, then in view of properties of L it must be $L_n = L_m$, and by the same argument as in the proof of Theorem 1 we get a contradiction.

Remark 5. Suppose that the assumptions of Theorem 2 hold. For any interval $\Delta_{n,j} = (c_{n,j-1}, c_{n,j})$ there exist two integers $k \leq 1 \leq n$ such that $f^k(c_{n,j-1})$, $f^l(c_{n,j}) \in C_1$ and

$$|f^l(c_{n,j-1}) - f^l(c_{n,j})| \geq \max\{\operatorname{dist}(C_1^0, V_1), \max_{1 \leq j \leq r_1} |c_{i,j-1} - c_{1,j}|\} \geq \rho > 0$$

for some ρ. Thus there exists a number $\bar{n}$ such that always $n - l \leq \bar{n}$. Without loss of generality we shall assume that always $l = n$.

Section 4.

Now we shall study the behavior of the derivatives $(f^n)'$ as n tends to $+\infty$. We suppose in this section that the assumptions A.1–A.5 of Theorem 2 are satisfied. Let

$$D = \{x : |f'(x)| < \lambda_2\},$$

where $\lambda_2 > 1$ and $D \cap V_1 = \varnothing$. Then of course $\operatorname{dist}(R, D) \geq \operatorname{dist}(R, V_1') = b$.

LEMMA 5. *Assume that the following condition holds*:

A.6. *there exist an integer* k_0 *and a number* $\lambda_3 > 1$ *such that if* $f^k(x) \in D$ *and* $k \geq k_0$ *then*

$$|(f^k)'(x)| \geq \lambda_3.$$

1) *Then there exist another integer* k_1 *and a number* $\lambda_4 > 1$ *such that for every sequence* $x_0, x_1, \cdots, x_r$ *such that* $x_0, x_r \in D - C_1{}^0$, $x_i \notin D$ *for* $i = 1, \cdots, r-1, r \geq k_1$, *we have*

$$\sqrt[r]{|(f^r)'(x)|} \geq \lambda_4. \tag{1}$$

2) There exists a number $\epsilon > 0$ such that if $r < k_0$ then

$$|(f^r)'(x)| \geq \epsilon. \tag{2}$$

Proof. Let $U_d = \{x : |f'(x)| < d\}$. If $x \notin U_d$, then $|f'(x)| \geq \delta(d)$ where $\delta(d)$ is a number depending on d. Assume $d < \min(d_0, \delta)$. Thus

$$|(f^r)'(x)| = |f'(x)| \cdot |(f^{r-1})'(x_1)| \geq \delta(d)\lambda_2{}^{r-1}.$$

Let k_1 be such that $\lambda_2{}^{k_1-1}\,\delta(d_0) \overset{df}{=} \lambda_5 > 1$. If $x \in U_d$, then by Lemma 2a

$$|(f^r)'(x)| = |(f^k)'(x)| \cdot |(f^{r-k})'(x_k)| \geq \frac{d_0}{|f'(x)|}\lambda_2{}^{r-k} \geq \lambda_2$$

because $r > k$. Therefore it is enough to take $\lambda_4 = \min(\lambda_2, \lambda_5)$. If $r < k_0$, then the number k from Lemma 2a is smaller than k_0 and x cannot be too close to $C_1{}^0$. Hence there exists an $\epsilon > 0$ such that (2) holds.

Let $k_2 > k_1$ be an integer number such that

$$\epsilon^{k_0}\lambda_4{}^{k_2-k_0} = \left[\epsilon^{\frac{k_0}{k_2}}\lambda_4{}^{\frac{k_2-k_0}{k_2}}\right]^{k_2} > 1. \tag{3}$$

We set

$$1 < \lambda_6 \overset{df}{=} \sqrt[k_2]{\epsilon^{k_0}\lambda_4{}^{k_2-k_0}} < \lambda_4.$$

LEMMA 6. *Assume that* A.6 *holds. There exist two numbers* n_0 *and* $\lambda_8 > 1$ *such that if* $f^n(x) \in D$ *for* $n \geq n_0$, *then*

$$|(f^n)'(x)| \geq \lambda_8{}^n. \tag{4}$$

Proof. Let $x_0, x_1, \cdots, x_n$ be the trajectory of a point x up to the moment n, $x_n \in D$. Let x_{n_i}, $n_i \leq n$, be the points which belong to D. We shall divide the trajectory in some blocks $B = (x_{i+1}, \cdots, x_{i+s})$ such that

$$\sqrt[s]{\textstyle\prod_{j=i+1}^{i+s}|f'(x_j)|} \geq \lambda_8 > 1$$

where λ_8 will be defined later on. Let $|B|$ denote the length of the block B. We construct the blocks B by induction. Suppose that the blocks $B_1, \cdots, B_t$ have been defined in such a way that the block B_t finishes at the place n_s or n_{s-1}. Now we have to discuss several possibilities:

1) Assume that B_t finishes at the place n_s. We take the closest n_{s+r} such that $n_{s+r} - n_s \geq k_0$. There are two possibilities:

a) $n_{s+r} - n_{s+r-1} < k_2$. Then we set $B_{t+1} = (x_{n_s+1}, \cdots, x_{n_{s+r}})$.

The length $|B_{t+1}|$ satisfies the inequalities $k_0 < |B_{t+1}| \leq k_2 + k_0$.

