Boletín de la Sociedad Matemática Mexicana Vol. 25, No. 1, 1980

# ON THE HURWITZ PROBLEM OVER AN ARBITRARY FIELD I

By José Adem

## 1. Introduction

In his celebrated paper [7], A. Hurwitz considered the following general problem. For what values of $p$, $q$ and $n$ do there exist identities of the form

$$(1.1) \qquad (x_1^2 + \cdots + x_p^2)(y_1^2 + \cdots y_q^2) = z_1^2 + \cdots + z_n^2,$$

where $z_1, \cdots, z_n$ are homogeneous bilinear forms in the two sets of variables $x_1, \cdots, x_p$ and $y_1, \cdots, y_q$?

Regarding this problem, he suggested the following, somewhat equivalent, two questions:

(1.2) *If $p$ and $q$ are given, find the minimal value of $n$.*

(1.3) *If $p$ and $n$ are given, find the maximal value of $q$.*

Years later, for the special case $p = n$, the answer to (1.3) was obtained, independently, by Hurwitz [8] and Radon [13], for the complex and real numbers as fields of coefficients. The maximal value of $q$ is given by the so-called Hurwitz-Radon function $\rho(n)$, defined as follows: $\rho(n) = 8a + 2^b$, for $n = 2^{4a+b} \cdot (2k + 1)$ where $0 \leq b \leq 3$.

Recently, D. B. Shapiro in [14] has extended these results to any field $F$ of characteristic not 2. In his generalization, he also considers, not only sum-of-squares forms, as in (1.1), but general nonsingular quadratic forms.

The author has also verified some of these results (sum-of-squares forms) for a field $F$ in [2], where an explicit construction of the identities, due to K. Y. Lam, is presented.

We restrict our attention only to identities of form (1.1) over a field of characteristic not 2. These, when $p = n$ and $q = \rho(n)$, will be called Hurwitz-Radon identities.

The assumption $p = n$ is essential in the different methods of proof used by the mentioned authors and without this assumption, the problem (1.2) is far from been solved in general. A very brief report of the situation for the case $p \neq n$, follows.

At present, the methods for constructing identities are mainly limited to two schemes. First, those obtained by taking restrictions on the Hurwitz-Radon identities. As an example of this, if $p = n = 8$, then $q = \rho(8) = 8$ and a suitable restriction of the variables gives an identity (1.1) with $p = 3$, $q = 5$ and $n = 7$. And second, the identities obtained from convenient restrictions on the multiplication of certain algebras constructed by the so-called Cayley-Dickson process. This method, studied by the author in [1], gives several new identities. As an illustration, there are identities for the values: $p = q = 10$, $n = 16$; $p = q = 12$, $n = 26$; $p = 18$, $q = 17$, $n = 32$, etc.

Clearly, these constructions provide some upper bounds for the respective minimal values of $n$.

For the *real numbers* as field of coefficients, a lower bound of $n$ is given by the following well known theorem of Hopf [6]:

(1.4) *Let $p$, $q$ and $n$ be as in* (1.1). *Then, the binomial coefficients* $\binom{n}{i}$ *are even numbers for all* $n - p < i < q$.

This theorem was proved by Hopf using algebraic topology techniques. Later, Behrend gave an algebraic proof [4], extending this result to identities over any *formally real* field.

With the use of (1.4), certain family of cases can be decided. Thus, for the identity mentioned above, where $p = 3$ and $q = 5$, over a formally real field, it follows that $n = 7$ is the minimal value, since the binomial coefficient $\binom{6}{4}$ is an odd number.

In this paper, problem (1.2) is solved for all $1 \leq p \leq 8$ and $1 \leq q \leq 8$, over any field of characteristic not 2. For each pair $(p, q)$, the minimal value of $n$ is shown to be the same as that obtained for the real field.

The identities are constructed directly and the hard part of the argument is to prove, in three cases (see (6.1), (7.1) and (8.1)), that the values of $n$ are minimal. To accomplish the latter, the original method of Hurwitz [8] is adapted here to convert (1.1) to an equivalent problem on the existence of a set of rectangular matrices, fulfilling certain conditions. Allowable transformations in this set are left and right multiplication by orthogonal matrices of the appropriate order. These transformations are used to obtain simple forms for the first two matrices of the set. Canonical forms for alternate matrices, based on orthogonal similarity, play an important role in the proofs. Finally, the low values of $p$ and $q$, makes it possible to settle the three needed cases.

## 2. Normed maps and rectangular matrices

Throughout this paper, a field $F$ will always be a field of characteristic different from 2.

Let $F^n$ denote the usual $n$-dimensional vector space over $F$ formed with the rows $x = (x_1, \cdots, x_n)$, where $x_i \in F$. Keep in mind that any $x \in F^n$, can be regarded as $1 \times n$ matrix.

The standard basis of $F^n$ is given by the vectors

$$e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 1).$$

Clearly, the $n$-columns also have the structure of an $n$-dimensional vector space over $F$. This vector space of columns will only be used in direct reference to $F^n$, throughout the transpose operation. Thus, if $x \in F^n$, the transpose of $x$, denoted by $x^t$, is the column vector

$$x^t = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

If $y = (y_1, \cdots, y_n)$ is another row vector of $F^n$, using the matrix multiplication, set

$$B(x, y) = xy^t = x_1y_1 + \cdots + x_ny_n. \tag{2.1}$$

Here, $x$ and $y$ can be viewed as variables over $F^n$. Then, $B:F^n \times F^n \to F$ becomes a symmetric bilinear pairing and it defines an inner product on $F^n$. The map $Q:F^n \to F$ given by $Q(x) = B(x, x)$ is a quadratic map and $(F^n, Q)$ (or with equivalent notation $(F^n, B)$), is a quadratic space, where

$$Q(x) = x_1^2 + \cdots + x_n^2,$$

is the quadratic form determined by $Q$, with respect to the standard basis of $F^n$. The $Q(x)$ is called the *norm* of $x$ and two vectors $x$ and $y$ are said to be *orthogonal* if $B(x, y) = xy^t = 0$. Trivially, it follows that $(F^n, Q)$ is a nonsingular quadratic space ([11; pp 3–6]).

Let $(F^p, Q_1)$, $(F^q, Q_2)$ and $(F^n, Q)$ be quadratic spaces as above, where $Q_1$, $Q_2$ and $Q$ are considered, respectively, for the values $p$, $q$ and $n$. A bilinear map $\phi:F^p \times F^q \to F^n$ is a *normed map* if

$$Q_1(x)Q_2(y) = Q(z), \tag{2.2}$$

for all $x \in F^p$, $y \in F^q$ and $z = \phi(x, y)$. Or equivalently, a map $\phi(x, y) = z = (z_1, \cdots, z_n)$ is a normed map if a formula (1.1) holds, where each $z_j$ is a homogeneous bilinear form in the coordinates of $x$ and $y$, with coefficients in $F$.

The same arguments used by Hurwitz in [8] (for the case $p = n$), can be applied to condition (2.2) to obtain an equivalent condition in terms of rectangular matrices over $F$, as follows.

For $1 \leq j \leq n$, write

$$z_j = x_1a_{1j} + \cdots + x_pa_{pj},$$

where each $a_{ij}$ is a linear homogeneous form in the variables $y_1, \cdots, y_q$. Hence, each $a_{ij}$ belongs to the ring $K = F[y_1, \cdots, y_q]$, of polynomial functions over $F$. Set

$$M = (a_{ij})$$

as a $p \times n$ matrix over $K$. Then

$$z = \phi(x, y) = xM,$$

and condition (2.2) becomes

$$zz^t = x(MM^t)x^t = (xx^t)(yy^t).$$

Now, if $A$ is a $p \times p$ symmetric matrix over $K$ (of characteristic $\neq 2$), it is easy to verify that

$$xAx^t = (xx^t)(yy^t) \Leftrightarrow A = (yy^t)I_p,$$

where $I_p$ is the identity matrix of order $p$.

