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SOME SPECTRAL PROPERTIES OF POSITIVE OPERATORS ON A BANACH SPACE

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§1. Introduction

The object of this work is to give a survey of some results concerning the spectral properties of operators leaving invariant a cone in a Banach space, including some comparison results for the spectral radii of positive operators. For simplicity of the exposition we restrict ourselves to linear operators, but we would like to point out that many of the results extend to the case when the operators are monotone and homogeneous as defined in [6]. There are some related results in the literature specially the well known results of Krein and Rutman [1], and also the works of Marek [2], Schaefer [4], Schneider and Turner [6], and Vandergraff [7]. So essentially most of the results shown here are slight extensions of results known in one form or another.

The originality of this work lies in that all the results we get are obtained by elementary ways. In fact, we only use very elementary properties of the resolvent operator, the Uniform Boundedness Principle, and the Spectral Mapping Theorem for polynominals. In this way we avoid the use of the machinery from analytic vector valued functions; and when dealing with the geometric properties of a cone, for example, in giving an alternate characterization of a strict B-cone the standard books on the subject [3], [5] go to the dual space together with its natural weak topologies to show that a (closed) reproducing cone is a strict B-cone. What we do here, is to give a direct proof of this fact by proving an Open Mapping Theorem for a class of maps which include the difference map $(x, y) \to x - y$. As a matter of fact, the proof goes on the same lines as the proof of the classical open mapping theorem for linear operators.

Another feature of this work is that we obtain the properties of positive irreducible operators on a cone satisfying the finite chain condition as corollaries of the properties of strongly positive operators. Unifying, in this way, both theories.

Finally, here we answer an open question concerning the order interior K^0 of a cone K, as defined by Schneider and Turner [6], and the topological interior K^i. It is well known that $K^i \subseteq K^0$ [6]. Here we show that if K is reproducing, then $K^i = K^0$. (Theorem 3.2).

To end this section we are going to give some notation and terminology. Let X be a real Banach space with norm $\|x\|$. If $S \subseteq X$ we denote its interior and boundary by S^i and S^b respectively. The vector space consisting of all bounded linear operators on X is denoted by $\mathscr{L}(X)$, and the vector subspace of $\mathscr{L}(X)$ consisting of all compact linear operators on X by $\mathscr{K}(X)$. By the spectrum $\sigma(A)$ of $A \in \mathscr{L}(X)$ we mean the spectrum $\sigma(\tilde{A})$ of the extension $\tilde{A}$ of A to the complexification $\tilde{X}$ of X; and by the spectral radius $r_\sigma(A)$ for A we mean the spectral radius $r_\sigma(\tilde{A})$ of $\tilde{A}$. Also we write $R(\lambda; A)$ for the resolvent $(\lambda I - \tilde{A})^{-1}$ of A.

§2. Basic properties of cones

Let X denote a real Banach space with norm $\|x\|$. A *linear semigroup* K in X is a set with the property that $x, y \in K$ and $\alpha, \beta \geqslant 0$ implies $\alpha x + \beta y \in K$. A *cone* is a closed linear semigroup. The cone K is a *proper* cone if $x \in K$ and $-x \in K$ implies $x = 0$. We say that a cone K is *normal* if there is a $\delta > 0$ such that $\|x + y\| \geqslant \delta \|x\|$ for all $x, y \in K$. It is clear that a normal cone is a proper cone. Finally we say that a cone K is reproducing if each $x \in X$ can be written in the form $x = x_1 - x_2$ with $x_1, x_2 \in K$. Unless we state it otherwise we will assume throughout this work that all cones are reproducing.

Each cone allows us to introduce in X a partial ordering; we shall write $x \leqslant y$ or $y \geqslant x$ if $y - x \in K$, and $x < y$ if $x \leqslant y$ and $x \neq y$. If A and B are linear operators on X we write $A \leqslant B$ if $B - A$ maps K into itself; we also write $A < B$ if $A \leqslant B$ and $A \neq B$. If A maps K into itself we say that A is a *positive operator.*

Let K be a cone in X and define

$$K(x) = \{y \in K \mid \exists\, \alpha > 0 \cdot \ni \cdot \alpha y \leqslant x\}$$

LEMMA 2.1. *For each $x \in K$, the set $K(x)$ is a linear semigroup which satisfies*: $y \in K(x)$, $z \in K$ and $y \geqslant z$, then $z \in K(x)$.

$K(x)$ is called the *face* generated by x, and contrary to what is assured in [6], $K(x)$ is not necessarily closed as the following example shows: Let $X = L^\infty(1, \infty)$ and $K = \{f \in X \mid f \geqslant 0\}$. Then K is a normal reproducing cone in X. Consider the face

$$K(e^{-t}) = \{f \in K \mid \exists\, \alpha > 0 \cdot \ni \cdot \alpha f(t) \leqslant e^{-t},\ 1\ l\ t < \infty\},$$

and define the sequence of functions

$$f_n(t) = \frac{1}{t}\chi_{(1,n)}(t) \qquad (n = 1, 2, \cdots)$$

Then, since

$$e^{-n}\frac{1}{t}\chi_{(1,n)}(t) \leqslant e^{-t}, \qquad 1 < t < \infty,$$

it follows that $\{f_n\} \subset K(e^{-t})$. Also

$$\left\| f_n - \frac{1}{t} \right\|_\infty = \left\| \frac{1}{t}\chi_{(n,\infty)} \right\|_\infty \leqslant \frac{1}{n} \to 0 \qquad \text{as } n \to \infty,$$

and the sequence of vectors $\{f_n\}$ in $K(e^{-t})$ converges to the function $f(t) = 1/t$.

