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NONIMMERSIONS OF REAL PROJECTIVE SPACES IMPLIED BY BP

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1. Discussion of Results

In [1] and [2] the first-named author proves the following result for $gd(2k\xi_{2m})$, the geometric dimension of the $2k$-fold Whitney sum of the Hopf bundle ξ over the real projective $2m$-space P^{2m}.

Let $\nu(a)$ denote the exponent of the largest power of 2 dividing a.

THEOREM 1.1. *If* $\nu\binom{k}{m-s-j} \geq s$ *for* $0 \leq j \leq s$ *and* $\nu\binom{k+s}{m-s} = s$, *then* $gd(2k\xi_{2m}) \geq 2m - 6s$.

The method of proof is to show that the $BP\langle 2\rangle$-Euler class of the bundle $2k\xi \otimes \xi$ over $P^{2m} \times P^{2k-2m+6s}$ is nontrivial, where $BP\langle 2\rangle$ is the spectrum related to the $p = 2$ Brown-Peterson spectrum which was studied in [9].

In [1] Theorem 1.1 is applied to the stable normal bundle η_{2m+1} of P^{2m+1} for certain values of m to obtain nonimmersion theorems (see (4.2) of [11]). Nonimmersion results for some P^{2m+2} are also obtained by applying (1.1) to $\eta_{2m+2} \oplus \xi$, noting that $gd(\theta) \geq gd(\theta \oplus \xi) - 1$.

In this paper we seek the most general nonimmersion theorem that can be derived from (1.1). Many of the results are obtained by the method of the preceeding paragraph. However, for some values of n the nonimmersion result for P^n thus obtained can be improved by applying Theorem 1.1 to $\eta \oplus L\xi$, where L is an appropriately chosen integer. We give a nonimmersion result for every even dimensional projective space. The results we obtain for P^n when n is odd are always implied by those for P^{n-1}.

Before stating the precise form of our results in Theorem 1.5, which is somewhat complicated, we make some general comments, which we hope are more illuminating than the precise formula.

Let $\alpha(m)$ denote the number of ones in the binary expansion of m, and let $q(m)$ be the largest power of 2 not exceeding m. Then every positive integer k can be written uniquely in the form $k = 2^iB + A$ with B odd, $0 \leq A < 2^i$ and $A - q(A) < \alpha(B) \leq A + 1$. Indeed, this decomposition of the binary expansion of k is simply the one having the smallest A among those decompositions satisfying $\alpha(B) \leq A + 1$. We have, for example,

$$23 = 2^2 \cdot 5 + 3$$

$$24 = 2^4 \cdot 1 + 8$$

$$25 = 2^3 \cdot 3 + 1.$$

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TABLE 1.2. *Values of* D *for which* $P^{2k} \not\subseteq R^{4k-2D}$ *with* $k = 2^iB+A$, *where* A *and* B *have the restrictions mentioned in the text. Vertical entries denote values of* α (B) *and horizontal entries denote values of* A.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1																			
1	3	4																		
2	4	6	7																	
3		7	9	10																
4	6	7	10	12	13															
5		9	10	13	15	16														
6			12	13	16	18	19													
7				15	16	19	21	22												
8	7	10	12	13	18	19	22	24	25											
9		11	14	15	16	21	22	25	27	28										
10			16	17	18	19	24	25	28	30	31									
11				19	20	21	22	27	28	31	33	34								
12					22	23	24	25	30	31	34	36	37							
13						25	26	27	28	33	34	37	39	40						
14							28	29	30	31	36	37	40	42	43					
15								31	32	33	34	39	40	43	45	46				
16	10	12	13	18	19	22	24	25	34	35	36	37	42	43	46	48	49			
17		14	15	17	21	23	26	27	28	37	38	39	40	45	46	49	51	52		
18			17	18	21	24	28	29	30	31	40	41	42	43	48	49	52	54	55	
19				20	23	25	30	31	32	33	34	43	44	45	46	51	52	55	57	58
20					23	26	32	33	34	35	36	37	46	47	48	49	54	55	58	60
21						27	34	35	36	37	38	39	40	49	50	51	52	57	58	61
22							36	37	38	39	40	41	42	43	52	53	54	55	60	61
23								39	40	41	42	43	44	45	46	55	56	57	58	63
24									42	43	44	45	46	47	48	49	58	59	60	61
25										45	46	47	48	49	50	51	52	61	62	63
26											48	49	50	51	52	53	54	55	64	65
27												51	52	53	54	55	56	57	58	67
28													54	55	56	57	58	59	60	61
29														57	58	59	60	61	62	63
30															60	61	62	63	64	65
31																63	64	65	66	67
32	12	13	18	19	22	24	25	34	35	36	37	42	43	46	48	49	66	67	68	69
33		15	17	21	23	26	27	29	37	38	39	41	45	47	50	51	52	69	70	71
34			18	21	24	28	29	30	33	40	41	42	45	48	52	53	54	55	72	73
35				23	25	30	31	32	35	37	43	44	47	49	54	55	56	57	58	75
36					26	32	33	34	35	38	41	46	47	50	56	57	58	59	60	61
37						34	35	36	37	39	43	45	49	51	58	59	60	61	62	63