By assumption A.6 we have

$$\sqrt[|B_{t+1}|]{\prod_{i\in B_{t+1}}|f'(x_i)|} \geq \sqrt[k_2+k_0]{\lambda_3} > 1. \tag{5}$$

b) $n_{s+r} - n_{s+r-1} \geq k_2$. We set $B_{t+1} = (X_{n_s+1}, \cdots, X_{n_{s+r}-1})$ and then

$$\sqrt[|B_{t+1}|]{\prod_{i\in B_{t+1}}|f'(x_i)|} = \sqrt[|B_{t+1}|]{\prod_{i=n_s+1}^{n_{s+r-1}}|f'(x_i)|\prod_{i=n_{s+r-1}+1}^{n_{s+r}-1}|f'(x_i)|}. \tag{6}$$

Since $n_{s+r-1} - n_s - 1 < k_0$, in view of the second part of Lemma 5 we have

$$\prod_{i=n_s+1}^{n_{s+r-1}}|f'(x_i)| \geq \epsilon^{k_0}.$$

In view of the first part of Lemma 5 we get

$$\prod_{i=n_{s+r-1}+1}^{n_{s+r}-1}|f'(x_i)| \geq \lambda_4^{n_{s+r}-1-n_{s+r-1}} > \lambda_4^{B|t+1|-k_0}.$$

Finally, by (3) we have

$$\begin{aligned}\sqrt[|B_{t+1}|]{\prod_{i\in B_{t+1}}|(f'(x_i)|} &\geq \sqrt[|B_{t+1}|]{\epsilon^{k_0}\lambda_4^{|B_{t+1}|-k_0}}\\ &\geq \sqrt[|B_{t+1}|]{\epsilon^{k_0}\lambda_4^{-k_0}\lambda_4^{k_2}\lambda_4^{|B_{t+1}|-k_2}}\\ &\geq \sqrt[|B_{t+1}|]{\lambda_6^{k_2}\lambda_4^{|B_{t+1}|-k_2}} \geq \lambda_6.\end{aligned} \tag{7}$$

2) Suppose that B_t finishes at x_{n_s-1}. In the same way we take the smallest n_{s+r} such that $n_{s+r} - n_s + 1 \geq k_0$.

Now, there are also two possibilities:

a) if $n_{s+r} - n_{s+r-1} < k_2$, we set $B_{t+1} = (x_{n_s}, \cdots, x_{n_{s+r}})$ and then

$$\sqrt[|B_{t+1}|]{\prod_{i\in B_{t+1}}|f'(x_i)|} \geq \sqrt[|B_{t+1}|]{\lambda_3} \geq \sqrt[k_2+k_0]{\lambda_3}. \tag{8}$$

b) if $n_{s+r} - n_{s+r-1} \geq k_2$, we set $B_{t+1} = (x_{n_s}, \cdots, x_{n_{r+s}} - 1)$. And once again we have two cases: if $n_{r+s-1} = n_s$, then by Lemma 5

$$\sqrt[|B_{t+1}|]{\prod_{i\in B_{t+1}}|f'(x_i)|} \geq \lambda_4. \tag{9}$$

If $n_{r+s-1} > n_s$, then by Lemma 5 and by (3) we get

$$\sqrt[|B_{t+1}|]{\prod_{i\in B_{t+1}} |f'(x_i)|} = \sqrt[|B_{t+1}|]{\prod_{i=n_s}^{n_{s+r}-1-1} |f'(x_i)| \prod_{i=n_{s+r-1}}^{n_{s+r}-1} f'(x_i)|}$$

$$\geq \sqrt[|B_{t+1}|]{\epsilon^{k_0}\lambda_4^{n_{s+r}-1-n_{s+r-1}}}$$

$$\geq \sqrt[|B_{t+1}|]{\epsilon^{k_0}\lambda_4^{|B_{t+1}|-k_0}} \tag{10}$$

$$= \sqrt[|B_{t+1}|]{\epsilon^{k_0}\lambda_4^{-k_0}\lambda_4^{k_2}\lambda_4^{|B_{t+1}|-k_2}}$$

$$\geq \sqrt[|B_{t+1}|]{\lambda_6^{k_2}\lambda_4^{|B_{t+1}|-k_2}} \geq \lambda_6.$$

3) The last block B_u begins at x_{n_u} or x_{n_u+1} and finishes at x_n. If $|B_u| \geq k_2 + k_0$, then (10) holds; if $k_0 \leq |B_u| \leq k_0 + k_2$, then (8) holds; finally, if $|B_u| < k_0$, then by Lemma 5

$$\sqrt[|B_u|]{\prod_{i\in B_u} |f'(x_i)|} \geq \sqrt[k_0]{\epsilon}. \tag{11}$$

The inequalities (5)-(11) imply

$$|(f^n)'(x)| = |(f^{n-|B_u|})'(x)| \prod_{i\in B_u} |f'(x_i)|$$

$$\geq \lambda_7^{n-|B_u|}\epsilon \geq \lambda_7^{n-k_0}$$

where $\lambda_7 \overset{df}{=} \min(\sqrt[k_2+k_0]{\lambda_3}, \lambda_6) > 1$. Let n_0 be such that $\lambda_8 \overset{df}{=} \lambda_7^{1-\frac{k_0}{n_0}}\epsilon^{\frac{1}{n_0}} > 1$. Then for $n \geq n_0$

$$|(f^n)'(x)| \geq \left(\lambda_7^{1-\frac{k_0}{n}}\epsilon^{\frac{1}{n}}\right)^n \geq \left(\lambda_7^{1-\frac{k_0}{n_0}}\epsilon^{\frac{1}{n_0}}\right)^n = \lambda_8^n.$$

From Lemma 6 follows easily the following

COROLLARY 2. *If $x \in \bigcup_{n=1}^{\infty} C_n$, then*

$$\lim_n\sup \sqrt[n]{|(f^n)'(x)|} > 1.$$

LEMMA 7. *There exists a constant number $\rho_1 > 0$ such that for every interval $I = (\alpha, \beta) \subset \Delta_{1,i}$, $i = 1, \cdots, r_1$, the following inequality holds*:

$$|f(I)| \geq \rho_1 |I|^2.$$

Lemma 7 is true due to the fact that $f''|_{C_1^0} \neq 0$. The proof is elementary, we omit it.

