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ABOUT A NONLINEAR SYSTEM IN R X R^2 WITH ONE DIMENSIONAL CONTROL*

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0. Abstract

Considering the system

(1) $$\dot{x}(t) = ua(t, x) + (1 - u)b(t, x),$$

where $0 \leq u \leq 1$, $x(t_0) = x^0$, with $x \in R^2$. Under certain assumptions in respect to the rotation of the segment $[a(t, x), b(t, x)]$ and the regularity of $a(t, x)$, $b(t, x)$, we give a local characterization of the structure of the family of trajectories. For instance we prove the uniform convexity of the accessible sets.

1. Introduction

We can consider the system (1) without controls in explicit form

(2) $$\dot{x}(t) \in [a(t, x); b(t, x)], x(t_0) = x^0,$$

where the set $[a; b]$ is the closed segment with end points a and b ($a, b \in R^2$). A function $x(t)$ is called a trajectory of (2) if it is absolutely continuous on any compact subinterval of I, where I is an open interval, and satisfies condition (2) for almost every $t \in I$.

By $\Omega(t)$, $t \geq t_0$, we denote the set of points $x(t)$ where x is a trajectory of (2), this set is called the accessible set of (2) at t.

We define

$$\Omega^t = \{(s, y) \in R^3 \mid y \in \Omega(s), t_0 \leq s \leq t\}$$

and call it zone of emission of (2) on $[t_0, t]$

Example 1. Let the system

(3) $$\dot{x}(t) \in [(0, 0); (dt + ex_2, 1)], x(0) = (0, 0).$$

It is easy to see that the accessible sets of (3) are like the four cases on fig. 1 that correspond to $d > e$, $d = e$, $d < e$ and $d = 0$, respectively

In this paper we are interested in systems with uniformly convex accessible sets like fig. 1(a).

Definition 1. A compact set $M \subset R$ is called a *lens* if its boundary consists of two simple arches $x(\tau), y(\tau) \in C^2$, $0 \leq \tau \leq 1$, $x(0) = y(0) \neq x(1) = y(1)$, with curvature of opposite sign and such that $x'(\tau)x'(0) > 0$, $y'(\tau)y'(0) > 0$ for each $\tau \in (0, 1]$.

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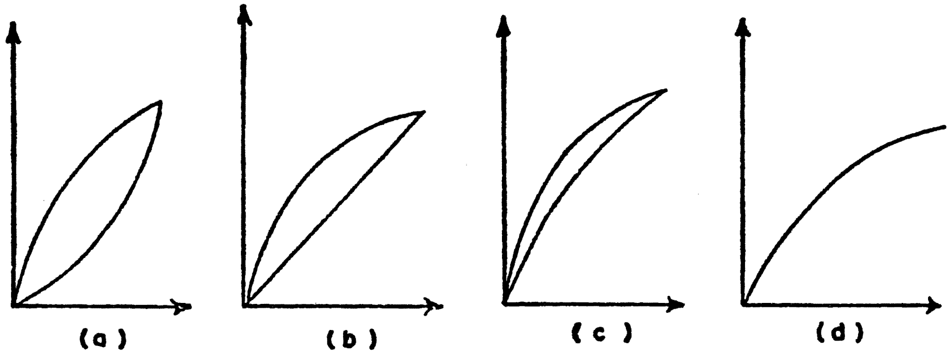

Fig. 1

We shall prove the following. For $\Omega(t)$ (for t close t_0) to be a lens set, it is enough to suppose that $a(t, x)$, $b(t, x) \in C^2$ and satisfy

Hypothesis G. In the point (t_0, x^0) the following inequality holds

$$\left|\det\left\{a-b; \frac{\partial(a-b)}{\partial x}\cdot(a-b)\right\}\right| < \left|\det\left\{a-b; \frac{\partial(a-b)}{\partial t} + \frac{\partial a}{\partial x}b - \frac{\partial b}{\partial x}\cdot a\right\}\right| \tag{4}$$

where, $\det\{v; u\} = v_1u_2 - v_2u_1$, $v = (v_1, v_2)$, $u = (u_1, u_2)$.

Remark 1. If the functions a, b do not depend on x, we get from (4)

$$0 < \left|\det\left\{a-b; \frac{\partial(a-b)}{\partial t}\right\}\right|. \tag{5}$$

This condition was obtained by S. Lojasiewicz Jr. (see [1]).