Theorem 1.5 states that P^{2k} does not immerse in R^{4k-2D}, in symbols $P^{2k} \not\subseteq R^{4k-2D}$, where D depends only upon A and $\alpha(B)$ in the above decomposition of k. The theorem, although difficult to state, is easy to tabulate, and Table 1.2 presents the early results of (1.5) in terms of A and $\alpha(B)$. In Table 1.3 we list our nonimmersion results in the case $k = 2^r + q$ with $0 \leq q < 2^r$ for all $q \leq 37$, along with the applicable values of A and $\alpha(B)$ and the best previously known nonimmersion result ([6], [4], [3], [7]). A star indicates that our result is new. The first of our many new results is $P^{30} \not\subseteq R^{46}$.

TABLE 1.3. *If* $k = 2^r + q$ *with* $0 \le q < 2^r$ *then* $P^{2k} \not\subseteq R^{4k-2D}$ *by* (1.5); *the result* $P^{2k} \not\subseteq R^{4k-E}$ *was previously known.*

q	A	α(B)	2D	E	q	A	α(B)	2D	E
0	0	1	2	2	19	3	2	14*	15
1	1	1	6	5	20	4	2	14	14
2	2	1	8	7	21	5	2	18	18
3	1	2	8	7	22	2	3	14*	22
4	4	1	12	11	23	3	3	18*	26
5	1	2	8	7	24	8	2	20	20
6	2	2	12	11	25	9	2	22*	24
7	3	2	14*	15	26	2	3	14*	28
8	8	1	14	14	27	3	3	18*	32
9	1	2	8	7	28	4	3	20	20
10	2	2	12	11	29	5	3	20*	24
11	3	2	14*	15	30	6	3	24*	28
12	4	2	14	14	31	3	4	20*	32
13	5	2	18	18	32	32	1	24	22
14	2	3	14*	22	33	1	2	8	7
15	3	3	18*	26	34	2	2	12	11
16	16	1	20	20	35	3	2	14*	15
17	1	2	8	7	36	4	2	14	14
18	2	2	12	11	37	5	2	18	18

We make no claims regarding the optimality of our results. Immersion results of this strength will probably be rather difficult to prove. However, it has recently been shown ([5]) how to obtain Theorem 1.1 from a slightly different obstruction theoretic perspective. This should enable one to utilize the theory of [10] to study higher order *BP* obstructions, which might possibly enable one to obtain some immersions. In most known cases, such as $n \equiv 7 \bmod 8$ and $\alpha(n) \le 7$, our results are close to best possible, but we do not come close to obtaining the nonimmersion results established in [8] and [6] for P^n when $n = 2^i - 1$ and $n = 2^i + 2^j + 1$. In the former case the obstructions to immersion are detected by the Adams operations in *K*-theory. This suggests that functional *BP* operations may give some stronger results, but the situation in that regard is not at all clear.

One very pleasing result that can be deduced from (1.5) is

THEOREM 1.4. *If* $n \ge 4$ *then* $P^n \not\subseteq R^{2n-E}$, *where* $E = 10(\alpha(n) + \ell(n) - [\log_2(\alpha(n) + \ell(n))]) - 12$ *and* $\ell(n)$ *is the length of the longest string of zeros occurring in the binary expansion of n.*

For most values of n Theorem 1.5. yields a result much stronger than that of (1.4), but (1.4) agrees with (1.5) for twenty-three values of n. These values are observed by tabulating the results of (1.4) and (1.5) in enough cases with small A and $\alpha(B)$ to see where equality might occur. The values are 12, 20, 36, 58, $\cdots$, 5462.