Therefore, with $A = MM^t$, it follows from the above expressions that $\phi(x, y) = xM$ is normed if and only if

$$MM^t = (y_1^2 + \cdots + y_q^2)I_p. \tag{2.3}$$

Consequently, the existence of $\phi$ is equivalent to the existence of a $p \times n$ matrix over $K$, such that any two different rows are orthogonal and the norm of any row is the same expression $yy^t$.

From (2.3) it readily follows that $n \geq \max(p, q)$. In fact, let $\bar{M}$ be the $p \times n$ matrix over $F$ obtained from $M$ by the substitutions $y_1 = 1$ and $y_2 = \cdots = y_q = 0$. The rows of $\bar{M}$ represent $p$ linearly independent vectors of $F^n$. Therefore, $n \geq p$. Since $p$ and $q$ can be interchanged from the beginning, it also follows that $n \geq q$. This ends the proof.

To proceed, decompose $M$ as follows:

$$M = y_1M_1 + y_2M_2 + \cdots + y_qM_q,$$

where each $M_j$ is $p \times n$ matrix over $F$. This is possible, since each element $a_{ij}$ of $M$ is a linear homogeneous form in the coordinates of $y$.

Substitution of this expression in (2.3), gives

$$(y_1M_1 + \cdots + y_qM_q)(y_1M_1^t + \cdots + y_qM_q^t) = (y_1^2 + \cdots + y_q^2)I_p. \tag{2.4}$$

This is an identity of polynomial functions with matrices as coefficients and it implies the following

LEMMA (2.5). *There exists a normed map $\phi: F^p \times F^q \rightarrow F^n$ if and only if there exists a set $M_1, \cdots, M_q$ of $q$ rectangular $p \times n$ matrices over $F$, such that*

$$M_iM_i^t = I_p \tag{2.6}$$

$$M_iM_j^t + M_jM_i^t = 0, \text{ for } i \neq j. \tag{2.7}$$

*Proof.* Clearly, the matrices of (2.5) imply the existence of $M$ fulfilling condition (2.3), and that is enough for the existence of $\phi$.

Given $\phi$, the identity (2.4) is established. Then, (2.6) follows by substituting there $y_i = 1$ and $y_k = 0$ if $k \neq i$. Now, with $i \neq j$, consider the values $y_i = y_j = 1$ and $y_k = 0$ if $k \neq i$ and $k \neq j$. Here, using (2.6) already established, the case (2.7) follows.

*Remark.* Expressions of the form (1.1) are considered here as identities on polynomial functions. They can also be viewed as relations in a polynomial ring in the $x_i$'s and $y_j$'s, regarded as *indeterminates* over $F$. In this last case, the corresponding (2.4) is also established. Hence, the same set of conditions

(2.6) and (2.7) hold in both cases. Therefore, the existence of an identity (1.1) in $x_i$'s and $y_j$'s as variables is equivalent to its existence in $x_i$'s and $y_j$'s as indeterminates, even if $F$ is finite (cf. [9; p 418]).

## 3. A generalization of the Hurwitz equations

Let $P$ and $Q$ be two orthogonal matrices over $F$, of orders $p$ and $n$, respectively. Then, $P$ and $Q$ are square matrices, such that

$$PP^t = I_p \quad \text{and} \quad QQ^t = I_n.$$

By transforming each $M_i$ of (2.5) into

$$\tilde{M}_i = PM_iQ, \quad (i = 1, \cdots, q), \tag{3.1}$$

a new set of matrices $\tilde{M}_1, \cdots, \tilde{M}_q$ is obtained. it follows trivially that this new set satisfies (2.6), (2.7) and it gives rise to the normed map $\tilde{\phi}(x, y) = xPMQ$.

Transformations such as the one in (3.1) are allowable, since they preserve the needed properties to assure existence of a normed map. They will be used to obtain some sort of canonical form for a given set of matrices.

Take the first matrix $M_1$ of (2.5) and consider its rows as vectors in $F^n$. That is,

$$M_1 = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}, \quad \text{where } u_i \in F^n.$$

Since $M_1M_1{}^t = I_p$, it follows that $u_iu_i{}^t = 1$ and $u_iu_j{}^t = 0$ if $i \neq j$. Hence, the rows of $M_1$ make up in $F^n$ a set of $p$ orthogonal vectors of norm 1.

Let $U_1$ and $U_2$ be two subspaces of $F^n$ generated as follows: $U_1$ by the first $p$ vectors $e_1, \cdots, e_p$ of the standard basis $e_1, \cdots, e_n$ of $F^n$ and $U_2$ by the vectors $u_1, \cdots, u_p$. Define a linear function $f: U_1 \to U_2$ by setting $f(e_i) = u_i$ for $i = 1, \cdots, p$. Let $B$ be the inner product of $F^n$ defined in (2.1). It readily follows that

$$B(e_i, e_j) = B(f(e_i), f(e_j)) = B(u_i, u_j).$$

Thus, $f$ is an isometry. Then, since $(F^n, B)$ is nonsingular, "Witt's extension theorem" implies that $f$ can be extended to an orthogonal transformation or isometry $f': F^n \to F^n$ (see [11; p 26], [9; p 351]).

The extension $f'$ will be used to complete the vectors $u_1, \cdots, u_p$ to an orthogonal matrix, as follows. For $p < k \leq n$, let $u_{p+1}, \cdots, u_n$ be vectors defined by $f'(e_k) = u_k$.

Set

$$P = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} M_1 \\ D \end{bmatrix}, \quad \text{where } D = \begin{bmatrix} u_{p+1} \\ \vdots \\ u_n \end{bmatrix}.$$

The rows of $P$ are the images under $f'$ of the standard basis. Therefore, $P$ is orthogonal.

From $PP^t = I_n$, it follows that

$$\begin{bmatrix} M_1 \\ D \end{bmatrix} P^t = \begin{bmatrix} M_1 P^t \\ DP^t \end{bmatrix} = I_n.$$

Hence, if $Q = P^t$,

$$M_1 Q = \begin{bmatrix} e_1 \\ \vdots \\ e_p \end{bmatrix} = [I_p, 0],$$

where 0 represents the $p \times (n - p)$ matrix of zeros.

Now, according to (3.1), transform all the other matrices of (2.5) by right multiplication by $Q$. If $\tilde{M}_i = M_i Q$, then a new set $\tilde{M}_1, \cdots, \tilde{M}_q$ is obtained with a very simple form for $\tilde{M}_1$.

To simplify notation, omit the tilde and suppose from the beginning, in the original set of matrices, that

$$M_1 = [I_p, 0] \tag{3.2}$$

In order to find the implications of (3.2) in the structure of the other matrices, decompose each $M_i$ as follows

$$M_i = [A_i, B_i] \tag{3.3}$$

where $A_i$ is a $p \times p$ matrix and $B_i$ a $p \times (n - p)$ matrix. The condition (2.6) becomes

$$[A_i, B_i] \begin{bmatrix} A_i^t \\ B_i^t \end{bmatrix} = I_p, \qquad \text{or}$$

$$A_i A_i^t + B_i B_i^t = I_p. \tag{3.4}$$

Similarly, from (2.7), for $i \neq j$, it follows that

$$A_i A_j^t + A_j A_i^t + B_i B_j^t + B_j B_i^t = 0. \tag{3.5}$$

Let $j = 1$ in (3.5). Since $A_1 = I_p$ and $B_1 = 0$, this relation reduces to

$$A_i + A_i^t = 0. \tag{3.6}$$

Therefore, each $A_i$ in (3.3) is an alternate matrix of order $p$, for $i = 2, \cdots, q$, and (3.4), (3.5) transform to

$$-A_i^2 + B_i B_i^t = I_p, \tag{3.7}$$

$$A_i A_j + A_j A_i = B_i B_j^t + B_j B_i^t. \tag{3.8}$$

Clearly, lemma (2.5) can be supplemented by the addition of (3.2), (3.6), (3.7) and (3.8). Moreover, if $p = n$, then $B_i = 0$, for all $i$, and these conditions reduce to the so-called *Hurwitz equations* (see [14; p 151]). Hence, they can be regarded as their generalization.