LEMMA 8. *Suppose that the assumptions* A.1–A.5 *of theorem* 2 *and the assumption* A.6 *of Lemma* 5 *are satisfied. Let U_1 be a small neighborhood of the set C_1^0. Then there exists a constant number $d_2 > 0$ such that for every* *interval I with $f^n(I) \cap U_1 = \varnothing$ for $n = 0, \cdots, k$, the following inequality holds*:

$$\sum_{n=0}^{k} |I_n| \leq d_2 |I_k|.$$

Proof. Assume for the time being that for every $n = 0, \cdots, k$, I_n is entirely contained in D or D'. Let $\delta_2 = \min_{x \in U_1} |f'(x)| > 0$.

If for every n, $I_n \subset D'$, then, we have

$$|I_{n+1}| = \int_{I_n} |f'(x)|\, dx \geq \lambda_2 |I_n|$$

since $|f'(x)| \geq \lambda_2$ for $x \in D'$, Thus

$$\sum_{n=0}^{k} |I_n| \leq \sum_{n=0}^{k} |I_k| \frac{1}{\lambda_2^{k-n}} \leq |I_k| \frac{\lambda_2}{\lambda_2 - 1}. \tag{12}$$

Suppose now that there exists an integer $n \leq k$ such that $I_n \subset D$. Let n_k be the biggest one which admits this property. Then in view of Lemma 6 we have for $n \leq n_k - n_0$.

$$|I_{n_k}| = \int_{I_n} |(f^{n_k-n})'(x)|\, dx \geq \min_{x \in I_n} |(f^{n_k-n})'(x)| \cdot |I_n| \geq \lambda_8^{n_k-n}| I_n | \tag{13}$$

For $n_k - n_0 < n < n_k$ we have

$$|I_{n_k}| = \int_{I_n} |(f^{n_k-n})'(x)|\, dx \geq \delta_2^{(n_k-n)}| I_n | \geq \delta_2^{n_0}| I_n | \tag{14}$$

Next,

$$|I_{n_k+1}| \geq \delta_2 |I_{n_k}|,$$

and for $n_{k+1} \leq n \leq k$, since $I_n \subset D'$, we have

$$|I_n| \leq \frac{1}{\lambda_2^{k-n}} |I_k| \tag{15}$$

Hence

$$|I_{n_k}| \leq \frac{1}{\delta_2 \lambda_2^{k-n_k-1}} |I_k| \leq \frac{1}{\delta_2} |I_k|. \tag{16}$$

Finally, by (13), (14), (15) and (16) we get

$$\sum_{n=0}^{k} |I_n| = \sum_{n=0}^{n_k-n_0} |I_n| + \sum_{n_k-n_0+1}^{n_k} |I_n| + \sum_{n=n_k+1}^{k} |I_n|$$

$$\leq \sum_{n=0}^{n_k-n_0} \frac{|I_{n_k}|}{\lambda_8^{n_k-n}} + n_0 \frac{1}{\delta_2^{n_0}} |I_{n_k}| + \sum_{n=n_k+1}^{k} \frac{|I_n|}{\lambda_2^{k-n}}$$

$$\leq |i_{n_k}| \left(\frac{1}{\lambda_8^{n_0}} \frac{\lambda_8}{\lambda_8^{-1}} + n_0 \frac{1}{\delta_2^{n_0}} \right) + |I_k| \frac{\lambda_2}{\lambda_2^{-1}}$$

$$\leq |I_k| \left[\frac{1}{\delta_2} \left(\frac{1}{\lambda_8^{n_0}} \frac{\lambda_8}{\lambda_8^{-1}} + n_0 \frac{1}{\delta_2^{n_0}} \right) + \frac{\lambda_2}{\lambda_2^{-1}} \right].$$

We define d_2 equal to the number in the square bracket. If the interval I_n is not contained in D or in D', for every n, then we divide I in some subintervals which already have this property. Summing over these subintervals we obtain our assertion.

THEOREM 3. *Suppose that the assumptions* A.1–A.5 *of Theorem* 2 *and the assumption* A.6 *of Lemma* 5 *are satisfied. Then there exists a constant number* $d_3 > 0$ *such that for every* $\Delta_{n,i}$ $i = 1, \cdots, r_n$, $n = 1, 2, \cdots$, *the following inequality holds*:

$$\sum_{s=0}^{n} | f^s(\Delta_{n,i}) | \leq d_3.$$

Proof. Let m be fixed. Every point $c_{1,j} \in C_1{}^0$ belongs also to C_m: let $c_{1,j} = c_{m,i}$. We set $U_1 = \bigcup_{j=1}^{r_m-1} (c_{m,i_j-1}, c_{m,i_j+1})$. Assume that m is such that $U_1 \subset U_\delta$ where U_δ is from Lemma 2a. Let $b_1 = \min_{1 \leq i \leq r_m} | \Delta_{m,i} | < \operatorname{dist}(C_1{}^0, D')$. Let $\Delta_{n,i} = I = (\alpha, \beta)$, and $f^k(\alpha), f^n(\beta) \in C_1$, $k \leq n$. We shall study the trajectory (I_s) of the interval I for $| I_s | < b_1$. Let r be the smallest integer such that $| I_r | < b_1$ but $| I_{r+1} | \geq b_1$. There are two possibilities: 1) $r \leq k$ 2) $r > k$. First we shall investigate the case 1). Since the end points of the components of U_1 belong to C_m, I_s is either disjoint with U_1 or is contained. If for every I_s, $s \leq r$, $I_s \cap U_1 = \emptyset$, then by Lemma 8.

$$\sum_{s=0}^{r} | I_s | \leq d_2 | I_r |. \tag{17}$$

Suppose that I_s is contained in U_1 for some s; let s_r be the biggest index such that $I_{s_r} \subset U_1$, $s_r \leq r$. Then in virtue of Lemma 6 for $s \leq s_r - n_0$

$$\begin{aligned} | I_{s_r} | &= \int_{I_s} | (f^{s_r-s})'(x) | \, dx \\ &\geq \min_{x \in I_s} | (f^{s_r-s})'(x) | \cdot | I_s | \geq \lambda_8{}^{s_r-s} | I_s |. \end{aligned} \tag{18}$$