An infinite family for which the value of E in (1.4) is only 10 greater than the value of $2D$ in (1.5) is given by $n = 2^{i+1}B + 2^i + 2^{i-1} + 8$ with $\alpha(B) = 2^{i-2} + 5$ and $\ell(B) \le i - 5$ and $i \ge 8$.

Gitler has asked whether there exists a number c such that $P^n \not\subseteq R^{2n-c\alpha(n)}$ for all n. If Theorem 1.4 reflects the behavior of the solution of the immersion problem it would suggest a negative answer to Gitler's question. Indeed, Theorem 1.4 would suggest that long strings of zeros near the end of the binary expansion of n should play a significant role in the ultimate answer to the immersion question.

In order to state our precise result, we let $p(t)$ denote the smallest power of 2 greater than or equal to t.

THEOREM 1.5. *Write* $k = 2^iB + 2^j + d$ *with* B *odd*, $0 \le d < 2^j < 2^i$ *and* $d < \alpha(B) \le 2^j + d + 1$. *Then* $P^{2k} \not\subseteq R^{4k-2D}$, *where* D *is defined below.*

i) *If* $\alpha(B) \ge 2^{j-1} - 1$ *and* $d \ne 0$, *then*

$$D = 2^{j+1} + 2d + \alpha(B) + 1 - p(2^j + d + 1 - \alpha(B)).$$

Otherwise, write $\alpha(B) + j - 2 = 2^{r+1} + r - t$ *with* $0 \le t \le 2^r$.

ii) *If* $r = j - 1$ *and* $d = 0$, *or if* $r < j - 1$ *and either* $t \le 1$ *or* $d \le p(t) - t$, *then*

$$D = 3\cdot 2^{r+1} + 2d + 2 - t - p(t).$$

iii) *If* $r < j - 1$ *and* $t \ge 2$ *and* $d > p(t) - t$, *then*

$$D = 3\cdot 2^{r+1} + 2d - d' - 3t + 4,$$

where d' *is the largest integer such that* $d' \le d$ *and* $\binom{2t-3+d'}{t-2}$ *is odd.*

2. Proof of the theorems

Parts (i) and (ii) of (1.5) will follow readily from

THEOREM 2.1. *If* $\ell = 2^e \cdot b + a$ *with* $0 \le a < 2^e$ *and* $\alpha(b) \le a + 2$, *then* $P^{2\ell+2} \not\subseteq R^{4\ell+4-2}$, *where* $D = 2a + \alpha(b) + 3 - p(a + 2 - \alpha(b))$.

Remark 2.2. Although (2.1) is similar to (1.5), and was in fact the main result in an earlier version of this paper, (1.5) has several advantages:

a) The decomposition of the binary expansion of ℓ in (2.1) is not unique.

b) In (2.1) the nonimmersion results implied by restriction are not considered. For example, if $n = 2^i + 2^j + 2^k + 2$ with $i > j > k > 4$, Theorem 2.1 gives $P^n \not\subseteq R^s$ and $P^{n-2} \not\subseteq R^t$ with $s < t$. Thus the best result for P^n is got by adding 2 to the value of D associated with $n - 2$. Theorem 1.5 incorporates these considerations.

c) Part (iii) includes results not implied by (2.1). These are obtained by applying (1.1) to $\eta \oplus L\xi$ with L odd and $L > 1$.

Proof of parts (i) *and* (ii) *of* (1.5). In part (ii) the cases with $d > 0$ are implied by the cases with $d = 0$ as in (b) of Remark 2.2, and these cases are (2.1) with $2^e \cdot b = 2^iB + 2^j - 2^{r+1}$ and $a = 2^{r+1} - 1$. Part (i) is (2.1) with $2^e \cdot b = 2^iB$ and $a = 2^j + d - 1$.