On the other hand, if $\alpha \geqslant 0$ is such that $\alpha \frac{1}{t} \leqslant e^{-t}$, $1 < t < \infty$, then $\alpha \leqslant t\, e^{-t} \to 0$ if $t \to \infty$; and hence $\alpha = 0$. Therefore $f(t) \notin K(e^{-t})$ and $K(e^{-t})$ is not closed.

Conditions that insure the closedness of the face $K(x)$ are given in Proposition 2.1.

Let $H(x)$ denote the linear subspace of X generated by $K(x)$, and denote by $K_H{}^b(x)$ and $K_H{}^i(x)$ the boundary and interior of $K(x)$ respectively, relative to the subspace $H(x)$. Then we have

PROPOSITION 2.1: (i) *If* $x \in K^i$, *then* $K(x) = K$. (ii) *If* $x \in K^b$, *then* $K(x) \subseteq K^b$. (iii) *If* $x \in K_H{}^i(x)$, *then* $K(x)$ *is closed.* (iv) *If* $x \in K_H{}^b(x)$, *then* $K(x) \subseteq K_H{}^b(x)$.

Proof. (i) If $x \in K^i$, then x-K contains a neighborhood of 0, and for each $y \in K$, αy is in this neighborhood for some $\alpha > 0$.

(ii) We are going to show that if there is a $y \in K(x)$ with $y \in K^i$, then $x \in K^i$. So assume that $y \in K(x) \cap K^i$, then there is an $\alpha > 0$ such that $\alpha y \leqslant x$. Also, there is an $\epsilon > 0$ such that $y + w \in K$ if $\| w \| < \epsilon$. Thus

$$x + \alpha w = (x - \alpha y) + \alpha(y + w) \in K \quad \text{if} \quad \| w \| < \epsilon,$$

and this says that $x \in K^i$.

(iii) Let $d = \text{dist}(x, K_H{}^b(x))$. Then, since $x \in K_H{}^i(x)$, $d > 0$. If $y \in K(x)$, then there is an $\alpha > 0$ such that $\alpha y \leqslant x$, and we take α to be the greatest number with this property. Then we must have $x - \alpha y \in K_H{}^b(x)$ and

$$\| \alpha y \| = \| x - (x - \alpha y) \| \geqslant d. \tag{2.1}$$

Now, let $y \in \overline{K(x)}$, $y \neq 0$, then there is a sequence $\{ y_n \}$ in $K(x)$ such that $y_n \to y$. But $y_n \in K(x)$ implies that there is an $\alpha_n > 0$ such that $\alpha_n y_n \leqslant x$, and from (2.1) we see that we can choose $\alpha_n > 0$ in such a way that

$$\| \alpha_n y_n \| \geqslant d. \tag{2.2}$$

Now, from (2.2) we see that the sequence $\{\alpha_n\}$ does not converge to zero. Hence, if we let $\alpha_n' = \min \{\alpha_n, 1\}$, then the sequence $\{\alpha_n'\}$ is bounded and does not converge to zero. Taking subsequence, if necessary, we can assume that $\alpha_n' \to \alpha$ where $\alpha > 0$. Since obviously one has $\alpha_n' y_n \leqslant x$, then $\alpha y \leqslant x$ where $\alpha > 0$. This shows that $y \in K(x)$ and hence that $K(x)$ is closed.

The proof of (iv) is analogous to the one given in (ii).

If in a cone K every chain $0 \subsetneqq K(x_1) \subsetneqq K(x_2) \subsetneqq \cdots$ of faces of K ends with K after a finite number of steps, we say that K satisfies the *finite chain condition* (f.c.c.). For example, if H is a Hilbert space, the cone

$$K = \{x \in H \mid (x, x_0) \geqslant \alpha \, \| x \| \, \| x_0 \|\}$$

for fixed $x_0 \neq 0$ and $0 < \alpha < 1$ satisfies the finite chain condition with maximum length being three. On the other hand, if we let $X = C[0, 1]$ with the usual "sup" norm and let $K = \{f \in C[0, 1] \mid f \geqslant 0\}$, then K is a normal reproducing cone which does not satisfy the finite chain condition. This can be seen as follows: consider the sequences of points $0 \leqslant s_n < t_n \leqslant 1$ given by $s_n = 1 - 1/n$, $t_n = 1 - 1/2n$. Define the sequence $\{f_n\}$ in K as follows; $f_n(t) = 1$, $0 \leqslant t \leqslant s_n$; $f_n(t) = 0$, $t_n \leqslant t \leqslant 1$; $f_n(t) = \text{linear}$, $s_n \leqslant t \leqslant t_n$. Then it is easy to check that

$$K(f_1) \subsetneq K(f_2) \subsetneq \cdots \subsetneq K(f_n) \subsetneq \cdots.$$

Following Schneider and Turner [6] we say that x is in the *order interior* of K, denoted by K°, if $K(x) = K$. From Proposition 2.1(i) we know that $K^i \subseteq K^\circ$, we are going to see in Theorem 3.2 that actually one has $K^i = K^\circ$. If $x \in K^\circ$ we write $x \gg 0$.