Example 2. We consider the system

$$\dot{y}(t) \in (y_2, -y_1) + [(0,0); (y_1\cos t\sin t + y_2\cos^2 t + \sin t, -y_1\sin^2 t - y_2\cos t\sin t + \cos t)] \tag{6}$$

where $y = (y_1, y_2) \in R^2$. This system satisfies (5) nevertheless, their accessible sets are like fig. 1 d.

We shall also obtain results of the following type:

Definition 2. We call $f(t, x)$ *a bang-bang control function of* (2) *with k changes in* $[t_0, t_0 + T]$, if there is a sequence $t_0 = s_0 < s_1 < \cdots < s_{k+1} = t_0 + T$ such that $f(t, x)$ is equal to an end point control $a(t, x)$ or $b(t, x)$, on each of the interval $[s_i, s_{i+1}]$, $i = 1, \ldots, k$, and such that any neighboring intervals equals different end points. We call the corresponding solutions *a bang-bang trajectory with k changes.*

For instance, considering the bang-bang controls without change $a(t, x)$ and $b(t, x)$, we obtain the system

$$\dot{x}(t) = a(t, x(t)), \qquad \dot{y}(t) = b(t, y(t)). \tag{7}$$

And then denote by $\alpha(t; t_1, x^1)$, $\beta(t; t_2, x^2)$ to the solutions of (7) that satisfies $\alpha(t_1; t_1, x^1) = x^2$ and $\beta(t_2; t_2, x^2) = x^2$.

The bang-bang controls with one change are

$$f(t, x, \tau) = \begin{cases} a(t, x) & \text{for} \quad t_0 \le t \le \tau \\ b(t, x) & \text{for} \quad t_0 \le \tau \le t \end{cases}$$

(8)

$$g(t, x, \tau) = \begin{cases} b(t, x) & \text{for} \quad t_0 \le t \le \tau \\ a(t, x) & \text{for} \quad t_0 \le \tau \le t \end{cases}$$

and their respective solutions are given by

(9)
$$x(t, \tau) = \beta(t; \tau, \alpha(\tau; t_0, x^0))$$
$$y(t, \tau) = \alpha(t; \tau, \beta(\tau; t_0, x^0)).$$

We shall prove that α, β, $x(t, \tau)$ and $y(t, \tau)$ are the unique optimal solution of (2) with respect to time.

2. Preliminary Results.

It is assumed subsequently, that t is near to t_0.

LEMMA 1. *A square $P(t)$ exists, continuous in the Hausdorff's sense, such that $\alpha(t; t_0, x^0)$, $\beta(t; t_0, x^0)$ are opposite vertices of $P(t)$ and it contains $\Omega(t)$.*

Proof. Let $\xi > 0$. Let d be a vector in R^2, we can define the set

$$D(\xi) = \{rd + v(r) \mid 0 \le r \le 1, \quad v(r)\cdot d = 0 \quad \text{and} \quad \| v(r) \| < r\xi\} \tag{10}$$

and the system

$$\dot{x}(t) \in a(t, x) + D(\xi), \; x(t_0) = x^0. \tag{11}$$

It is easy to see (since $a(t, x)$ is Lipschitzian) that the accessible set of (11) is contained in the set $\alpha(t; t_0, x^0) + (t - t_0)D(\xi')$, where $\xi' > \xi$ and for t close to t_0.

If $d = (1 + \xi)\{b(t_0, x^0) - a(t_0, x^0)\}$, there exist positive numbers T, M, such that

$$[a(t, x), b(t, x)] \subset a(t, x) + D(\xi), \tag{12}$$

for $t \in [t_0, t_0 + T]$, $\| x - x^0 \| \le M$. From (12) we have that $\Omega(t)$ is contained in the accessible set of (11) and hence

$$\Omega(t) \subset \alpha(t; t_0, x^0) + D(\xi'). \tag{13}$$

We can do the same for b and obtain

$$\Omega(t) \subset \beta(t; t_0, x^0) + D'(\xi'). \tag{14}$$

Completing the proof.