Lemma 2.3 below follows easily from the well known facts that

$$\nu\binom{m}{n} = \alpha(n) + \alpha(m-n) - \alpha(m)$$

and

$$\alpha(2^L - 1 - m) = L - \alpha(m).$$

LEMMA 2.3. *If* $\ell = 2^e \cdot b + a$ *with* $0 \leq a < 2^e$ *and* L *is sufficiently large, then*

$$\nu\binom{2^L-\ell-1}{\ell-j} = \alpha(b) + \begin{cases} \nu\binom{2a-j}{a} & \text{if } 0 \leq j \leq a \\ \text{a positive integer} & \text{if } a < j \leq 2a \\ \nu\binom{j-a-1}{a} - 1 & \text{if } 2a < j \leq a + 2^e \end{cases}$$

Proof of (2.1). Let η be the normal bundle of $P^{2\ell+2}$; then $\eta \oplus \xi = 2(2^L - \ell - 1)\xi$ by [11]. We apply (1.1) using the identity

$$\binom{k+s}{m-s} = \sum_{j=0}^{s} \binom{s}{j}\binom{k}{m-s-j} \tag{2.4}$$

which is easily proved by expanding $(1+x)^{k+s} = (1+x)^s(1+x)^k$ and comparing coefficients.

If $\alpha(b) \leq a$ take $s = \alpha(b)$ and let g be an integer such that $\alpha(b) - 1 \leq g \leq a - 1$ and the sum

$$\sum_{j=0}^{\alpha(b)} \binom{\alpha(b)}{j}\binom{2a-g-j}{a} \tag{2.5}$$

is odd. Letting $m - s = \ell - g$ and using (2.3) we obtain

$$gd(2(2^L - \ell - 1)\xi_{2m}) \geq 2m - 6s$$

which implies

$$gd(\eta) > gd(\eta \oplus \xi) - 2 \geq 2\ell - 2g - 4\alpha(b) - 2$$

and hence $P^{2\ell+2} \not\subseteq R^{4\ell+4-2(g+2\alpha(b)+2)}$. It only remains to select the most adequate integer g. Working modulo 2 we have that $\binom{2a-g-j}{a}$ is the coefficient of x^{a-g-j} in $(1+x)^{-a-1}$, so that the sum (2.5) is $\binom{2a-\alpha(b)-g}{a-\alpha(b)}$, the coefficient of x^{a-g} in $(1+x)^{-(a-\alpha(b))-1}$. Thus we may take $g = 2a - \alpha(b) - h$ where h is the largest integer such that $h \leq 2(a - \alpha(b)) + 1$ and $\binom{h}{a-\alpha(b)}$ is odd. It is easily proved that $h = p(a + 2 - \alpha(b)) - 1$, and this yields (2.1) when $\alpha(b) \leq a$.

If $\alpha(b) = a + 1$ or $a + 2$ we take $s = \alpha(b) - 1$ and $m - 2s = \ell - 2a - 1$. This yields

$$gd(2(2^L - \ell - 1)\xi) \geq 2\ell - 4a - 2\alpha(b)$$

and consequently

$$gd(\eta) > 2\ell + 2 - 2(2a + \alpha(b) + 2)$$

which implies (2.1) in this case.

Proof of part (iii) *of* (1.5). The cases with $d' < d$ are implied by the cases with $d' = d$ as in (b) of Remark 2.2; the condition $d > p(t) - t$ guarantees the existence of an odd binomial coefficient to go back to. So we shall show that if $r < j - 1$ and $\binom{2t-3+d}{t-2}$ is odd then $P^n \not\subseteq R^{2n-2D}$, where $D = 3\cdot 2^{r+1} + d - 3t + 4$.

Set $s = \alpha(B) + j - r - 1$. Then $s = 2^{r+1} + 1 - t$ and we have

$$0 < s - d < 2^{r+1} \leq 2s - d < 2^{r+2} \leq 2^j$$

as follows: The second and fourth inequalities are immediate from the definitions, while the first is a consequence of $d < \alpha(B) < s$. Now, $d < \alpha(B)$ implies $2t - 3 + d < 2^{r+1} + t - 2$, and because $\binom{2t-3+d}{t-2}$ is odd we must have $2t - 3 + d < 2^{r+1}$, which implies the third of the above inequalities.

It follows from (2.3) that if $0 < q \leq 2^j$ then

$$\nu\binom{2^L - 2^i.B - 2^j}{2^i.B + 2^j - q} = \alpha(B) + j - \nu(q).$$

This, together with the inequalities above, implies that for $s \leq u \leq 2s$ we have

$$\nu\binom{2^L - 2^iB - 2^j}{k - u} \geq s$$

with equality obtained only for $u = d + 2^{r+1}$. Thus the sum of (2.4) in this case becomes

$$2^s\binom{s}{d + 2^{r+1} - s}$$

modulo 2^{s+1}, and since

$$\binom{s}{d + 2^{r+1} - s} = \binom{2^{r+1} + 1 - t}{d + t - 1} \equiv \binom{2t - 3 + d}{t - 2},$$

the congruence being modulo 2, Theorem 1.1 gives

$$gd(2(2^L - 2^i\cdot B - 2^j)\xi_{2k}) \geq 2k - 6s$$

and hence

$$gd(\eta) > gd(\eta \oplus (2d + 1)\xi) - 2d - 2 \geq 2k - 2(3\cdot 2^{r+1} + d - 3t + 4),$$

which establishes (1.5)(iii).