By Lemma 8 we have

$$\sum_{s_r+1}^{r} | I_s | \leq d_2 | I_r |.$$

Therefore

$$\begin{aligned} \sum_{s=0}^{r} | I_s | &= \sum_{s=0}^{s_r-n_0} | I_s | + \sum_{s_r-n_0+1}^{s_r} | I_s | + \sum_{s_r+1}^{r} | I_s | \\ &\leq | I_{s_r} | \sum_{s=s_r-n_0}^{\infty} \frac{1}{\lambda_8{}^s} + \frac{n_0}{\delta_2{}^{n_0}} | I_{s_r} | + d_2 | I_r | \\ &\leq | I_{s_r} | \left(\frac{\lambda_8}{\lambda_8 - 1} + \frac{n_0}{\delta_2{}^{n_0}} \right) + d_2 | I_r | \\ &\leq b_1 \left(\frac{\lambda_8}{\lambda_8 - 1} + \frac{n_0}{\delta_2{}^{n_0}} + d_2 \right) = d_4 \end{aligned} \tag{19}$$

since $| I_{s_r} | \cdot | I_r | \leq 1$; δ_2 is as in Lemma 8. In view of Theorem 2 there exists a number m_1 such that if $| I_{r+1} | \geq b_1$ then $I_{r+1} \in \Delta_m$ for some $m \leq m_1$. Thus

$$\sum_{s=r+1}^{n} | I_s | \leq \sum_{m=1}^{m_1} \operatorname{diam} \Delta_m \stackrel{df}{=} d_5.$$

Therefore

(20) $$\sum_{s=0}^{n} |I_s| \leq d_4 + d_5.$$

2) Assume $r > k$. Note that for $I_{k+1} = (\alpha_{k+1}, \beta_{k+1})$ we have $\alpha_{k+1} \in R$, and therefore I_s for $s \geq k + 1$ cannot be contained in U_1. Using Lemma 8 and (19), where we replace r by k, we have

$$\sum_{s=0}^{n} |I_s| = \sum_{s=0}^{k} |I_s| + \sum_{s=k+1}^{r} |I_s| + \sum_{s=r+1}^{n} |I_s|$$
$$\leq d_4 + d_2 |I_r| + d_5 \leq d_4 + d_2 + d_5 \overset{df}{=} d_3.$$

By (20) and the last inequality we see that d_3 is a number which satisfies our assertion.

Section 5.

In this section we give a sufficient condition for existence an f-invariant measure absolutely continuous with respect to the Lebesgue measure. We assume through this section that A.1–A.5 of Theorem 2 hold, plus the following new assumption:

A.7. there exists a constant number $d_3 > 0$ such that for every interval $\Delta_{n,i}$, $i = 1, \cdots, r_n$, $n = 1, 2, \cdots$ the following inequality holds:

$$\sum_{s=0}^{n} |f^s(\Delta_{n,i})| < d_3.$$

LEMMA 9. *There exists a number $w_0 > 0$ such that for every interval $\Delta_{n,i} = (\alpha, \beta)$ where $f^k(\alpha), f^n(\beta) \in C_1$, $k \leq n$, the following inequality holds*:

$$\frac{\max\{|f'(x)|: x \in f^j(\Delta_{n,i})\}}{\min\{|f'(x)|: x \in f^j(\Delta_{n,i})\}} \leq w_0 \quad \text{for} \quad j = 1, \cdots, k.$$

Proof. Let U_δ be as in Lemma 2a, let $U_1 \subset U_\delta$, be an open neighborhood of C_1^0 as in the proof of Theorem 3. First we shall consider the intervals $f^j(\Delta_{n,i})$ such that their lengths are smaller than b_1 (see the proof of Theorem 3). Those $f^j(\Delta_{n,i})$ are either disjoint with U_1 or contained in it. If $f^j(\Delta_{n,i}) \cap U_1 = \emptyset$, then

(1) $$\frac{\max\{|f'(x)|: x \in f^j(\Delta_{n,i})\}}{\min\{|f'(x)|: x \in f^j(\Delta_{n,i})\}} \leq \frac{\lambda}{\delta_2}$$

(δ_2 — see the proof of Lemma 8). Suppose now that for every $\tau > 0$ there exists an interval $f^j(\Delta_{n,i}) = I = (\alpha, \beta) \subset U_1$ such that

$$\frac{\max\{|f'(x)|: x \in I\}}{\min\{|f'(x)|: x \in I\}} \geq \tau.$$

Assume $a < \alpha < \beta$ where $a \in C_1^0$. Since $f''(x) \neq 0$ for $x \in U_1$,

$$\max\{|f'(x)|: x \in I\} = |f'(\beta)|, \min\{|f'(x)|: x \in I\} = |f'(\alpha)|.$$

Using the same argument as in the proofs of Lemma 2a and 2b we show that

(2) $$\frac{\vartheta_2}{\vartheta} \frac{|f'(\beta)|}{|f'(\alpha)|} \leq \frac{|\beta - a|}{|\alpha - a|} \leq \frac{\vartheta}{\vartheta_2} \frac{|f'(\beta)|}{|f'(\alpha)|},$$

and that

$$\vartheta_2 | \alpha - a |^2 \leq | \alpha_1 - a_1 | \leq \vartheta | \alpha - a |^2, \qquad (3)$$
$$\vartheta_2 | \beta - a |^2 \leq | \beta_1 - a_1 | \leq \vartheta | \beta - a |^2.$$

We set $N = (a, \beta)$, $M = (a, \alpha)$. Let k be the same as in (4) of the proof of Lemma 2a, with $x = \beta$. Then by Lemma 1 we get

$$\begin{aligned} b &> \textstyle\int_{N_1} | (f^{k-1})'(x) | \, dx \geq | N_1 | \min_{x \in N_1} | (f^{k-1})'(x) | \\ &\geq | N_1 | \frac{1}{\delta} \max_{x \in N_1} | (f^{k-1})'(x) |. \end{aligned} \qquad (4)$$