LEMMA 2. *Let the functions* Σ: $(t, \tau) \to (t, x(t, \tau))$, $\tilde{\Sigma}$: $(t, \tau) \to (t, y(t, \tau))$ *where* $x(t, \tau)$, $y(t, \tau)$ *are given by* (9) *and also for* $\tau > t$. *Then the image of the functions* Σ *and* $\tilde{\Sigma}$ *are surfaces, and their normal vectors satisfy*

$$N(t, x(t, \tau)) \cdot (1, ua(t, x(t, \tau)) + (1 - u)b(t, x(t, \tau))) \neq 0 \text{ for } u \neq 0, \tag{15}$$
$$\tilde{N}(t, y(t, \tau)) \cdot (1, ua(t, y(t, \tau)) + (1 - u)b(t, y(t, \tau))) \neq 0 \text{ for } u \neq 1.$$

Proof. If we denote $a(t) = a(t, x(t, \tau))$, $b(t) = b(t, x(t, \tau))$ we have

$$\frac{\partial x(t, \tau)}{\partial t} = b(t) \tag{16}$$
$$\frac{\partial x(t, \tau)}{\partial \tau} = a(\tau) - b(\tau) + (t - \tau)\frac{\partial b(\tau)}{\partial x} \cdot [a(\tau) - b(\tau)] + o(t - \tau)$$

where $\lim_{\tau, t \to 0} \dfrac{o(t - \tau)}{t - \tau} = 0$. Then

$$\frac{\partial \Sigma}{\partial t} = (1, b(t)) \tag{17}$$
$$\frac{\partial \Sigma}{\partial \tau} = \left(0, a(\tau) - b(\tau) + (t - \tau)\frac{\partial b(\tau)}{\partial x} | a(\tau) - b(\tau) | + o(t - \tau)\right),$$

and they are linearly independent vectors. Hence the image of Σ is a surface.

We define $N = \dfrac{\partial \Sigma}{\partial t} \times \dfrac{\partial \Sigma}{\partial \tau}$, so that

$$N(t, x(t, \tau)) \cdot (1, ua(t) + (1 - u)b(t)) = u \det\{a(\tau) - b(\tau) + (t - \tau)\frac{\partial b(\tau)}{\partial x}(a(\tau) - b(\tau)) + o(t - \tau); a(t) - b(t)\}, \tag{18}$$

and if we do just the same for $y(t, \tau)$, we find

$$\tilde{N}(t, y(t, \tau)) \cdot (1, u\tilde{a}(t) + (1 - u)\tilde{b}(t)) = (1 - u)\det\{a(\tau) - b(\tau) + (t - \tau)\frac{\partial a(\tau)}{\partial x}(a(\tau) - b(\tau)) + \tilde{o}(t - \tau); a(t) - b(t)\}. \tag{19}$$

Finally, we get from hypothesis G

$$\det\{a(\tau) - b(\tau) + (t - \tau)\frac{\partial b(\tau)}{\partial x}(a(\tau) - b(\tau)); a(t) - b(\tau)\}$$
$$= \det\{a(\tau) - b(\tau) + (t - \tau)\frac{\partial b(\tau)}{\partial x}(a(\tau) - b(\tau))$$
$$+ (t - \tau)\frac{d}{dt}(a(t) - b(t))\,|_{t=\tau} + o(t - \tau)\}$$

$$(20)\qquad = \det\{a(\tau) - b(\tau); (t-\tau)\frac{d}{dt}(a(t) - b(t))|_{t=\tau} - \frac{\partial b(\tau)}{\partial x}(a(\tau) - b(\tau)) + o(t-\tau)$$

$$= (t-\tau)\det\{a(\tau) - b(\tau); \frac{\partial}{\partial t}(a-b)(\tau) + \frac{\partial a(\tau)}{\partial x}\cdot b(\tau) - \frac{\partial b(\tau)}{\partial x}\cdot a(\tau) + o(t-\tau) \neq 0,$$

and the proof is completed.

LEMMA 3. *Let* $x^t(\tau) = x(t, \tau)$, $y^t(\tau) = y(t, t_0 + t - \tau)$ *where* x, y *given by* (9). *Then the curves* x^t, y^t (*for* $\tau \in [t_0, t]$) *have curvature of different sign.*

Proof. If $\gamma(\tau)$ is a curve in R^2 and $C(\gamma(\tau))$ *its curvature, we have that*

$$\operatorname{sign} C(\gamma(\tau)) = \operatorname{sign}(\dot\gamma_1\ddot\gamma_2 - \dot\gamma_2\ddot\gamma_1).$$