Outline of proof of Theorem 1.4. We shall use the following lemma, which is easy to verify.

LEMMA 2.6. *For* $0 \leq d < 2^{j-1}$ *let* $f_j(d) = j - \alpha(d) - \ell(2^j + 2d)$. *Then*

$$f_j(2^i) = \begin{bmatrix} j - i - 2 & if \quad j \leq 2i + 3 \\ i + 1 & if \quad j \geq 2i + 3 \end{bmatrix}$$

and $f_j(d) \leq i$ *if* $2^i < d < 2^{i+1}$.

We use the notation of (1.5) and write $m^* = m - [\log_2 m]$.

1st Case: $\alpha(B) \geq 2^{j-1} + d + 1$. We need only consider $\alpha(B) = 2^{j-1} + d + 1$, for increasing $\alpha(B)$ above this increases D by no more than two times the difference in α's. In this case the statement $D \leq 5(\alpha(n) + \ell(n))^* - 6$ reduces to

$$5f_{j+1}(d) \leq 2^{j-1} + 2d + 2$$

for $0 \leq d < 2^j$, which follows easily from (2.6)

2nd Case: $2^{j-1} \leq d + 1 \leq \alpha(B) \leq 2^{j-1} + d$. We need only consider $\alpha(B) = d + 1$, for the difference of D's equals the difference of α's in this range. When $\alpha(B) = d + 1$, the statement $D \leq 5(\alpha(n) + \ell(n))^* - 6$ reduces to

$$5f_{j+1}(d) \leq 2(d - 2^{j-1}) + 8.$$

Since $f_{j+1}(2^{j-1} - 1) = 1$ and $f_{j+1}(2^{j-1} + e) = f_j(e)$, we need to show $5f_j(e) \leq 2e + 8$, and this follows from (2.6).

3rd Case: $0 \leq d \leq 2^{j-1} - 2 < \alpha(B) \leq 2^{j-1} + d$. As in the previous cases, it suffices to consider $\alpha(B) = 2^{j-1} - 1$. The desired result is

$$5f_{j+1}(d) \leq 2^j - 2d + 11,$$

which follows from (2.6) if $d < 2^{j-2}$. If $d \geq 2^{j-2}$ we let $c = 2^{j-1} - d$, so that we need

$$5\alpha(c - 1) - 5\ell(2^{j+1} + 2^j - 2c) \leq 2c + 1,$$

which is clear.

4th Case: $\alpha(B) < 2^{j-2}$ *or* $d = 0$ *and* $\alpha(B) \leq 2^{j-1}$. The result is implied by the case in which j is decreased by one and $\alpha(B)$ is increased by one.

5th Case: $2^{j-2} \leq \alpha(B) < 2^{j-1} - 1$ *and* $d \leq \alpha(B) - 2^{j-2}$. Let $\alpha(B) = 2^{j-2} + e$. We use the case $d = 0$ to estimate

$$D \leq 5\cdot 2^{j-2} + 2d + e + 2 - p(2^{j-2} - e).$$

The condition of (1.4) becomes

$$5f_{j+1}(d) + 2d - p(2^{j-2} - e) \leq 4e + 14,$$

which follows easily from (2.6), since $d \leq e$.

6th Case: $2^{j-2} \leq \alpha(B) < 2^{j-1} - 1$ *and* $d > \alpha(B) - 2^{j-2}$. Let $\alpha(B) = 2^{j-2} + e$. The case $d = e + 1$ has an odd coefficient in (1.5)(iii), and for larger d we use this case to estimate

$$D \leq 3\cdot 2^{j-2} + 2d + 2e + 3.$$

The condition of (1.4) reduces to

$$5f_{j+1}(d) + 2d \leq 2^{j-1} + 3e + 6.$$

This is satisfied except in a few cases (such as when $d = 2^{j+2} + 8$ with $j \geq 9$ and $e = 9$, 10 or 11), and for these cases (1.4) can be verified directly from (1.5)(iii).

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