Using the last inequality we estimate $| \alpha_k - a_k |$:

$$\begin{aligned} | \alpha_k - a_k | &= \textstyle\int_{M_1} | (f^{k-1})'(x) | \, dx \leq | M_1 | \max_{x \in M_1} | (f^{k-1})'(x) | \\ &\leq | M_1 | \max_{x \in N_1} | (f^{k-1})'(x) | \leq | M_1 | \frac{\delta b}{| N_1 |}. \end{aligned} \qquad (5)$$

Hence by (2) and (3) we obtain

$$\begin{aligned} | \alpha_k - a_k | &\leq \frac{\delta b \vartheta}{\vartheta_2^3} \frac{| \alpha - a |}{| \beta - a |} \leq \frac{\delta b \vartheta^3}{\vartheta_2^3} \frac{| f'(\alpha) |^2}{| f'(\beta) |^2} \\ &\leq \frac{\delta b \vartheta^3}{\vartheta_2^3} \frac{1}{\tau^2}. \end{aligned} \qquad (6)$$

If τ satisfies the condition: $\tau^2 > \dfrac{2\lambda \delta \vartheta^3}{\theta_2^3}$, then

$$| \alpha_k - a_k | \leq \frac{b}{2\lambda}.$$

Thus

$$| \alpha_{k+1} - a_{k+1} | = \textstyle\int_{I_k} | f'(x) | \, dx \leq \lambda | I_k | \leq \dfrac{b}{2}.$$

On the other hand (see (4), in the proof of Lemma 2a)

$$| \beta_{k+1} - a_{k+1} | > b.$$

Therefore

$$| I_{k+1} | = | \beta_{k+1} - \alpha_{k+1} | \geq | \beta_{k+1} - a_{k+1} | - | \alpha_{k+1} - a_{k+1} | \geq \frac{b}{2}.$$

In view of Theorem 2 there exists an integer m_1 such that if $f^s(\alpha_{k+1})$, $f^r(\beta_{k+1}) \in C_1$ for $s \leq r \leq m_1$. But in view of (6)

$$\begin{aligned} | \alpha_{k+1+s} - a_{k+1+r} | &= \textstyle\int_{M_1} | (f^{k+s})'(x) | \, dx \leq \int_{M_k} | (f^s)'(x) | \, dx \\ &\leq \lambda^s | M_k | \leq \lambda^{m_1} \frac{\delta b \theta^3}{\vartheta_2^3} \frac{1}{\tau^2}. \end{aligned}$$

If τ is big enough, for instance if $\tau^2 > \lambda^{m_1}\frac{\delta\vartheta^3}{\vartheta_2^3}$, then $\alpha_{k+1+r} \in V_1$, which contradicts $f^s(\alpha_{k+1}) \in C_1$. In this way we have proved that

$$\sup_{|f^j(\Delta_{n,i})|<b_1}\left\{\frac{\max\{|f'(x)|: x \in f^j(\Delta_{n,i})\}}{\min\{|f'(x)|: x \in f^j(\Delta_{n,i})\}}\right\} < +\infty.$$

But there exist only finite number of intervals $f^j(\Delta_{n,i})$ such that their lengths are not smaller than b_1. This completes the proof.

LEMMA 10. *There exist two constant numbers u_1 and u_2 such that for every interval $\Delta_{n,i} = (\alpha, \beta)$ where $f^k(\alpha), f^n(\beta) \in C_1$, $k \le n$, the following inequalities hold*:

(a) $\left|\frac{(f^j)'(x)}{(f^j)'(y)}\right| \le u_1$ *for every $x, y \in \Delta_{n,i}$*, $\quad j = 1, \cdots, k$

(b) $\left|\frac{(f^j)'(x)}{f^j(y)}\right| \le u_2$ *for every $x, y \in f^{k+1}(\Delta_{n,i})$ and for every $j = 1, \cdots, n-k-1$.*

Proof. (a) By the same argument as in Lemma 1 we get

$$\left|\frac{(f^j)'(x)}{(f^j)'(y)}\right| \le \prod_{s=0}^{j-1}\left(1+\frac{\vartheta|x_s - y_s|}{|f'(y_s)|}\right) \tag{7}$$
$$\le \exp\left\{\vartheta\sum_{s=0}^{j-1}\frac{|f^s(\Delta_{n,i})|}{v_s}\right\}, \quad j = 1, \cdots, k,$$

where $v_s = \min\{|f'(x)|: x \in f^s(\Delta_{n,i})\}$. For simplicity we set $I = \Delta_{n,i}$, $I_s = f^s(\Delta_{n,i})$.

Let δ and U_δ be as in Lemma 2a. For fixed m we define U_1 as the union of those intervals $\Delta_{m,j}$ which have at least one point in C_1^0. We assume $U_1 \subset U_\delta$. Let $b_1 = \min_j|\Delta_{m,j}|$ and let r be an integer such that $|I_s| \le b_1$ for $s \le r$ and $|I_{r+1}| > b_1$. For every interval I_s, $s \le r$, there are two possibilities: (1) $I_s \cap U_1 = \varnothing$; (2) $I_s \subset U_1$. We shall estimate $\sum_{s=0}^{r}\frac{|I_s|}{v_s}$ separately for the groups (1) and (2). Let $\delta_2 = \min_{x\in U_1'}|f'(x)|$. By the assumption A.7 we have

$$\sum_{(1)}\frac{|I_s|}{v_s} \le \frac{1}{\delta_2}\sum_{(1)}|I_s| \le \frac{1}{\delta_2}\sum_{s=0}^{n}|I_s| \le \frac{d_3}{\delta_2}. \tag{8}$$