For later use and as an application of (3.7) the following result will be considered:

$$\text{rank} A_i \geq 2p - n. \tag{3.9}$$

*Proof*: In fact,

$$\text{rank} A_i + \text{rank} B_i \geq \text{rank}(A_i^2) + \text{rank}(B_i B_i^t) \geq p,$$

and since $n - p \geq \text{rank} B_i$, the inequality (3.9) follows.

## 4. Canonical forms for alternate matrices

Let us briefly recall some well-known facts about the elementary divisors of a matrix (see [5], [12]).

Let $A$ be a square matrix of order $p$ over $F$. The characteristic matrix $\lambda I - A$ is *equivalent* over $F[\lambda]$ to a diagonal matrix

$$\text{diag}[f_1(\lambda), \cdots, f_p(\lambda)],$$

where each $f_i(\lambda)$ is a monic polynomial in $\lambda$ with coefficients in $F$, such that

$$f_i(\lambda) \quad \text{divides} \quad f_{i+1}(\lambda) \qquad (i = 1, \cdots, p-1). \tag{4.1}$$

The polynomials $f_1(\lambda), \cdots, f_p(\lambda)$ are the *similarity invariants* of $A$. Two given matrices are *similar* over $F$ if and only if they have the same similarity invariants.

The *characteristic polynomial* of $A$ is

$$f(\lambda) = \det[\lambda I - A] = f_1(\lambda) \cdots f_p(\lambda), \tag{4.2}$$

and if $m(\lambda)$ denotes the *minimum polynomial* of $A$, then

$$m(\lambda) = f_p(\lambda),$$

is the last similarity invariant.

Set,

$$f(\lambda) = p_1(\lambda)^{a_1} \cdots p_r(\lambda)^{a_r}, \tag{4.3}$$

where $p_1(\lambda), \cdots, p_r(\lambda)$ are distinct, monic polynomials which are irreducible over $F$.

For each similarity invariant, write

$$f_i(\lambda) = p_1(\lambda)^{a_{i1}} \cdots p_r(\lambda)^{a_{ir}} \qquad (i = 1, \cdots, p). \tag{4.4}$$

Then (4.1) implies that $a_{i+1,j} \geq a_{i,j}$, and those polynomials $p_j(\lambda)^{a_{ij}}$, for all $i$ and $j$, which appear in (4.4) with nonzero exponents, including repetitions, are the *elementary divisors* of $A$ over $F$. Clearly, their product is the characteristic polynomial and two given matrices over $F$ are similar if and only if they have the same elementary divisors.

Before proceeding the following trivial observation is appropriate. The results of this section will be used to prove nonexistence of certain normed maps. According to (2.5) this is equivalent to prove nonexistence of a set of matrices over $F$, fulfilling certain conditions. Let $K$ be an algebraically closed field containing $F$. Clearly, a given set of matrices over $F$ can also be regarded as a set of matrices over $K$. Therefore, if this set of matrices cannot exist over $K$, they too cannot exist over $F$. The assumption $F \subset K$ will be made, and this type of reasoning will be used.

Most of the results that will be presented in the last part of this section come from two sources. The paper of J. Wellstein [15] and the book of F. R. Gantmacher [5; chapter XI]. Although the results that will be quoted are originally stated for the field of complex numbers, it readily follows that they also hold for an algebraically closed field $K$. The proofs will be omitted.

The characteristic polynomial of an alternate matrix $A$ over $F$, factors over $K$, as follows

$$f(\lambda) = \lambda^{a_0}(\lambda - \lambda_1)^{a_1}(\lambda + \lambda_1)^{a_1} \cdots (\lambda - \lambda_h)^{a_h}(\lambda + \lambda_h)^{a_h}.$$

THEOREM (4.5). *The nonzero characteristic values of an alternate matrix $A$ appear in pairs $\pm \lambda_j (j = 1, \cdots, h)$ and if the elementary divisors corresponding to $\lambda_j$ are $(\lambda - \lambda_j)^{q_k}$ $(k = 1, \cdots, t)$, then the elementary divisors corresponding to $-\lambda_j$ are exactly $(\lambda + \lambda_j)^{q_k}$, for the same set of exponents.*

*If zero is a characteristic value of $A$, then in the system of elementary divisor corresponding to zero, all those of even degree are repeated an even number of times.*

THEOREM (4.6). *There exists an alternate matrix over $K$, with any given set of elementary divisors which fulfill the conditions of* (4.5).

This last result is established by the following explicit construction. Let $i$ be a fixed element of $K$ such that $i^2 = -1$. Define the matrices $U^{(p)}$ and $V_a^{(p)}$ each of order $p$, by the equalities

$$U^{(p)} = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ -1 & 0 & 1 & & & & \cdot \\ \cdot & -1 & 0 & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & -1 & 0 \end{bmatrix},$$

$$V_a^{(p)} = i \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 1 & 2a \\ \cdot & & & & 1 & 2a & 1 \\ \cdot & & & \cdot & 2a & 1 & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ 1 & 2a & \cdot & & & & \cdot \\ 2a & 1 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix},$$

then construct the alternate matrix $W_a^{(p,p)}$ of order $2p$, as follows

$$(4.7) \qquad W_a^{(1,1)} = i\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \quad \text{and} \quad W_a^{(p,p)} = \frac{1}{2}\begin{bmatrix} U^{(p)} & V_a^{(p)} \\ -V_a^{(p)} & -U^{(p)} \end{bmatrix} \quad \text{if} \quad p > 1.$$

If $a \neq 0$, it is proved (*loc. cit.*) that the elementary divisors of $W_a{}^{(p,p)}$ are $(\lambda - a)^p$ and $(\lambda + a)^p$. Moreover, if $a = 0$ and $p$ is even, the elementary divisors of the corresponding matrix are $\lambda^p$ and $\lambda^p$.

Now for the elementary divisors $\lambda^q$, where $q$ is an odd number, define the alternate matrix $W^{(q)}$ of odd order $q$, as follows: $W^{(1)} = [0]$, for $q = 1$, and for $q > 1$ by the equality

$$2W^{(q)} = \begin{bmatrix} 0 & 1 & & & \cdot & \cdot & \cdot & \cdot & & & 0 \\ -1 & 0 & 1 & & & & & & & & \\ & -1 & \cdot & \cdot & & & & & & & \\ & & \cdot & \cdot & \cdot & & & & & & \cdot \\ & & & \cdot & 0 & 1 & & & & & \cdot \\ & & & & -1 & 0 & -1 & & & & \cdot \\ & & & & & 1 & \cdot & \cdot & & & \cdot \\ & & & & & & \cdot & \cdot & \cdot & & \\ & & & & & & & \cdot & 0 & -1 & \\ & & & & & & \cdot & & 1 & 0 & -1 \\ 0 & & & \cdot & \cdot & \cdot & & & & 1 & 0 \end{bmatrix} \tag{4.8}$$

$$+\, i \begin{bmatrix} 0 & & & \cdot & \cdot & \cdot & \cdot & & & 1 & 0 \\ & & & & & & & & 1 & 0 & 1 \\ & & & & & & & \cdot & \cdot & 1 & \\ \cdot & & & & & & \cdot & \cdot & \cdot & & \\ \cdot & & & & & 1 & 0 & \cdot & & & \cdot \\ \cdot & & & & -1 & 0 & 1 & & & & \cdot \\ \cdot & & & \cdot & \cdot & -1 & & & & & \cdot \\ & & \cdot & \cdot & \cdot & & & & & & \cdot \\ & -1 & 0 & \cdot & & & & & & & \\ -1 & 0 & -1 & & & & & & & & \\ 0 & -1 & & & \cdot & \cdot & \cdot & \cdot & & & 0 \end{bmatrix}$$

As before, it is proved (*loc. cit.*) in this case that $W^{(q)}$ has a single elementary divisor $\lambda^q$.