A linear operator A on X is said to be *strongly positive*, if for every $x > 0$ there is an integer $n = n(x)$ such that $A^n x \gg 0$. If this is the case we shall write $A \gg 0$.

§3. An open mapping theorem

Let X, Y and Z be three real Banach spaces. Let K and L be cones in X and Y respectively. We say that a mapping $\Phi: K \times L \to Z$ is a *difference mapping* if it satisfies,

$$\Phi(\alpha_1 x_1 + \alpha_2 x_2, \quad \alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 \Phi(x_1, y_1) + \alpha_2 \Phi(x_2, y_2),$$

for every $x_1, x_2 \in K$, $y_1, y_2 \in L$ and $\alpha_1, \alpha_2 \geqslant 0$.

For example, if $A: X \to Z$ and $B: Y \to Z$ are linear operators, and if we define $\Phi(x, y) = Ax - By$, then Φ is a difference mapping.

If Φ is a difference mapping, then $\Phi(0, 0) = 0$; and if we let $M = \Phi(K \times L)$, then M is a linear semigroup in Z.

We say that a difference mapping Φ is *closed* if $x_n \in K$, $y_n \in L$, $x_n \to x$, $y_n \to y$ and $\Phi(x_n, y_n) \to z$, then $\Phi(x, y) = z$. It is obvious that every continuous difference mapping is closed. The converse statement will follow from Theorem 3.1.

We denote by X_α, Y_α and Z_α the open balls with radius α centered at the origin in X, Y and Z respectively.

The main result of this section is the following:

THEOREM 3.1. *If $\Phi: K \times L \to Z$ is a closed difference mapping, with K and L not necessarily reproducing, such that its range $R(\Phi)$ is of the second category in Z and is balanced, then Φ is an open mapping. That is, for every $\alpha > 0$ there is a $\beta > 0$ such that*

$$Z_\beta \subseteq \Phi(K \cap X_\alpha \times L \cap Y_\alpha). \tag{3.1}$$

Proof. First we are going to show that for every $\alpha > 0$ there is a $\beta > 0$ such that

$$Z_\beta \subseteq \overline{\Phi(K \cap X_\alpha \times L \cap Y_\alpha)}. \tag{3.2}$$

So let $\alpha > 0$ be given. Then

$$R(\Phi) = \cup_n \overline{\Phi(E_n)},$$

where $E_n = \underline{K \cap X_{n\alpha} \times L \cap Y_{n\alpha}} = n(\underline{K \cap X_\alpha \times L \cap Y_\alpha})$, and n is a positive integer. Hence, some $\overline{\Phi(E_{n_0})} = \overline{n_0 \Phi(K \cap X_\alpha \times L \cap Y_\alpha)}$ must have a nonempty interior by the Baire Category Theorem; and

$$C_\alpha = \overline{\Phi(K \cap X_\alpha \times L \cap Y_\alpha)}$$

itself must have a nonempty interior. Let z_0 be in the interior of C_α. Since $\Phi(K \times L)$ is balanced and $\Phi(K \times L) = \cup_{n=1}^{\infty} \overline{\Phi(E_n)}$ there is an integer n such that $-z_0 \in \overline{\Phi(E_n)}$. Hence $C_\alpha{}^i - z_0 \subset C_\alpha - z_0 \subset \overline{\Phi(E_n)} - z_0$ as $\overline{\Phi(E_n)} - z_0 \subset \overline{\Phi(E_{2n})}$ (3.2) holds.

Now, we are going to show that given $\alpha > 0$ there is a $\beta > 0$ satisfying (3.1). From the first part of the proof we know that given $\alpha > 0$ there is a $\beta > 0$ such that

$$Z_\beta \subseteq \overline{\Phi(K \cap X_{\alpha/2} \times L \cap Y_{\alpha/2})}. \tag{3.3}$$

Choose a sequence $\{\epsilon_n\}$ such that $\epsilon_n > 0$ and $\Sigma_1^\infty \epsilon_n < \alpha/2$. Again, from the first part of the proof, for each integer n there is a positive number δ_n such that

$$Z_{\delta_n} \subseteq \overline{\Phi(K \cap X_{\epsilon_n} \times L \cap Y_{\epsilon_n})}, \tag{3.4}$$

and we can take the sequence $\{\delta_n\}$ so that it converges to zero.