To find the curvature sign of x^t and y^t, we write

$$(21)\qquad \begin{aligned}
\frac{\partial}{\partial\tau}x^t(\tau) &= a - b + O^1(t-t_0)\\
\frac{\partial}{\partial\tau}y^t(\tau) &= a - b + O^2(t-t_0),\\
\frac{\partial^2}{\partial\tau^2}x^t(\tau) &= \frac{\partial}{\partial_t}(a-b) + \frac{\partial(a-b)}{\partial x}\cdot a - \frac{\partial b}{\partial x}\cdot(a-b) + O^3(t-t_0)\\
\frac{\partial^2}{\partial\tau^2}y^t(\tau) &= \frac{\partial}{\partial_t}(b-a) + \frac{\partial(b-a)}{\partial x}\cdot b - \frac{\partial a}{\partial x}\cdot(b-a) + O^4(t-t_0)
\end{aligned}$$

so that

$$(22)\qquad \begin{aligned}
\dot x_1^t\ddot x_2^t - \dot x_2^t\ddot x_1^t &= A + B + C + O^5(t-t_0)\\
\dot y_1^t\ddot y_2^t - \dot y_2^t\ddot y_1^t &= A - B + D + O^6(t-t_0)
\end{aligned}$$

where

$$(23)\qquad \begin{aligned}
A &= \det\left\{a - b; \frac{\partial a}{\partial x}\cdot a + \frac{\partial b}{\partial x}\cdot b\right\}\\
B &= \det\left\{a - b; \frac{\partial a}{\partial t} - \frac{\partial b}{\partial t}\right\}\\
C &= \det\left\{a - b; -2\frac{\partial b}{\partial x}\cdot a\right\}\\
D &= \det\left\{a - b; -2\frac{\partial a}{\partial x}\cdot b\right\}
\end{aligned}$$

and where the functions are valued on (t_0, x^0) and $\lim_{t\to\infty} O^j(t - t_0) = 0, j = 1, 2, \ldots, 6$.

From hypothesis G it follows

$$|A + (C + D)/2| < |B + (C - D)/2|$$

and from (22), $C(x^t(\tau))$ and $C(y^t(\tau))$ must be of different sign for t near to t_0.

3. The Main Theorems

Let $E(t)$ be the accessible set by bang-bang trajectories. Putting all of the above results together we can state.

THEOREM 1. *For t close to t_0 we have:*

a) *There exists a lens set $M(t)$, continuous in the Hausdorff sense, such that its boundary is composed of the curves x^t and y^t*

b) *If a bang-bang trajectory has more than one change, then it is in the interior of $M(t)$.*

c) *If p is in the interior of $M(t)$, then there are two different trajectories with two changes x and y, such that $x(t) = y(t) = p$,*

d) $M(t) = E(t)$.

Proof.

a) It follows from lemmas 1, 3 and the continuity of $C(x^t(\tau))$, $C(y^t(\tau))$

b) Let $t \in (t_0, t_0 + T)$ *be fixed. It is sufficient to prove it for trajectories with two changes. For instance let*

$$H^{\tau,\nu}(s, x) = \begin{cases} b(s, x) & \text{for} \quad t_0 \le s \le \nu \\ a(s, x) & \text{for} \quad \nu \le s \le \tau \\ b(s, x) & \text{for} \quad \tau \le s \le t \end{cases} \tag{24}$$

be a two changes control, then $x(t; \tau, \nu) = \beta(t; \tau, \alpha(\tau; t_0, x^0)))$ it is the correspondent solution. If we fix $\bar{t}_0 = \nu$ and $\bar{x}^0 = \beta(\bar{t}_0; t_0, x^0)$ we can define as in lemma 3

$$\begin{aligned} x^{t,\bar{t}_0}(\tau) &= x(t, \tau, \bar{t}_0) = \beta(t; \tau, \alpha(\tau; \bar{t}_0, \bar{x}^0)) \\ y^t(\tau) &= y(t, \bar{t}_0 + t - \tau) = \alpha(t; \bar{t}_0 + t - \tau, \beta(\bar{t}_0 + t - \tau; \bar{t}_0, \bar{x}^0)). \end{aligned} \tag{25}$$

From lemma 3, $x^{t,\bar{t}_0}$ has curvature of different sign that $y^t(\tau)$ and of equal sign to $x^t(\tau)$.

It is enough to prove that $x^{t,\bar{t}_0}$ and x^t, do not have different intersection that $\beta(t; \bar{t}_0, \bar{x}^0)$ (see fig. 2). Suppose that it does not hold, then there exist τ_1, τ_2 such that $\bar{t}_0 < \tau_1 < t$, $\tau_2 < t$ and

$$x^{t,\bar{t}_0}(\tau_1) = x^t(\tau_2) = z. \tag{26}$$

We obtain from the uniqueness of the system (7) with the initial condition (t, z) that

$$x^{s,\bar{t}_0}(\tau_1) = x^s(\tau_2) \qquad \text{for } s \in [l, t] \tag{27}$$