Form a list of all the elementary divisors of $A$ which appear in the expressions (4.4) with nonzero exponents, as follows:

$$(\lambda - \lambda_j)^{c_j},\ (\lambda + \lambda_j)^{c_j} \quad \text{and} \quad \lambda^{d_k},$$

where $j = 1, \cdots, r$; $k = 1, \cdots, s$ and each $d_k$ an odd number. Moreover, in this new arrangement some of the numbers $\lambda_j$ may be repeated and some may also be equal to zero. In fact, if $\lambda_j = 0$ then $c_j$ will be even.

The block diagonal alternate matrix

$$W = \operatorname{diag}[W_{\lambda_1}{}^{(c_1,c_1)}, \cdots, W_{\lambda_r}{}^{(c_r,c_r)};\ W^{(d_1)}, \cdots, W^{(d_s)}] \tag{4.9}$$

has the same elementary divisors as the alternate matrix $A$ of (4.5). Therefore, they are similar. Now, if two alternate matrices $A$ and $W$ are similar, it follows from a wellknown result that they are orthogonally similar (see [3; p 408], [10; p 79]). That is, there exists an orthogonal matrix $Q$ over $K$ such that

$$W = QAQ^{-1} = QAQ^t \tag{4.10}$$

Hence, the matrix $W$ gives a canonical form for $A$. This will be used to improve some of the conditions considered in the preceding section.

## 5. A restriction on a second matrix

Let $M_i = [A_i, B_i]$ be the $p \times n$ matrices of (3.3) where $M_1 = [I_p, 0]$ and each $A_i$ is an alternate matrix of order $p$ for $i = 2, \cdots, q$. Consider $M_2 = [A_2, B_2]$ and let $W = QA_2Q^t$ be a canonical matrix for $A_2$, determined as in (4.10), where $Q$ is an orthogonal matrix of order $p$. Now transform each $M_i$, as in (3.1), into

$$\tilde{M}_i = QM_iS$$

where

$$S = \begin{bmatrix} Q^t & 0 \\ 0 & I_{(n-p)} \end{bmatrix}$$

is an orthogonal matrix of order $n$.

It easily follows that

$$\tilde{M}_i = [QA_iQ^t, QB_i].$$

Therefore, if $\tilde{A}_i = QA_iQ^t$ and $\tilde{B}_i = QB_i$, the collection of $p \times n$ matrices $\tilde{M}_i = [\tilde{A}_i, \tilde{B}_i]$, besides fulfilling the conditions of (2.5), has $M_1 = \tilde{M}_1 = [I_p, 0]$ and $\tilde{M}_2 = [\tilde{A}_2, \tilde{B}_2]$ where $\tilde{A}_2 = W$ is a canonical form.

As before, omit the tilde and suppose that $M_1$ and $M_2$ already have the form

$$M_1 = [I_p, 0] \quad \text{and} \quad M_2 = [W, B_2], \tag{5.1}$$

where $W$ is one of the several possible canonical forms for the alternate matrix $A_2$ of order $p$.

In the next sections all these results will be used in order to prove nonexistence of normed maps for certain values of $p$, $q$ and $n$.

## 6. Nonexistence of normed maps for $p = 5$, $q = 3$ and $n = 6$.

Here the following will be established.

THEOREM (6.1). *For any field $F$, no normed map $F^5 \times F^3 \to F^6$ can exist.*

*Proof.* Suppose a normed map $F^5 \times F^3 \to F^6$ exists over some field $F \subset K$, where $K$ is an algebraically closed field. Let $M_1$, $M_2$ and $M_3$ be the $5 \times 6$ matrices of (2.5) associated with the map, where $M_1 = [I_5, 0]$, $M_2 = [W, B_2]$ and $M_3 = [A_3, B_3]$, according to (3.3) and (5.1).

An analysis of the possible forms of $W$ will be made. First, $W$ is an alternate matrix of order 5 and from (3.9), it follows that rank $W \geqslant 4$. Hence, rank $W$

= 4. Now, the characteristic polynomial of $W$ has the form

$$f(\lambda) = \lambda(\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b),$$

with 0, $\pm a$ and $\pm b$ as characteristic values.

The different sets of elementary divisors that have to be considered give rise to the following nine cases:

I) $a \neq b$ with $a \neq 0$ and $b \neq 0$.

Here, $m(\lambda) = f_5(\lambda) = f(\lambda)$ since (4.1) must hold. Then, $f_1(\lambda) = \cdots = f_4(\lambda) = 1$. Therefore,

$$\lambda,\ (\lambda - a),\ (\lambda + a),\ (\lambda - b),\ (\lambda + b)$$

are the elementary divisors in this case. Hence, from (4.9), it follows that

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ia & 0 & 0 \\ 0 & -ia & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ib \\ 0 & 0 & 0 & -ib & 0 \end{bmatrix} \tag{6.2}$$

II) $a \neq 0$ and $b = 0$.

The characteristic polynomial is

$$f(\lambda) = \lambda^3(\lambda - a)(\lambda + a),$$

with 0 and $\pm a$ as characteristic values. From (4.5) it follows that $\lambda^2$ cannot be an elementary divisor and (4.1) implies that $(\lambda - a)$, $(\lambda + a)$ can only appear in the last similarity invariant. Then there are only two possibilities for the similarity invariants:

II$_1$) $f_1(\lambda) = f_2(\lambda) = 1$, $f_3(\lambda) = f_4(\lambda) = \lambda$ and $f_5(\lambda) = \lambda(\lambda - a)(\lambda + a)$. Thus, the elementary divisors are

$$\lambda,\ \lambda,\ \lambda,\ (\lambda - a),\ (\lambda + a)$$

and from (4.8) it follows that

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ia \\ 0 & 0 & 0 & -ia & 0 \end{bmatrix} \tag{6.3}$$

II$_2$) $f_1(\lambda) = \cdots = f_4(\lambda) = 1$ and $f_5(\lambda) = \lambda^3(\lambda - a)(\lambda + a)$. Here, the elementary divisors are

$$\lambda^3,\ (\lambda - a),\ (\lambda + a)$$

and again (4.9) gives

$$W=\frac{1}{2}\begin{bmatrix}0 & 1+i & 0 & 0 & 0\\ -1-i & 0 & -1+i & 0 & 0\\ 0 & 1-i & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2ia\\ 0 & 0 & 0 & -2ia & 0\end{bmatrix} \tag{6.4}$$

III) $a = b \neq 0$. Then $f(\lambda) = \lambda(\lambda - a)^2(\lambda + a)^2$, with 0 and $\pm a$ as characteristic values. There are two possibilities for the similarity invariants:

III$_1$) $f_1(\lambda) = \cdots = f_4(\lambda) = 1$ and $f_5(\lambda) = f(\lambda)$. In this case the elementary divisors are

$$\lambda,\ (\lambda - a)^2,\ (\lambda + a)^2,$$

and from (4.9) it follows that

$$W=\frac{1}{2}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & i & 2ia\\ 0 & -1 & 0 & 2ia & i\\ 0 & -i & -2ia & 0 & -1\\ 0 & -2ia & -i & 1 & 0\end{bmatrix} \tag{6.5}$$