Now, let $z \in Z_\beta$. From (3.3) we see that there are points $x_0 \in K \cap X_{\alpha/2}$, $y_0 \in L \cap Y_{\alpha/2}$, such that $z_0 = \Phi(x_0, y_0)$ satisfies

$$\| z - z_0 \| < \delta_1.$$

Then $z - z_0 \in Z_{\delta_1}$ and from (3.4) we see that there are points $x_1 \in K \cap X_{\epsilon_1}$, $y_1 \in L \cap Y_{\epsilon_1}$ such that

$$\| z - (z_0 + z_1) \| < \delta_2.$$

Proceeding in this way, we can define sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that $x_n \in K \cap X_{\epsilon_n}$, $Y_n \in L \cap Y_{\epsilon_n}$ and $z_n = \Phi(x_n, y_n)$ satisfy

$$\| z - \textstyle\sum_0^n z_k \| < \delta_{n+1}, \tag{3.5}$$

and since

$$\textstyle\sum_0^n z_k = \sum_0^n \Phi(x_k, y_k) = \Phi(\sum_0^n x_k, \sum_0^n y_k),$$

we get from (3.5)

$$\| z - \Phi(\textstyle\sum_0^n x_k, \quad \sum_0^n y_k) \| < \delta_{n+1}. \tag{3.6}$$

Since $\delta_n \to 0$, we see from (3.5) that

$$z = \textstyle\sum_0^\infty z_n. \tag{3.7}$$

Now, $\| x_n \| < \epsilon_n$, $\| y_n \| < \epsilon_n$ and

$$\textstyle\sum_0^\infty \| x_n \| < \alpha/2 + \sum_1^\infty \epsilon_n < \alpha, \qquad \sum_0^\infty \| y_n \| < \alpha/2 + \sum_1^\infty \epsilon_n < \alpha.$$

Since X and Y are complete, K and L are closed, then each of the series $\sum_0^\infty x_n$ and $\sum_0^\infty y_n$ converges to points $x \in K$ and $y \in L$ respectively, and we have

$$x = \textstyle\sum_0^\infty x_n,\ y = \sum_0^\infty y_n. \tag{3.8}$$

Moreover

$$\| x \| \leqslant \| x_0 \| + \textstyle\sum_0^\infty \epsilon_n < \alpha, \qquad \| y \| \leqslant \| y_0 \| + \sum_0^\infty \epsilon_n < \alpha.$$

Thus $x \in K \cap X_\alpha$, $y \in L \cap Y_\alpha$, and from (3.6), (3.8), and the fact that Φ is closed we conclude that $z = \Phi(x, y)$. Therefore $z \in \Phi(K \cap X_\alpha \times L \cap Y_\alpha)$, and (3.1) holds.

COROLLARY 3.1. *There is a constant $M > 0$ such that each $x \in X$ has a representation $x = x_1 - x_2$ where $x_1, x_2 \in K$ and $\| x_i \| \leqslant M \| x \|$ $(i = 1, 2)$.*

Proof. It is clear that $\Phi(x, y) = x - y (x, y \in K)$ is a difference mapping. Since K is reproducing $R(\Phi) = X$, and from Theorem 3.1 it follows that Φ is an open mapping. Thus, there is a $\beta > 0$ such that $X_\beta \subseteq K \cap X_1 - K \cap X_1$. If we write $1 = \beta M$ we obtain the desired result.

THEOREM 3.2. *If K is a cone in X, then $K^i = K^\circ$.*

Proof. If $K^\circ = \varnothing$, then there is nothing to prove, because we already know that $K^i \subseteq K^\circ$. So now we assume that $K^\circ \neq \varnothing$.

Let $x_0 \in K^\circ$ and, for each positive integer n, define the closed set

$$A_n = \{ y \in K \mid y \leqslant n x_0 \}.$$

Then, clearly $K = \cup_n A_n$. Since K is complete, it follows from the Baire Category Theorem that there is an integer m such that A_m has nonempty interior relative to K. Hence, there is a $u_0 \in K$ and an $\epsilon > 0$ such that $\| x - u_0 \| < \epsilon$ and $x \in K$ implies $x \in A_m$. In particular, we have that if $w \in K$ and $\| x \| < \epsilon$, then $u_0 + w \in A_m$ i.e., $m x_0 \geqslant u_0 + w$; and from this it follows that if $w \in K$, $\| w \| < \epsilon$, then

$$z \geqslant w, \tag{3.9}$$

where

$$z = m x_0 - u_0. \tag{3.10}$$

Now, from Corollary 3.1 we know that there is a constant M such that each $x \in X$ can be written in the form $x = x_1 - x_2$ with $x_1, x_2 \in K$ and $\| x_i \| \leqslant M \| x \|$ $(i = 1, 2)$. So if $x \in X$ and $\| w \| < \epsilon/M$, then $x = x_1 - x_2$ with $\| x_2 \| \leqslant M \| x \| < \epsilon$; and if z is given by (3.10) it follows from (3.9) with $w = x_2$, that $z \geqslant x_2$, and hence that $z + x = (z - x_2) + x_1 \geqslant 0$. Thus we have shown that if $\| x \| < \epsilon/M$, then $z + x \geqslant 0$ i.e., $z + X_{\epsilon/M} \subseteq K$. But from (3.10) we obtain $x_0 + X_{\epsilon/Mm} \subseteq K$. Therefore $x_0 \in K^i$ and $K^\circ \subseteq K^i$.