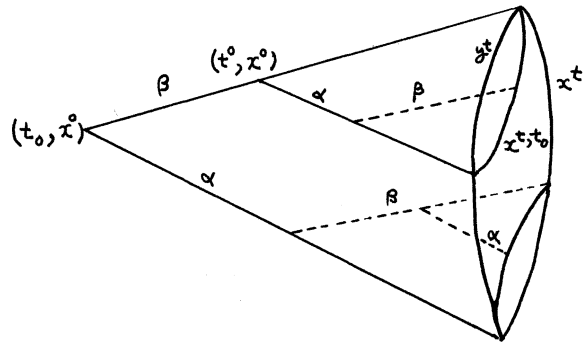

Fig. 2

where $l = \max\{\tau_1, \tau_2\}$. If $l = \tau_1$, then we get

$$(28) \qquad y^{\tau_1}(\tau_1\} = \alpha(\tau_1; \bar{t}_0, \bar{x}^0) = x^{\tau_1, \bar{t}_0}(\tau_1) = x^{\tau_1}(\tau_2)$$

which contradicts (a). If $l = \tau_2$ then

$$(29) \qquad x^{\tau_2, \bar{t}_0}(\tau_1) = x^{\tau_2}(\tau_2) = \alpha(\tau_2; t_0, x^0) = y(\tau_2, t_0)$$

and with

$$(30) \qquad \begin{aligned} x^{\tau_2, \bar{t}_0}(\tau_2) &= \alpha(\tau_2; \bar{t}_0, \bar{x}^0) = y(\tau_2, \bar{t}_0) \\ x^{\tau_2, \bar{t}_0}(\bar{t}_0) &= \beta(\tau_2; \bar{t}_0, \bar{x}^0) = y(\tau_2, \tau_2) \end{aligned}$$

we find that the curves $x^{\tau_2, \bar{t}_0}$ and y^{τ_2} have three different intersections, and from lemmas 1, 3 and the continuity of $C(x^t(\tau))$, $C(y^t(\tau))$, we can see that it is a contradiction for t close to t_0. The proof of (b) is completed.

c) If we define F: $[t_0, t] \times [-1, 1] \to R^2$ by

$$F(\bar{t}_0, \tau) = \begin{cases} x^{t, \bar{t}_0}((t_0 - t)\tau + \bar{t}_0) & \text{for } \tau \in [-1, 0] \\ y^t((t - \bar{t}_0)\tau + \bar{t}_0) & \text{for } \tau \in [0, 1] \end{cases}$$

it is easy to prove that F is a homotopy of the boundary of $M(t)$ on $\beta(t; t_0, x^0)$, where $F(\bar{t}_0, \tau) \in M(t)$, for all t_0, τ. From here, for each $p \in M(t)$ there exists $\bar{t}_0$, τ, such that $p = \underline{F}(\bar{t}_0, \tau)$, but also $y^t \in \partial M(t)$, from where, if p is in interior of $M(t)$ then $p = x^{t, t_0}((t_0 - t)\tau + \bar{t}_0)$ for $\tau \in [-1, 0]$.

Repeating the procedure for y instead of x and α instead of β, (c) is proved.

d) Its proof follows from (a), (b) and (c).

THEOREM 2. *For t close to t_0 we have*

e) $E(t) = \Omega(t)$

f) *Let A and B the graphic of $\alpha(t; t_0, x^0)$ and $\beta(t; t_0, x^0)$ respectively. Then the set $\partial\Omega^t - (A \cup B)$ is a smooth set.*

g) *The solution $x(t)$ of* (2) *is on the boundary of Ω^t if and only if $x(t)$ is a bang-bang trajectory with one or none changes.*

h) *Each point of the boundary of Ω^t is reached by one and only one trajectory. At each point in the interior of Ω^t arrive at least two different trajectories.*

Proof.

e) It is well known (see [2]).

f) It follows from lemma 2, and from (a), (d) and (e).

g) Suppose that $\gamma(s)$ is a solution of (2) on $\partial\Omega^t - (A \cup B)$. Then from (15) we have that $\gamma'(s) = b(s, \gamma(s))$ (or $a(s, \gamma(s))$), for each s such that $\gamma'(s)$ there exists. Thus there exist $\bar{t}_0$, $\bar{x}^0$ such that $\gamma(s) = \beta(s; \bar{t}_0, \bar{x}^0)$, and if s is decreasing, there exists $\bar{s}$ such that $(\bar{s}, \gamma(\bar{s})) \in A$ (or B), concluding the proof.

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