III$_2$) $f_1(\lambda) = f_2(\lambda) = f_3(\lambda) = 1$, $f_4(\lambda) = (\lambda - a)(\lambda + a)$ and $f_5(\lambda) = \lambda(\lambda - a)(\lambda + a)$. The elementary divisors are

$$\lambda,\ (\lambda - a),\ (\lambda + a),\ (\lambda - a)(\lambda + a),$$

and (4.9) implies that

$$W=\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & ia & 0 & 0\\ 0 & -ia & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & ia\\ 0 & 0 & 0 & -ia & 0\end{bmatrix} \tag{6.6}$$

IV) $a = b = 0$. Then $f(\lambda) = \lambda^5$. There are four possibilities:

IV$_1$) $f_1(\lambda) = \cdots = f_5(\lambda) = \lambda$. Then

$$W = [0_5]\,. \tag{6.7}$$

IV$_2$) $f_1(\lambda) = f_2(\lambda) = 1$, $f_3(\lambda) = \lambda$ and $f_4(\lambda) = f_5(\lambda) = \lambda^2$. The elementary divisors are $\lambda$, $\lambda^2$, $\lambda^2$ and

$$W=\frac{1}{2}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & i & 0\\ 0 & -1 & 0 & 0 & i\\ 0 & -i & 0 & 0 & -1\\ 0 & 0 & -i & 1 & 0\end{bmatrix} \tag{6.8}$$

IV$_3$) $f_1(\lambda) = f_2(\lambda) = 1$, $f_3(\lambda) = f_4(\lambda) = \lambda$ and $f_5(\lambda) = \lambda^3$. The elementary divisors are $\lambda$, $\lambda$, $\lambda^3$ and

(6.9)
$$W = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+i & 0 \\ 0 & 0 & -1-i & 0 & -1+i \\ 0 & 0 & 0 & 1-i & 0 \end{bmatrix}$$

IV$_4$) $f_1(\lambda) = \cdots = f_4(\lambda) = 1$ and $f_5(\lambda) = \lambda^5$. There is only one elementary divisor $\lambda^5$ and

(6.10)
$$W = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & i & 0 \\ -1 & 0 & 1+i & 0 & i \\ 0 & -1-i & 0 & -1+i & 0 \\ -i & 0 & 1-i & 0 & -1 \\ 0 & -i & 0 & 1 & 0 \end{bmatrix}$$

Since rank $W = 4$, the cases (6.3), (6.7), (6.8) and (6.9) are ruled out. For the remaining cases let

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_5 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} b_1 \\ \vdots \\ b_5 \end{bmatrix}$$

where $w_i \in K^5$ and $b_i \in K$. As can easily be seen, the condition (3.4) turns into

(6.11)
$$w_i w_j^t + b_i b_j = \delta_{ij}$$

where $\delta_{ij}$ is the Kronecker delta.

To take care first of the less complicated cases, let $W$ be as in (6.4). Since $w_2 w_2^t = 0$, condition (6.11) implies that $b_2 = \pm 1$. As $w_2 w_1^t = 0$ and $b_2 \neq 0$, again condition (6.11) implies that $b_1 = 0$. Hence $w_1 w_1^t + b_1^2 = i$, and this contradicts (6.11). Therefore, case (6.4) is eliminated.

To continue, let $W$ be as in (6.5). Here $w_1 = 0$ and $b_1 = \pm 1$. Because $w_1 w_j^t = 0$ and $b_1 \neq 0$, from (6.11) it follows that $b_j = 0$ for all $j \neq 1$. Then, from (6.5), (6.11) and $b_2 = 0$, it follows that $w_2 w_2^t = -a^2 = 1$ and $w_2 w_3^t = -a = 0$. But this is a contradiction and, consequently, matrix (6.5) is also eliminated.

Now consider $W$ as in (6.10). From $w_1 w_1^t = 0$ and (6.11) it follows that $b_1 = \pm 1$. Since $w_1 w_2^t = 0$ and $b_1 \neq 0$, it follows that $b_2 = 0$, and this last result in (6.11) gives $w_2 w_2^t = 1$. Now directly from (6.10) it is obtained that $w_2 w_2^t = i$. Hence, there is a contradiction. So, this case must also be left out.

The two remaining cases (6.2) and (6.6) will be represented by the same matrix (6.2) where the possibility $a = b$ is considered. Let $W$ be such matrix. As before, since $w_1 = 0$ it follows that $b_1 = \pm 1$. From $w_1 w_j^t = 0$, $b_1 \neq 0$ and (6.11), it follows that $b_j = 0$ for all $j \neq 1$. Now from (6.2), (6.11) and $b_2 = b_4 = 0$, it follows that $w_2 w_2^t = -a^2 = 1$ and $w_4 w_4^t = -b^2 = 1$. Therefore, $a = \pm i$ and $b = \pm i$.

For the purpose of deciding the existence of a map as in (6.1), all these cases with different signs can be reduced to

$$(6.12) \qquad M_2 = [W, B_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In fact, if $ia = -1$, a permutation of the second and third rows followed by the same permutation on the columns of $W$, will fix the signs. This transformation is given by $QWQ$, where $Q = Q^t$ is the symmetric orthogonal matrix obtained by performing the indicated permutations on the rows and columns of the identity matrix. A similar consideration holds if $ib = -1$. Therefore, the transformed matrix (cf. (3.1))

$$Q[W, B_2]\begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} = [QWQ, B_2]$$

can be regarded as in (6.12).

Now write the matrices $A_3$ and $B_3$ explicitly as

$$(6.13) \qquad A_3 = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & a_5 & a_6 & a_7 \\ -a_2 & -a_5 & 0 & a_8 & a_9 \\ -a_3 & -a_6 & -a_8 & 0 & a_{10} \\ -a_4 & -a_7 & -a_9 & -a_{10} & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{bmatrix},$$

then take $M_2 = [W, B_2]$ as in (6.12), and substitute these expressions in the equation (3.8):

$$A_3W + WA_3 = B_3B_2^t + B_2B_3^t,$$

to obtain a matric equality. From this equality it readily follows that

$$(6.14) \qquad [A_3, B_3] = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & 0 & a_6 & a_7 \\ -a_2 & 0 & 0 & a_7 & -a_6 \\ -a_3 & -a_6 & -a_7 & 0 & 0 \\ -a_4 & -a_7 & a_6 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -a_2 \\ a_1 \\ -a_4 \\ a_3 \end{bmatrix}$$

Let $u_i$, for $i = 1, \cdots 5$, denote the rows of $M_3$ where each $u_i \in K^6$. Clearly, condition $M_3M_3^t = I_5$ is equivalent with $u_iu_j^t = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta.

Using the rows of (6.14) write conditions $u_iu_i^t = 1$ for $i = 1, 2, 4$. From these three equations it easily follows that

$$(6.15) \qquad a_1^2 + a_2^2 = a_3^2 + a_4^2 = \frac{1}{2}.$$

Now from $u_2u_4^t = 0$ and $u_2u_5^t = 0$ the following pair of equations are obtained

$$\begin{cases} a_1a_3 + a_2a_4 = 0, \\ a_1a_4 - a_2a_3 = 0. \end{cases} \tag{6.16}$$

Relation (6.15) implies that $(a_1, a_2) \neq (0, 0)$. On the other hand, given $a_3$ and $a_4$, a pair $(a_1, a_2) \neq (0, 0)$ satisfies (6.16) if and only if $a_3^2 + a_4^2 = 0$. Hence, there is a contradiction with (6.15) and therefore $M_3$ cannot exist. Since all the possibilities for $W$ have been analyzed, this ends the proof of (6.1).