COROLLARY 3.2. *If Φ: $K \times L \to Z$ is a closed difference mapping, then Φ is continuous.*

Proof: Let Γ be the graph of Φ. That is $\Gamma = \{(k, \ell, z): (k, \ell) \in K \times L$ and $z = \Phi(k, \ell)\}$. Since Φ is a closed difference mapping Γ is a cone. Let π be the map $\pi: \Gamma \times \Gamma \to X \times Y$ defined by $\pi((k, \ell, z), (k', \ell', z')) = (k, \ell) - (k', \ell')$. Since K and L are reproducing $R(\pi) = X \times Y$ and so by Theorem 3.1 there is a $\beta > 0$ such that

$$\bar{B}_1 \subset \pi(\Gamma \cap V_\beta \times \Gamma \cap V_\beta), \tag{3.11}$$

where V_β is the open ball in $X \times Y \times Z$ with radius β and $\bar{B}_1$ is the closed unit ball in $X \times Y$.

Define π_0: $\Gamma \to X \times Y$ by $\pi_0(k, \ell, z) = (k, \ell)$. Let $(k, \ell) \in K \times L$ be a unit vector. By (3.11) we have $(k, \ell) = \pi_0(\gamma_1) - \pi_0(\gamma_2)$ with $\gamma_1, \gamma_2 \in \Gamma \cap V_\beta$. As Φ is a difference mapping

$$\Phi(\pi_0(\gamma_1)) = \Phi((k, \ell) + \pi_0(\gamma_2)) = \Phi(k, \ell) + \Phi(\pi_0(\gamma_2)).$$

So

$$\| \Phi(k, \ell)\|_Z \leqslant \| \Phi(\pi_0(\gamma_1))\|_Z + \| \Phi(\pi_0(\gamma_2))\|_Z \leqslant \| \gamma_1 \|_{X\times Y\times Z} + \| \gamma_2 \|_{X\times Y\times Z} < 2\beta.$$

Therefore, for any $(k, \ell) \in K \times L$

$$\| \Phi(k, \ell)\| \leqslant 2\beta \|(k, \ell)\|. \tag{3.12}$$

Since $K \times L$ is reproducing, extend Φ to the whole $X \times Y$. That is, if $(x, y) = (k, \ell) - (k', \ell')$ we define $\bar{\Phi}$: $X \times Y \to Z$ by $\bar{\Phi}(x, y) = \Phi(k, \ell) - \Phi(k', \ell')$. It is easily seen that this map is well defined. By Corollary 3.1 and from (3.12) $\bar{\Phi}$ is a bounded linear map. Hence its restriction Φ is continuous.

§4. Irreducible and Strongly Positive Operators

From now on we shall assume that K is a normal reproducing cone with nonempty interior.

Following Schneider and Turner [6], we say that a positive linear operator A is *irreducible* if $x > 0$ and $Ax \in K(x)$ imply that $K(x) = K$.

We start with the following result due to Schneider and Turner [6].

LEMMA 4.1. *Suppose that A is an irreducible operator and $x > 0$. Then $F_k = K((I + A)^k x) \subseteq F_{k+1}$, with strict inclusion unless $F_k = K$.*

Proof. We let $x_k = (I + A)^k x$. First

$$x_{k+1} = x_k + Ax_k \geqslant x_k, \tag{4.1}$$

so $F_k \subseteq F_{k+1}$. If $F_k = F_{k+1}$, then for some $\alpha > 0$, $\alpha x_{k+1} \leqslant x_k$, and from (4.1) we obtain $\alpha Ax_k \leqslant (1 - \alpha)x_k$ which, since A is irreducible implies $F_k = K$.

PROPOSITION 4.1. *Suppose $A > 0$ and K is a cone satisfying the* f.c.c. *Then A is irreducible if and only if $I + A$ is a strongly positive operator.*

Proof. Suppose first that A is an irreducible operator. Let $x > 0$, and $F_k = K((I + A)^k x)$. Since K satisfies the f.c.c. and A is irreducible, it follows from Lemma 4.1 that there is an integer n such that $K((I + A)^n x) = K$ i.e., $(I + A)^n x \gg 0$. Therefore $I + A$ is a strongly positive operator.

Suppose now that $I + A$ is a strongly positive operator, and let $x > 0$ be such that $Ax \in K(x)$. We have to show that $K(x) = K$. Now, since $I + A$ is strongly positive there is an integer n such that

$$(I + A)^n x \gg 0. \tag{4.2}$$

On the other hand, since $Ax \in K(x)$, then $(I + A)x \in K(x)$ so there is an $\alpha > 0$ such that $\alpha(I + A)x \leqslant x$, and hence that $\alpha^n(I + A)^n x \leqslant x$. Thus $(I + A)^n x \in K(x)$ and there is a $\beta > 0$ such that

$$x \geqslant \beta(I + A)^n x. \tag{4.3}$$

From (4.2) and (4.3) we conclude that $x \gg 0$, and hence that $K(x) = K$. Therefore A is an irreducible operator.

Given a positive linear operator A we can associate to it a number $\lambda(A)$ which plays the role of the spectral radius when A is compact, and this is done as follows: If A is a positive linear operator we define the set

$$\Lambda(A) = \{\lambda > 0 \mid \exists\, x > 0 \cdot\ni\cdot Ax \geqslant \lambda x\},$$

and the number

$$\lambda(A) = \sup \Lambda(A).$$

LEMMA 4.2. *If $A > 0$ and there is an $\eta \geqslant 0$ such that $\eta I + A \gg 0$, then $\lambda(A) > 0$.*

Proof. Pick an $x > 0$ with $Ax > 0$. Since $\eta I + A \gg 0$, there is an integer n such that $(\eta I + A)^n Ax \gg 0$. Hence there is an $\alpha > 0$ such that

$$\alpha(\eta I + A)^n x \leqslant (\eta I + A)^n Ax = A(\eta I + A)^n x.$$

Therefore $\lambda(A) \geqslant \alpha > 0$.