In view of Lemma 2b for every I_s of group (2) there exists an interval $I_{s+k(s)}$ such that

$$\frac{|I_s|}{u_s} \le \frac{1}{d_1}|I_{s+k(s)}|, \tag{9}$$

where $u_s = \max\{|f'(x)|: x \in I_s\}$, and the numbers $s + k(s)$ are distinct. Thus

$$\sum_{(2)}\frac{|I_s|}{u_s} \le \frac{1}{d_1}\sum_{s=0}^{n}|I_s| \le \frac{d_3}{d_1} \tag{10}$$

In virtue of Lemma 8 we get

$$\sum_{(2)}\frac{|I_s|}{v_s} \le w_0 \sum_{(2)}\frac{|I_s|}{u_s} \le \frac{w_0 d_3}{d_1} \tag{11}$$

In view of Theorem 2 there exists a number p_1 such that if $|\Delta_{s,j}| > b_1$ then $s \le p_1$. Hence for $s > r$ $I_s \in \Delta_p$ where $p \le p_1$. Thus setting $v_{p,j} = \inf\{|f'(x)| : x \in \Delta_{p,j}\}$ we have

$$\sum_{s=r+1}^{k}\frac{|I_s|}{v_s} \le p_1 \max\left\{\frac{|\Delta_{p,j}|}{v_{p,j}} : v_{p,j} \ne 0, j \le r_p, p \le p_1\right\} \overset{df}{=} \bar{u}. \tag{12}$$

The inequalities (8), (11) and (12) give

$$\sum_{s=0}^{k}\frac{|I_s|}{v_s} \le \frac{d_3}{\delta_2} + \frac{w_0 d_3}{d_1} + \bar{u} \overset{df}{=} u_1.$$

The proof of the part (b) is similar.

LEMMA 11. *There exists a constant number $u_3 > 0$ such that for every set $A \subset I \subset f(\Delta_{1,i})$, where I is an interval, $i = 1 \cdots, r_1$, the following inequality holds*:

$$\frac{|A_1|}{|I_1|} \le u_3 \sqrt{\frac{|A|}{|I|}}$$

where

$$A_1 = f^{-1}(A) \cap \Delta_{1,i}, \qquad I_1 = f^{-1}(I) \cap \Delta_{1,i}.$$

The proof is elementary, so we omit it.

LEMMA 12. *There exists a constant number $u_4 > 0$ such that for every interval $\Delta_{n,i}$ and for every set A the following inequality holds*:

$$\frac{|f^{-n}(A) \cap \Delta_{n,i}|}{|\Delta_{n,i}|} \le u_4 \sqrt[4]{\frac{|A|}{|f^n(\Delta_{n,i})|}}$$

Proof. Denote $I = f^n(\Delta_{n,i})$. The map $f^n: \Delta_{n,i} \to I$ is $1-1$ (see Remark 5). Let f_i^{-n} denote the inverse mapping: $f_i^{-n}: I \to \Delta_{n,i}$. Assume $\Delta_{n,i} = (a, c)$ where $f^k(a)$, $f^n(c) \in C_1$, $k \le n$. By definition of f_i^{-n} we have

$$|f^{-n}(A) \cap \Delta_{n,i}| = \int_A |f_i^{-n}(y)|\, dy. \tag{13}$$

We decompose f_i^{-n} as follows: $f_i^{-1} = f_i^{-1} \circ f_i^{-(n-k-2)} \circ f_i^{-1} \circ f_i^{-k}$. Let $A_1 = f_i^{-1}(A)$. Then

$$|A_1| = \int_A |(f^{-1})'(y)|\, dy. \tag{14}$$

In view of Lemma 11 we have

$$\frac{|A_1|}{|f_i^{-1}(I)|} \le u_3 \sqrt{\frac{|A|}{|I|}}. \tag{15}$$

Now we set $A_2 = f_i^{-(n-k-2)}(A_1)$. Then

$$|A_2| = \int_{A_1} |(f_i^{-(n-k-2)})'(y)|\, dy$$

and in virtue of Lemma 10 we get

$$\begin{aligned}
\frac{|A_2|}{|f_i^{-(n-k-1)}(I)|} &= \frac{|A_2|}{|f_i^{-(n-k-2)}(f_i^{-1}(I))|} \\
&= \frac{\int_{A_1} |(f_i^{-(n-k-2)})'(y)|\, dy}{\int_{f^{-1}(I)} |(f^{-(n-k-2)})'(y)|\, dy} \\
&\leq \frac{|A_1| \max_{y\in f^{-1}(I)} |(f_i^{-(n-k-2)})'(y)|}{|f_i^{-1}(I)| \min_{y\in f_i^{-1}(I)} |(f_i^{-(n-k-2)})'(y)|} \\
&\leq u_2 \frac{|A_1|}{|f_i^{-1}(I)|}
\end{aligned} \tag{16}$$

Let $A_3 = f_i^{-1}(A_2)$. By Lemma 11 we have

$$\frac{|A_3|}{|f_i^{-(n-k)}(I)|} = \frac{|A_3|}{|f_i^{-1}(f_i^{-(n-k-1)}(I))|} \leq u_3 \sqrt{\frac{|A_2|}{|f_i^{-(n-k-1)}(I)|}}. \tag{17}$$

Finally, we set $A_4 = f_i^{-k}(A_3) = f_i^{-n}(A)$. Then by Lemma 10 (the argument is similar as in (16)). We obtain

$$\frac{|A_4|}{f_i^{-n}(I)|} = \frac{|A_4|}{|f_i^{-k}(f_i^{-(n-k)}(I))|} \leq u_1 \frac{|A_3|}{|f_i^{-(n-k)}(I)}. \tag{18}$$

The inequalities (15)–(18) imply

$$\begin{aligned}
\frac{|f^{-n}(A) \cap \Delta_{n,i}|}{|\Delta_{n,i}|} &= \frac{|A_4|}{|f_i^{-n}(I)|} \leq u_1 \frac{|A_3|}{|f_i^{-(n-k)}(I)|} \\
&\leq u_1 u_3 \sqrt{\frac{|A_2|}{|f^{-(n-k-1)}(I)|}} \leq u_1 u_3 \sqrt{u_2 \frac{|A_1|}{|f_i^{-1}(I)|}} \\
&\leq u_1 u_3 \sqrt{u_2\, u_3 \sqrt{\frac{|A|}{|I|}}} = u_1 \sqrt{u_2}\, u_3^{3/2} \sqrt[4]{\frac{|A|}{|I|}};
\end{aligned}$$

the constant u_4 is equal to $u_1 \sqrt{u_2}\, u_3^{3/2}$.