## 7. Nonexistence of normed maps for $p = 5$, $q = 4$ and $n = 7$.

Using some of the results already developed in the preceding section, the next theorem will be proved.

THEOREM (7.1). *For any field $F$, no normed map $F^5 \times F^4 \to F^7$ can exist.*

*Proof.* As in the proof of (6.1), suppose that such map exists over some field $F \subset K$. Let $M_1 = [I_5, 0]$, $M_2 = [W, B_2]$, $M_3 = [A_3, B_3]$ and $M_4 = [A_4, B_4]$ be the $5 \times 7$ matrices associated with the map.

Again, from (3.9), it follows that $\operatorname{rank} W \geqslant 3$. Since $W$ is an alternate matrix of order 5, its rank must be even. Therefore, $\operatorname{rank} W = 4$ and as before, only the cases (6.2), (6.4), (6.5), (6.6) and (6.10), have to be considered for $W$.

Set

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_5 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} b_1 & c_1 \\ \vdots & \vdots \\ b_5 & c_5 \end{bmatrix} \tag{7.2}$$

where $w_i \in K^5$ and $b_i, c_i \in K$. The condition (3.4) becomes

$$w_iw_j^t + b_ib_j + c_ic_j = \delta_{ij} \tag{7.3}$$

Suppose $w_kw_k^t = 0$ *for some index $k$*. Then $b_k^2 + c_k^2 = 1$. Let $T$ and $S$ be orthogonal matrices, respectively of order 2 and 7, defined by

$$T = \begin{bmatrix} b_k & c_k \\ c_k & -b_k \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} I_5 & 0 \\ 0 & T \end{bmatrix}.$$

It follows that

$$\tilde{M}_2 = M_2S = [W, B_2]S = [W, B_2T],$$

with

$$\tilde{B}_2 = B_2T = \begin{bmatrix} \beta_1 & \gamma_1 \\ \vdots & \vdots \\ \beta_5 & \gamma_5 \end{bmatrix},$$

where $\beta_j = b_jb_k + c_jc_k$ and $\gamma_j = b_jc_k - c_jb_k$.

Hence, $\beta_k = 1$ and $\gamma_k = 0$, and (5.1) is preserved. Therefore, without a change in notation (to omit the tilde) this assumption will be made, according with each case, for a suitable fixed choice of the index $k$. That is,

(7.4) let $w_k w_k^t = 0$ and $k$ the suitable index, then $b_k = 1$ and $c_k = 0$.

Moreover, from (7.3), it follows that

(7.5) if $w_j w_k^t = 0$ and $k$ is as in (7.4), then $b_j = 0$.

Let $W$ be as in (6.4) and observe that $w_j w_2^t = 0$ for $j = 1, \cdots, 5$. Then, from (7.4) and (7.5), it follows that $b_2 = 1$, $c_2 = 0$ and $b_j = 0$ for all $j \neq 2$.

From the norm of the first and third rows of $M_2 = [W, B_2]$, it follows

$$c_1^2 = 1 - w_1 w_1^t = 1 - \frac{i}{2},$$

$$c_3^2 = 1 - w_3 w_3^t = 1 + \frac{i}{2}.$$

Then, $c_1^2 c_3^2 = \frac{5}{4}$. On the other hand, since the first and third rows of $M_2$ are orthogonal,

$$c_1 c_3 = -w_1 w_3^t = -\frac{1}{2}.$$

Hence, $c_1^2 c_3^2 = \frac{1}{4}$. This contradiction eliminates the case (6.4) as a possible option for $W$.

Next, consider $W$ as in (6.5). Since $w_1 = 0$, from (7.4), and (7.5), it follows that $b_1 = 1$, $c_1 = 0$ and $b_j = 0$, for all $j \neq 1$.

From the norm of the second and fifth rows of $M_2$, it follows that $c_2^2 = c_5^2 = 1 + a^2$. Now, as $w_2 w_5^t = 0$, condition (7.3) implies that $c_2 c_5 = 0$. Then, $c_2 = c_5 = 0$ and $a^2 = -1$. But then, since $w_2 w_3^t = -a$, the condition (7.3) is not fulfilled. In other words, the second and third rows of $M_2$ are not orthogonal. This rules out (6.5).

Suppose $W$ as in (6.10). Since $w_1 w_1^t = 0$, choose $b_1 = 1$ and $c_1 = 0$. Then, from $w_2 w_1^t = w_5 w_1^t = 0$ it follows $b_2 = b_5 = 0$. Now, $w_5 w_5^t = 0$ and $b_5 = 0$ implies that $c_5 = \pm 1$. Since $w_2 w_5^t = 0$ and $b_5 = 0$, from (7.3) it follows that $c_2 c_5 = \pm c_2 = 0$. Hence, the norm of the second row of $M_2$ becomes $w_2 w_2^t = \frac{i}{2}$. But this is contrary to condition (7.3). Therefore, the case (6.10) is also excluded.

As before, the two left cases (6.2) and (6.6) will be represented only by (6.2), where the possibility of $a = b$ is added.

Because $w_1 w_1^t = 0$, choose $b_1 = 1$ and $c_1 = 0$. Then, since $w_j w_1^t = 0$ it follows that $b_j = 0$ for $j = 2, \cdots, 5$. From the expressions for the norm of the second and third rows of $M_2$, it is obtained that $c_2^2 = c_3^2 = 1 + a^2$. Now a substitution of $w_2 w_3^t = 0$ and $b_2 = 0$ in (7.3) gives $c_2 c_3 = 0$. Consequently, $c_2 = c_3 = 0$ and $a^2 = -1$. Similarly, it follows that $c_4 = c_5 = 0$ and $b^2 = -1$.

This determines the matrix $M_2$ and as in (6.12), the signs can be fixed so that

$$M_2 = [W, B_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.6}$$

Let $M = [A, B]$ denote a $5 \times 7$ matrix that can be regarded to represent either $M_3$ or $M_4$, where $A$ and $B$ are, respectively, $5 \times 5$ and $2 \times 5$ matrices. Use for $A$ the same expression given in (6.13) for $A_3$ and let

$$B = \begin{bmatrix} b_1 & c_1 \\ \vdots & \vdots \\ b_5 & c_5 \end{bmatrix}.$$

Like in case (6.14), from the matric equality

$$AW + WA = BB_2^t + B_2B^t,$$

it easily follows that

$$M = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & 0 & a_6 & a_7 \\ -a_2 & 0 & 0 & a_7 & -a_6 \\ -a_3 & -a_6 & -a_7 & 0 & 0 \\ -a_4 & -a_7 & a_6 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & c_1 \\ -a_2 & c_2 \\ a_1 & c_3 \\ -a_4 & c_4 \\ a_3 & c_5 \end{bmatrix}$$

Recall that $MM^t = I_5$ means that all rows of $M$ have norm 1 and different rows are orthogonal. Then, from the norm of the second and third rows, it follows that

$$c_2^2 = c_3^2 = 1 - (a_1^2 + a_2^2 + a_6^2 + a_7^2), \tag{7.7}$$

and the norm of the fourth and fifth rows, gives

$$c_4^2 = c_5^2 = 1 - (a_3^2 + a_4^2 + a_6^2 + a_7^2). \tag{7.8}$$

On the other hand, orthogonality of the second and third rows and of the fourth and fifth rows, implies

$$c_2 c_3 = 0 \qquad \text{and} \qquad c_4 c_5 = 0.$$

Hence, this together with (7.7) and (7.8), implies that

$$c_2 = c_3 = c_4 = c_5 = 0.$$

Therefore, the terms to the right of the expressions (7.7) and (7.8) are equal and this yields the relation

$$a_1^2 + a_2^2 = a_3^2 + a_4^2. \tag{7.9}$$

Also with the above information the expression for $M$ becomes

$$M = \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ -a_1 & 0 & 0 & a_6 & a_7 \\ -a_2 & 0 & 0 & a_7 & -a_6 \\ -a_3 & -a_6 & -a_7 & 0 & 0 \\ -a_4 & -a_7 & a_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & c_1 \\ -a_2 & 0 \\ a_1 & 0 \\ -a_4 & 0 \\ a_3 & 0 \end{bmatrix} \tag{7.10}$$