COROLLARY 4.1. *If $A \gg 0$, then $\lambda(A) > 0$.*

COROLLARY 4.2. *If $A > 0$ is irreducible and K satisfies the* f.c.c., *then $\lambda(A) > 0$.*

Proof. Immediate from Proposition 4.1 and Lemma 4.2 with $\eta = 1$.

The relation between the number $\lambda(A)$ and the spectral radius $r_\sigma(A)$ of a positive operator $A \in \mathscr{L}(X)$ is given by the following.

PROPOSITION 4.2. If $A \in \mathscr{L}(X)$ is a positive operator, then $\lambda(A) \leqslant r_\sigma(A)$.

Proof. Let $\lambda \in \Lambda(A)$. Then there is an $x > 0$ such that $Ax \geqslant \lambda x$; and hence $A^n x \geqslant \lambda^n x (n = 1, 2, \cdots)$. Now, since K is normal, there is a $\delta > 0$ such that

$$\delta \| \lambda^n x \| \leqslant \|(A^n x - \lambda^n x) + \lambda^n x \| = \| A^n x \|.$$

Thus

$$\lambda^n \leqslant \frac{\| A^n x \|}{\delta \| x \|} \leqslant \frac{1}{\delta} \| A^n \|,$$

and

$$\lambda \leqslant \left(\frac{1}{\delta}\right)^{1/n} \| A^n \|^{1/n}.$$

Taking the limit as $n \to \infty$, w obtain $\lambda \leqslant r_\sigma(A)$, and since this holds for every $\lambda \in \Lambda(A)$ we conclude that $\lambda(A) \leqslant r_\sigma(A)$.

As a consequence of Corollaries 4.1, 4.2 and Proposition 4.2 we obtain the following two results.

THEOREM 4.1. *If $A \in \mathscr{L}(X)$, $A > 0$ and there is an $\eta \geqslant 0$ such that $\eta I + A \gg 0$, then $r_\sigma(A) > 0$.*

THEOREM 4.2. *If $A \in \mathscr{L}(X)$, $A > 0$ is irreducible and K satisfies the* f.c.c., *the $r_\sigma(A) > 0$.*

§5. The Results of Krein and Rutman

We start by recalling some basic facts about the resolvent $R(\lambda; A)$ of an operator $A \in \mathscr{L}(X)$. First we have the "resolvent equation"

$$R(\lambda; A) - R(\mu; A) = (\mu - \lambda) R(\lambda; A) R(\mu; A), \qquad \lambda, \mu \in \rho(A), \tag{5.1}$$

where $\rho(A)$ denotes the resolvent set of A. Also, if $|\lambda| > r_\sigma(A)$, then the resolvent of A is given by the convergent power series

$$R(\lambda; A) = \sum_{n=0}^{\infty} \lambda^{-n-1} A^n. \tag{5.2}$$

In particular, it is clear from (5.2) that if A is also positive then so is $R(\lambda; A)$ for $\lambda > r_\sigma(A)$.

Using the resolvent equation (5.1) it is easy to prove the following well known result.

LEMMA 5.1. *Let $A \in \mathscr{L}(X)$, and assume that $\{\lambda_n\}$ is a sequence in $\rho(A)$ converging to some λ. Then $\lambda \in \sigma(A)$ if and only if $\lim_{n\to\infty} \|R(\lambda_n; A)\| = \infty$.*

Now we prove a well known result about the spectral radius of a positive operator; the proof we present here is due to Schneider and Turner [6].

THEOREM 5.1. *Let $A \in \mathscr{L}(X)$ be a positive operator. If $r_\sigma(A) > 0$, then $r_\sigma(A) \in \sigma(A)$.*

Proof. We can assume, without loss of generality, that $r_\sigma(A) = 1$. Suppose, by contradiction, that $1 \notin \sigma(A)$. Then from the spectral mapping theorem we have $\sigma(I + A) = 1 + \sigma(A)$, and hence $r_\sigma(I + A) > 2$.

Thus, for $\alpha > 0$ small enough.

$$((1-\alpha)I-A)^{-1}=((2-\alpha)I-(I+A))^{-1}=\sum_{k=0}^{\infty}\frac{(I+A)^k}{(2-\alpha)^{k+1}},$$

exists and maps K into itself. Since

$$((1-\alpha)I-A)^{-1}=\sum_{k=0}^{N}\frac{A^k}{(1-\alpha)^{k+1}}+\frac{A^{N+1}}{(1-\alpha)^{N+1}}((1-\alpha)I-A)^{-1},$$

it follows that

$$0\leqslant\frac{A^k}{(1-\alpha)^{k+1}}\leqslant R(1-\alpha;A),$$

for all k. Since K is normal, there is a $\delta>0$ such that

$$\left\|\frac{A^kx}{(1-\alpha)^{k+1}}\right\|\leqslant\delta^{-1}\,\|\,R(1-\alpha;A)x\,\|$$

for all $x\in K$; from which, using Corollary 3.1, follows that

$$\|\,A^k\,\|\leqslant 2\delta^{-1}M(1-\alpha)^kc_\alpha,$$

where $c_\alpha=(1-\alpha)\|R(1-\alpha;A)\|$. But then $r_\sigma(A)\leqslant 1-\alpha$, contradiction.