Denote by ν_o the Lebesque measure on the interval $\langle 0, 1\rangle$, and let $\nu_n = \nu_o \circ f^n$, i.e. $\nu_n(\mathrm{A}) = \nu_o(f^{-n}(A))$. It is obvious that $\nu_n \ll \nu_o$, then we set $g_n = \frac{d\nu_n}{d\nu_o}$, $n = 1, 2, \cdots$. The functions g_n belong to $L_1\langle 0, 1\rangle$ and

$$\int_0^1 g_n(x)\, dx = 1 \quad \text{for} \quad n = 1, 2, \cdots.$$

LEMMA 13. *The set* $\{g_n\}_1^\infty$ *is weakly sequentially compact in the space* $L_1\langle 0, 1\rangle$.

Proof. Given an interval $\Delta_{n,i} = (\alpha, \beta)$, let $f^k(\alpha), f^n(\beta) \in C_1$. We note that

$$|f^n(\Delta_{n,i})| \geq \min\{\min_{i\neq j} |c_{1,i} - c_{1,j}|, \operatorname{dist}(C_1^\circ, V_1)\} \overset{df}{=} d_4 > 0$$

Indeed, since $(f^n)'(x) \neq 0$ for every $x \in \Delta_{n,i}$, either $f^n(\alpha), f^n(\beta) \in C_1$ and then $|f^n(\alpha) - f^n(\beta)| \leq \min_{i\neq j} |c_{1,i} - c_{1,j}|$, or $f^n(\alpha) \in V_1$ and $f^n(\beta) \in C_1$ which gives $|f^n(\alpha) - f^n(\beta)| \geq \operatorname{dist}(C_1, V_1)$. Let $A \subset \langle 0, 1\rangle$ be arbitrary. Then for every $i = 1, \cdots, r_n$

$$\frac{|A \cap f^n(\Delta_{n,i})|}{|f^n(\Delta_{n,i})|} \leq \frac{|A|}{d_4}.$$

In virtue of Lemma 12 we have

$$\nu_n(A) = \nu_o(f^{-n}(A)) = \textstyle\sum_{i=1}^{r_n} |f^{-n}(A) \cap \Delta_{n,i}|$$

$$\leq \sum_{i=1}^{r_n} \frac{|f^{-n}(A) \cap \Delta_{n,i}}{|\Delta_{n,i}|} |\Delta_{n,i}| \leq \sum_{i=1}^{r_n} u_4 \sqrt[4]{\frac{|A|}{d_4}} |\Delta_{n,i}| \leq u_4 \sqrt[4]{\frac{|A|}{d_4}}.$$

Given $\epsilon > 0$, if $|A| < \dfrac{\epsilon^4 d_4}{u_4^4}$, then $\nu_n(A) < \epsilon$, $n = 1, 2, \cdots$ which means

$$\textstyle\int_A g_n(x)\, dx < \epsilon$$

It means that the set $\{g_n\}_1^\infty$ is weakly sequentially compact in $L_1\langle 0, 1\rangle$ (see [2], Ch.IV).

THEOREM 4. *Let* f: $\langle 0, 1\rangle \to \langle 0, 1\rangle$ *satisfy the assumption* A.1–A.5 *of Theorem* 2 *and the assumption A.*7 (*see the beginning of this section*). *Then there exists an f-invariant measure absolutely continuous with respect to the Lebesgue measure.*

Proof. We set

$$\mu_n = \frac{1}{n} \textstyle\sum_{k=0}^{n-1} \nu_k.$$

Since $\nu_k(A) = \int_A g_k\, dx$, we have

$$\mu_n(A) = \int_A \left(\frac{1}{n} \textstyle\sum_{k=0}^{n-1} g_k(x)\right) dx.$$

By Lemma 13 the set $\{g_k\}_{k=1}^\infty$ is weakly sequentially compact, so is $\left\{\frac{1}{n} \sum_{k=0}^{n-1} g_k\right\}_{n=1}^\infty$. Therefore there exists a function $g_o \in L_1\langle 0, 1\rangle$ and an increasing sequence of integers $(n_s)_{s=1}^\infty$ such that

$$\frac{1}{n_s} \textstyle\sum_{k=0}^{n_s-1} g_k \underset{w}{\rightarrow} g_o$$

("$\underset{w}{\rightarrow}$" denotes the weak convergence). Hence

$$\mu(A) \stackrel{df}{=} \lim_s \mu_{n_s}(A) = \lim_s \int_A \left(\frac{1}{n_s} \sum_{k=0}^{n_s-1} g_k\right) dx = \int_A g_o \, dx$$

for every $A \subset \langle 0, 1\rangle$. It is obvious that the measure μ is f-invariant.

Section 6.

Example (1).