The next equations, conveniently arranged in pairs, follow from orthogonality among the rows of $M$ in (7.10).

$$\begin{cases} a_1a_3 + a_2a_4 = 0, \\ a_2a_3 - a_1a_4 = 0, \end{cases} \tag{7.11}$$

$$\begin{cases} a_6a_1 + a_7a_2 = 0, \\ a_7a_1 - a_6a_2 = 0, \end{cases} \tag{7.12}$$

$$\begin{cases} a_6a_3 + a_7a_4 = 0, \\ a_7a_3 - a_6a_4 = 0. \end{cases} \tag{7.13}$$

These expressions will be used to prove that

$$a_1 = a_2 = a_3 = a_4 = 0. \tag{7.14}$$

Suppose the pair $(a_1, a_2) \neq (0, 0)$. Then from (7.12) it follows that $a_6^2 + a_7^2 = 0$ and similarly (7.11) implies that $a_3^2 + a_4^2 = 0$. But this gives the value zero for the norm of the last two rows of $M$. Consequently, $a_1 = a_2 = 0$. Now from (7.9) it follows that $a_3^2 + a_4^2 = 0$, yet it can be that $(a_3, a_4) \neq (0, 0)$. However, if the last inequality holds, from (7.13), it implies again $a_6^2 + a_7^2 = 0$ and then the norm of the second and third rows of $M$ becomes zero. Therefore, $a_3 = a_4 = 0$ and (7.14) is proved.

The norm of the first row implies that $c_1^2 = 1$ and by the same argument used in (6.12), the signs can be fixed so tht $c_1 = 1$. The final form for $M$ is the following

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 & a_7 \\ 0 & 0 & 0 & a_7 & -a_6 \\ 0 & -a_6 & -a_7 & 0 & 0 \\ 0 & -a_7 & a_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.15}$$

Remember that $M$ contains the possible forms for the matrices $M_3$ and $M_4$, in the case that they exist simultaneously. Assuming they exist, it follows from (7.15) that they must have the same first row shown above. On the other hand, the condition $M_3M_4^t + M_4M_3^t = 0$ implies, in particular, that the first row of $M_3$ is orthogonal to the first row of $M_4$. Hence, only $M_3$ can exist and this ends the proof of (7.1).

**8. Nonexistence of normed maps for $p = 6$, $q = 3$ and $n = 7$.**

Now the following theorem will be proved.

THEOREM (8.1). *For any field $F$, no normed map $F^6 \times F^3 \to F^7$ can exist.*

*Proof.* Assume that the map exists over some field $F \subset K$ where $K$ is algebraically closed. Let $M_1$, $M_2$ and $M_3$ be the $6 \times 7$ matrices, associated with the map, where $M_1 = [I_6, 0]$, $M_2 = [W, B_2]$ and $M_3 = [A_3, B_3]$.

From (3.9) it follows that $\operatorname{rank} W \geqslant 5$ and since $W$ is an alternate matrix of order 6, its rank must be even, therefore $\operatorname{rank} W = 6$.

Over $K$ the characteristic polynomial of $W$ has the form

$$f(\lambda) = (\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b)(\lambda - c)(\lambda + c),$$

where $\operatorname{rank} W = 6$ excludes zero as a characteristic value.

As before, different cases will be considered for the elementary divisors associated with the characteristic polynomial.

I) Suppose $a \neq b$, $a \neq c$ and $b \neq c$. Then (4.1) implies that $f_6(\lambda) = f(\lambda)$ and $f_i(\lambda) = 1$ for $i = 1, \cdots, 5$. Hence, the elementary divisors are

$$(\lambda - a), (\lambda + a), (\lambda - b), (\lambda + b), (\lambda - c), (\lambda + c). \tag{8.2}$$

II) Let $a = c$ with $a \neq b$. The characteristic polynomial is

$$f(\lambda) = (\lambda - a)^2(\lambda + a)^2(\lambda - b)(\lambda + b).$$

Compatible with (4.1) there are two possibilities for the similarity invariants:

II$_1$) $f_i(\lambda) = 1$ for $i = 1, \cdots, 4$, $f_5(\lambda) = (\lambda - a)(\lambda + a)$ and $f_6(\lambda) = (\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b)$. The elementary divisors are

$$(\lambda - a), (\lambda + a), (\lambda - a), (\lambda + a), (\lambda - b), (\lambda + b). \tag{8.3}$$

II$_2$) $f_i(\lambda) = 1$ for $i = 1, \cdots, 5$ and $f_6(\lambda) = f(\lambda)$. The elementary divisors are

$$(\lambda - a)^2, (\lambda + a)^2, (\lambda - b), (\lambda + b). \tag{8.4}$$

III) Assume $a = b = c$. Then $f(\lambda) = (\lambda - a)^3(\lambda + a)^3$. In this case there are three possibilities for the similarity invariants:

III$_1$) $f_i(\lambda) = 1$ for $i = 1, \cdots, 5$ and $f_6(\lambda) = f(\lambda)$. The elementary divisors are

$$(\lambda - a)^3, (\lambda + a)^3. \tag{8.5}$$

III$_2$) $f_1(\lambda) = 1$ for $i = 1, \cdots, 4$, $f_5(\lambda) = (\lambda - a)(\lambda + a)$ and $f_6(\lambda) = (\lambda - a)^2(\lambda + a)^2$. The elementary divisors are

$$(\lambda - a), (\lambda + a), (\lambda - a)^2, (\lambda + a)^2. \tag{8.6}$$

III$_3$) $f_i(\lambda) = 1$ for $i = 1, 2, 3$, and $f_4(\lambda) = f_5(\lambda) = f_6(\lambda) = (\lambda - a)(\lambda + a)$. The elementary divisors are

$$(\lambda - a), (\lambda + a), (\lambda - a), (\lambda + a), (\lambda - a), (\lambda + a). \tag{8.7}$$

If repeated characteristic values are allowed, the three cases (8.2), (8.3) and

(8.7) can be represented by (8.2). Similarly, the two cases (8.4) and (8.6) can be represented by (8.4). Moreover, this is compatible with the construction of W. Hence, only the cases (8.2), (8.4) and (8.5) will be considered.