The next result of this section is a well known result of Krein and Rutman [1] about the existence of "positive eigenvectors" for compact positive operators. For completeness we give its proof here.

THEOREM 5.2. *Let $A\in\mathscr{K}(X)$ be a positive operator. If $r_\sigma(A)>0$, then $r_\sigma(A)\in\sigma(A)$, and there is a $u>0$ such that $Au=r_\sigma(A)u$.*

Proof. Let $r=r_\sigma(A)>0$. Then from Theorem 5.1 $r_\sigma(A)\in\sigma(A)$. Pick a sequence $\epsilon_n>0$, $\epsilon_n\to 0$. Since $r\in\sigma(A)$, it follows from Lemma 5.1 that

$$\lim_{n\to\infty}\|\,R(r+\epsilon_n;A)\|=\infty. \tag{5.3}$$

Then, there is an $x_o>0$ such that

$$c_n=\|\,R(r+\epsilon_n;A)x_o\|$$

becomes unbounded as $n\to\infty$. For otherwise, if

$$\sup_n\|\,R(r+\epsilon_n;A)x\,\|<\infty, \tag{5.4}$$

for every $x\in K$, then since K is reproducing (5.4) holds for all $x\in X$; and from the Uniform Boundedness Principle we conclude that $\|R(r+\epsilon_n;A)\|$ is bounded, which is in contradiction with (5.3). Therefore, there is an $x_o>0$ such that $\{c_n\}$ is inbounded and taking subsequences, if necessary, we can assume that

$$\lim_{n\to\infty}c_n=\infty. \tag{5.5}$$

Since $R(r+\epsilon_n;A)$ is a positive operator, if we let

$$x_n=c_n{}^{-1}R(r+\epsilon_n;A)x_o, \tag{5.6}$$

then $x_n \in K$ and $\| x_n \| = 1$; and from (5.5), (5.6) we see that

$$rx_n - Ax_n = c_n^{-1}x_o - \epsilon_n x_n \to 0$$

as $n \to \infty$. Thus we see that there are unit vectors x_n in K for which $(rI - A)x_n \to 0$ as $n \to \infty$. Since the sequence $\{x_n\}$ is bounded and A is compact, then, taking subsequences, if necessary, we can assume that $Ax_n \to y$ if $n \to \infty$. Let $z_n = rx_n - Ax_n$ then $z_n \to 0$ and

$$x_n = r^{-1}z_n + r^{-1}Ax_n \to r^{-1}y$$

as $n \to \infty$; and since K is closed $y \in K$. If we let $u = r^{-1}y$, then $u \in K$, $\| u \| = 1$, and $Au = ru$.

COROLLARY 5.1. *If $A \in \mathscr{K}(X)$ and there is an $\eta \geqslant 0$ such that $\eta I + A \gg 0$, then $r_\sigma(A) > 0$ and there is a $w \in K^i$ such that $Aw = r_\sigma(A)w$.*

Proof. From Lemma 4.1 and Proposition 4.2 we know that $r_\sigma(A) \geqslant \lambda(A) > 0$. Hence, from Theorem 5.1 there is a $u > 0$ such that $Au = r_\sigma(A)u$. On the other hand there is an integer n such that $(\eta I + A)^n u \gg 0$. So, if we let $w = (\eta I + A)^n u$, then $w \gg 0$ and $Aw = A(\eta I + A)^n u = (\eta I + A)^n Au = r_\sigma(A)w$.

COROLLARY 5.2. *If $A \in \mathscr{K}(X)$, $A > 0$ is irreducible and K satisfies the* f.c.c., *then $r_\sigma(A) > 0$, and there is a $w \in K^i$ such that $Aw = r_\sigma(A)w$.*

Proof. Immediate from Corollary 5.1 and Proposition 4.1.

COROLLARY 5.3. *If $A \in \mathscr{K}(X)$ is a positive operator, then*

$$r_\sigma(\eta I + A) = \lambda(\eta I + A) \tag{5.7}$$

for every $\eta \geqslant 0$.

Proof. From Proposition 4.2 we know that

$$\lambda(\eta I + A) \leqslant r_\sigma(\eta I + A). \tag{5.8}$$

Also, from The Spectral Mapping Theorem and Theorem 5.1 we have

$$r_\sigma(\eta I + A) = \eta + r_\sigma(A). \tag{5.9}$$

Now, from (5.8) and (5.9) we obtain

$$\eta \leqslant \lambda(\eta I + A) \leqslant r_\sigma(\eta I + A) = \eta + r_\sigma(A).$$

So if $r_\sigma(A) = 0$, then

$$\lambda(\eta I + A) = r_\sigma(\eta I + A) = \eta,$$

and this is (5.7) in this case.

If $r_\sigma(A) > 0$, then from Theorem 5.1 there is a $u > 0$ such that $Au = r_\sigma(A)u$, and hence that $(\eta I + A)u = (\eta + r_\sigma(A))u$. From the definition of $\lambda(\eta I + A)$ we obtain

$$\lambda(\eta I + A) \geqslant \eta + r_\sigma(A).$$

This last inequality together with (5.8) and (5.9) give (5.7).