PROPOSITION 4. *Assume that* $f\colon \langle 0, 1\rangle \rightarrow \langle 0, 1\rangle$ *is of class* C^3 *and satisfies the assumptions* A.2–A.5 *of Theorem* 2. *Suppose that* $Sf \leq 0$. *Then the condition* A.6 *of Lemma* 5 *is satisfied.*

Proof. In view of Proposition 2 for every n $|f^n(x)|$ has no positive local minima. Suppose that $|(f^n)'(x_0)| \leq 1$, $f^n(x_0) \in D$, and n is large. Let $\Delta_{n,i} = (\alpha, \beta) \ni x_0$. Then either $|(f^n)'(x)| \leq 1$ for $x \in (x_0, \beta)$ or $|(f^n)'(x)| \leq 1$ for $x \in (\alpha, x_0)$. Thus $|f^n(\beta) - f^n(x_0)| \leq |\Delta_{n,i}|$ or $|f^n(\alpha) - f^n(x_0)| \leq |\Delta_{n,i}|$. By assumption A.4 $|f^n(\beta) - f^n(x_0)|$, $|f^n(\alpha) - f^n(x_0)| \geq \operatorname{dist}(D, V_0)$ for large n, therefore we get $|\Delta_{n,i}| \geq \operatorname{dist}(D, V_0) > 0$. On the other hand by Theorem 2 $|\Delta_{n,i}| \rightarrow 0$ as $n \rightarrow +\infty$, which contradicts the previous inequality.

By Theorems 3 and 4 we get

COROLLARY 3. *If f satisfies the assumptions of Proposition* 4, *then there exists an f-invariant measure absolutely continuous with respect to the Lebesgue measure.*

By the same argument we prove

PROPOSITION 5. *Assume that f satisfies the assumptions* A.1–A.5 *of Theorem* 2 *and moreover that* $\dfrac{f''}{f'}$ *is strongly decreasing* (*see Definition* 2) *on the intervals where it is continuous. Then the condition* A.6 *is satisfied and there exists an f-invariant measure absolutely continuous with respect to the Lebesgue measure.*

Remark 6. Misiurewicz has proved [5] that if f satisfies the following condition:

A.8. the condition A.5 holds and there exists an open set $V_2 \supset C_1$ such that $R \subset C_1 \subset V_2'$,
then there exists an integer m such that f^m satisfies the condition A.4.
As Singer noticed, it also follows easily from Lemmas 1a and 1b of [3].

Thus, if f is of class C^3 and satisfies A.2, A.3, A.8 and $Sf \leq 0$, then there exists an f-invariant measure absolutely continuous with respect to the Lebesgue measure. This theorem has been proved by Misiurewicz [5] under some weaker assumptions about f. Moreover, he has studied some properties of this measure.

Example (2). Lasota and York have proved [4] that if $f:\langle 0,1\rangle \to \langle 0,1\rangle$ is piecewise C^2 and expanding (i.e. $|f'(x)| \geq 1+\epsilon$ for every x such that $f'(x)$ exists), then there exists an f-invariant measure absolutely continuous with respect to the Lebesgue measure.

Suppose f is a Lasota-York mapping, let $0 = c_0 < \cdots < c_r = 1$ be the points such that $f|_{\langle c_{i-1}, c_i\rangle}$ is of class C^2 and expanding. Let $\mathscr{I}_i \ni c_i$, $i = 0, \cdots, r$, be a collection of small open intervals. We set $U = \bigcup_{i=0}^{r} \mathscr{I}_i$. Let g be a function of class C^2 such that $g(x) = f(x)$ if $x \notin U$ and $g(c_i^-) = f(c_i^-)$ (or $g(c_i^+) = f(c_i^+)$). The function g is close to f in the metric

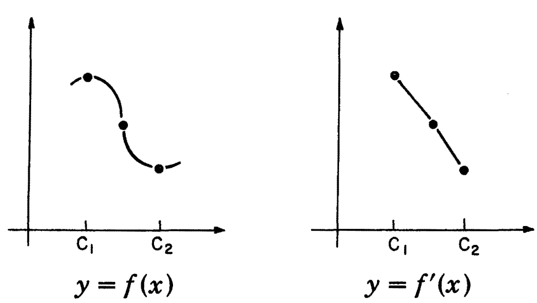
$$(1) \qquad \rho(f,g) = \int_0^1 |f'(x) - g'(x)|\,dx + \int_0^1 |f(x) - g(x)|\,dx.$$

PROPOSITION 6. *If f satisfies the condition* A.4 *with respect to* $C_1 = \{c_i\}_{i=0}^{r}$, *and g has the property: if $|g'(x)| \leq 1$ then $|g''(x)|$ is large, then there exists a g-invariant measure absolutely continuous with respect to the Lebesgue measure.*

Proof. If $\mathscr{I}_i$ are small enough, then the set U satisfies the conditions of the set U_δ in Lemma 2. In view of the formula (7) we see that if $|g''(x)|$ is big enough for $|g'(x)| \leq 1$ then $d_0 > 1$ (δ – depends on the behavior of g on the set V_1, b – is a constant which depends on f, $\vartheta_2 = \min\{|g''(x)| : |g'(x) \leq 1\}$. Thus for every $x \in D$ if $f^n(x) \in D$, $n \geq 1$, then $|(f^n)'(x)| \geq \lambda_3 > 1$. Hence, by Theorems 3 and 4 there exists a g-invariant measure absolutely continuous with respect to the Lebesgue measure.

COROLLARY 4. *If $f:\langle 0,1\rangle \to \langle 0,1\rangle$ is a Lasota-York mapping which satisfies the condition* A.4, *then f can be approximated in metric* (1) *by some C^2 – mappings which also admitted an invariant measure absolutely continuous with respect to the Lebesgue measure.*

Example (3). Let $f:\langle 0,1\rangle \to \langle 0,1\rangle$ be a piecewise quadratic mapping (i.e. f is piecewise polynomial of the second degree) of class C^1, such that every parabolic piece contains a critical point of f.



Assume that the conditions A.4 and A.5 are fulfilled. It is easy to see that the Theorems 2 and 3 hold: all what we need is that f' satisfies the Lipschitz condition and that for every critical point $c : f'(c) = 0$ there exists a neighborhood of x such that $|f'(x)| \geq \gamma |x - c|$ for some $\gamma > 0$. Moreover, f can be approximated by some mappings g of class C^2 such that $\frac{g''}{g'}$ is strongly decreasing on the intervals where it is continuous. Thus the condition A.6 holds. Therefore, in view of Theorems 3 and 4, there exists an invariant measure absolutely continuous with respect to the Lebesgue measure.

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