§6. Comparison Results for Spectral Radii of Positive Operators

We start with the following:

THEOREM 6.1. *Let $A \in \mathscr{K}(X)$ and $B \in \mathscr{L}(X)$ be positive operators. If $A \leqslant B$, then*

$$r_\sigma(A) \leqslant r_\sigma(B). \tag{6.1}$$

Proof. Let us note first that $A \leqslant B$ implies $\Lambda(A) \subseteq \Lambda(B)$ and hence that $\lambda(A) \leqslant \lambda(B)$. Now, from Corollary 5.3 and Proposition 4.2 we have

$$r_\sigma(A) = \lambda(A) \leqslant \lambda(B) \leqslant r_\sigma(B). \tag{6.2}$$

If $A, B \in \mathscr{K}(X)$ are positive operators with $A < B$, then it does not necessarily follows that $r_\sigma(A) < r_\sigma(B)$, as the following example shows: Let $X = \mathbf{R}^n (n > 1)$ and $K = \{(x_1, \cdots, x_n) | \, x_i \geqslant 0, 1 \leqslant i \leqslant n\}$. Consider the diagonal matrices $A = \operatorname{diag}(1, 2, \cdots, n)$ and $B = \operatorname{diag}(n, n, \cdots, n)$. Then A, B are positive operators such that $A < B$. But clearly $r_\sigma(A) = n = r_\sigma(B)$. Thus, in order to insure the holding of the strict inequality we have to impose an extra condition on the operator A or on the operator B. This is done in the following three results.

THEOREM 6.2. *Let $A \in \mathscr{K}(X)$ and $B \in \mathscr{L}(X)$ be positive operators with $0 < A < B$. If $r_\sigma(A) > 0$ and there is an $\eta \geqslant 0$ such that $\eta I + B \gg 0$, then*

$$r_\sigma(A) < r_\sigma(B). \tag{6.3}$$

Proof. Let $r = r_\sigma(A) > 0$. Then Theorem 5.2 implies the existence of a $u > 0$ such that $Au = ru$. We are going to show that, in this case, the equality $Bu = Au$ is impossible. So assume

$$Bu = Au = ru. \tag{6.4}$$

Since $\eta I + B \gg 0$, there is an integer n such that $(\eta I + B)^n \, u \gg 0$. But from (6.4) we obtain $(\eta I + B)u = (\eta + r)u$, and hence that $(\eta + r)^n u = (\eta I + B)^n u \gg 0$. Therefore $u \gg 0$. Let $x \in K$ be arbitrary then there is an $\alpha > 0$ such that $y = u - \alpha x \gg 0$.

Now,

$$\begin{aligned} 0 &\geqslant A(\alpha x) - B(\alpha x) \\ &= A(\alpha x) - B(u - y) \\ &= A(\alpha x) - Bu + By \\ &= A(\alpha x) - Au + By \\ &= By - A(u - \alpha x) \\ &= By - Ay \\ &\geqslant 0. \end{aligned}$$

Since K is a proper cone we must have $A(\alpha x) - B(\alpha x) = 0$, and hence $Ax = Bx$. Thus we have shown that $Ax = Bx$ for all $x \in K$, and since K is reproducing $A = B$, which is a contradiction. Showing in this way that $Bu > Au$.

Now, since $Bu > Au = ru$, then $Bu - ru > 0$. But $\eta I + B \gg 0$ implies that $(\eta I + B)^m(Bu - ru) \gg 0$, for some integer m. Let $v = (\eta I + B)^m(Bu - ru) \gg 0$. Hence, there is an $\epsilon > 0$ such that $\epsilon v \leqslant Bv - rv$, and hence $(r + \epsilon)v \leqslant Bv$. It follows from the definition of $\lambda(B)$ that $r + \epsilon \leqslant \lambda(B)$. Using this last inequality together with (6.2) we obtain (6.3).

COROLLARY 6.1. *Let $A \in \mathscr{K}(X)$ and $B \in \mathscr{L}(X)$ be positive operators with $A < B$. If there is an $\eta \geqslant 0$ such that $\eta I + B \gg 0$, then $r_\sigma(A) < r_\sigma(B)$.*

Proof. If $r_\sigma(A) > 0$, this follows from Theorem 6.2. If $r_\sigma(A) = 0$, this is immediate, because from Theorem 4.1 we have $r_\sigma(B) > 0$.

COROLLARY 6.2. *Suppose that $A \in \mathscr{K}(X)$ and $B \in \mathscr{L}(X)$ are positive operators and K satisfies the* f.c.c. *If $0 < A < B$ and either A or B are irreducible, then $r_\sigma(A) < r_\sigma(B)$.*

Proof. The result for B irreducible follows from the preceding corollary and Proposition 4.1. If $A > 0$ is irreducible Theorem 4.2 implies that $r_\sigma(A) > 0$. From Proposition 4.1 we know that $I + A \gg 0$ and since$I + A < I + B$, $I + B \gg 0$. If we apply Theorem 6.2 with $\eta = 1$ we obtain the desired result.

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