Let the elementary divisors be as in (8.2). Then (4.9) implies that

$$W=\begin{bmatrix} 0 & ia & 0 & 0 & 0 & 0 \\ -ia & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ib & 0 & 0 \\ 0 & 0 & -ib & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ic \\ 0 & 0 & 0 & 0 & -ic & 0 \end{bmatrix} \tag{8.8}$$

Now for (8.4) the construction (4.9) gives

$$W=\frac{1}{2}\begin{bmatrix} 0 & 1 & i & 2ia & 0 & 0 \\ -1 & 0 & 2ia & i & 0 & 0 \\ -i & -2ia & 0 & -1 & 0 & 0 \\ -2ia & -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2ib \\ 0 & 0 & 0 & 0 & -2ib & 0 \end{bmatrix} \tag{8.9}$$

Finally, for the case (8.5), from (4.9) it follows that

$$W=\frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & i & 2ia \\ -1 & 0 & 1 & i & 2ia & i \\ 0 & -1 & 0 & 2ia & i & 0 \\ 0 & -i & -2ia & 0 & -1 & 0 \\ -i & -2ia & -i & 1 & 0 & -1 \\ -2ia & -i & 0 & 0 & 1 & 0 \end{bmatrix} \tag{8.10}$$

As in section 6 let

$$W=\begin{bmatrix} w_1 \\ \vdots \\ w_6 \end{bmatrix} \quad \text{and} \quad B_2=\begin{bmatrix} b_1 \\ \vdots \\ b_6 \end{bmatrix},$$

where $w_i \in K^6$ and $b_j \in K$. First, it will be shown that (8.9) and (8.10) fail to satisfy condition (3.4) (or its equivalent (6.11)). Let $W$ be as in (8.10). Since $w_1 w_6{}^t = w_2 w_5{}^t = w_3 w_4{}^t = 0$, it follows from (6.11), that $b_1 b_6 = b_2 b_5 = b_3 b_4 = 0$. Then, $b_1 = 0$ or $b_6 = 0$, etc. A direct computation in (8.10), gives $w_j w_j^t = -a^2$ for all $j$. Hence, $w_j w_j^t + b_j^2 = 1$ implies that $b_j^2 = 1 + a^2$ and that $b_1{}^2 = b_6{}^2$, etc. Consequently, $b_j = 0$ for all $j$ and $a = \pm i$. But then, $w_1 w_2{}^t + b_1 b_2 = -a \neq 0$, and this contradicts (6.11). Therefore (8.10) is eliminated as a possible matrix for $W$.

Suppose $W$ is as in (8.9). Using only the first four rows of $W$, a very similar argument to the one given for (8.10), shows that $W$ fails to satisfy (6.11). Therefore, (8.9) should not be considered.

Now let $W$ be of the form (8.8). A direct inspection of $W$ shows that $w_i w_j^t$

$= 0$ for all $i \neq j$ and that

$$w_1 w_1^t = w_2 w_2^t = -a^2, \qquad w_3 w_3^t = w_4 w_4^t = -b^2, \qquad w_5 w_5^t = w_6 w_6^t = -c^2.$$

From (6.11) and from this information, it readily follows that $b_i b_j = 0$ for all $i \neq j$ and that

$$b_1^2 = b_2^2 = 1 + a^2, \qquad b_3^2 = b_4^2 = 1 + b^2, \qquad b_5^2 = b_6^2 = 1 + c^2.$$

Consequently, $b_j = 0$ for all $j$ and $a^2 = b^2 = c^2 = -1$.

With the same argument used in (6.12) the signs can be arranged so that

$$\text{(8.11)} \qquad W = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

To study the possible existence of $M_3 = [A_3, B_3]$ assume, with the usual notation, that

$$A_3 = (a_{ij}) \qquad \text{and} \qquad B_3 = \begin{bmatrix} b_1 \\ \vdots \\ b_6 \end{bmatrix},$$

where $A_3$ is an alternate matrix of order 6. Because of $B_2 = 0$, the equation (3.8) becomes $WA_3 + A_3 W = 0$, where $W$ is the matrix of (8.11). Like in (6.14), this equality imposes some conditions on the elements of $A_3$. Taking into account these conditions, the matrix $A_3$ becomes

$$\text{(8.12)} \qquad A_3 = \begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 0 & a_{14} & -a_{13} & a_{16} & -a_{15} \\ -a_{13} & a_{14} & 0 & 0 & a_{35} & a_{36} \\ -a_{14} & a_{13} & 0 & 0 & a_{36} & -a_{35} \\ -a_{15} & -a_{16} & -a_{35} & -a_{36} & 0 & 0 \\ -a_{16} & a_{15} & -a_{36} & a_{35} & 0 & 0 \end{bmatrix}$$

As before, let $w_i$ denote the $i^{th}$-row of $A_3$. Let $i = 1, 3, 5$, then the following relations are quickly verified:

$$w_i w_{i+1}^t = 0 \qquad \text{and} \qquad w_i w_i^t = w_{i+1} w_{i+1}^t.$$

Then, from (6.11), it follows that $b_i b_{i+1} = 0$ and that $b_i^2 = b_{i+1}^2$. Hence, $b_i = 0$ for all $i$ and consequently, $B_3 = 0$.

Like in the case (6.14), a further analysis will show that the matrix $M_3 = [A_3, 0]$ fails to satisfy (3.4). However, a short cut in the proof is given by the following argument. Assume that $A_3$ is compatible with (3.4). Since $B_i = 0$ for $i = 1, 2, 3$, it follows that the matrices $M_1 = I_6$, $M_2 = W$ and $M_3 = A_3$ determine a normed map $F^6 \times F^3 \rightarrow F^6$ and its restriction $F^5 \times F^3 \rightarrow F^6$ is also a normed map. But this implies a contradiction with (6.1). Therefore, the original $M_3 = [A_3, B_3]$ cannot exist as required. This ends the proof of (8.1).

## 9. The main theorem

The results of the last three sections will be used to prove the following theorem that constitutes the main result of this paper.

THEOREM (9.1). *Let F be a field of characteristic not 2 and let $1 \leqslant p, q \leqslant 8$. Then the minimal n for the existence of a normed map $F^p \times F^q \to F^n$ is independent of F and its value is given by the following table:*

| q \ p | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | | 2 | 4 | 4 | 6 | 6 | 8 | 8 |
| 3 | | | 4 | 4 | 7 | 8 | 8 | 8 |
| 4 | | | | 4 | 8 | 8 | 8 | 8 |
| 5 | | | | | 8 | 8 | 8 | 8 |
| 6 | | | | | | 8 | 8 | 8 |
| 7 | | | | | | | 8 | 8 |
| 8 | | | | | | | | 8 |

*Proof.* Since the table is symmetric only half of it is presented. The maps are constructed out of the classical products $F^k \times F^k \to F^k$ where $k = 1, 2, 4, 8$, by taking direct sums and restrictions. As an illustration, the direct sum of the maps $F^4 \times F^3 \to F^4$ and $F^1 \times F^3 \to F^3$ gives a map $F^5 \times F^3 \to F^7$. The construction of all other maps is readily obtained. Hence, the details are omitted.

Let $F^p \times F^q \to F^n$ be a normed map. Then the following properties hold:

(9.2) $$n \geqslant \max(p, q),$$

(9.3) $$\text{if } p = n \text{ then } q \leqslant \rho(n),$$

(9.4) $$\begin{cases} \text{if } p = 5 \text{ and } q = 3 \text{ then } n \geqslant 7, \\ \text{if } p = 5 \text{ and } q = 4 \text{ then } n \geqslant 8, \\ \text{if } p = 6 \text{ and } q = 3 \text{ then } n \geqslant 8. \end{cases}$$

Recall that (9.2) was established in section 2 and that (9.3) is essentially the result mentioned in the introduction, where the Hurwitz-Radon function $\rho(n)$ was defined. The use of (9.3) is restricted here to its very weak form: $\rho(n) = 1$ if $n$ is odd. Finally, (9.4) collects the statements (6.1), (7.1) and (8.1) of the last sections.

Now given $p$ and $q$, the proof that the value $n$ in the table is minimal follows easily using in each case a suitable condition out of (9.2), (9.3) and (9.4). The details of the verification are trivial and therefore they are omitted. This ends the proof of (9.1).

CENTRO DE INVESTIGACION DEL IPN, MEXICO 14, D.F.

REFERENCES

[1] J. ADEM, *Construction of some normed maps*, Bol. Soc. Mat. Mexicana **20**(1975), 59–75.
[2] J. ADEM, Algebra Lineal, campos vectoriales e inmersiones, III ELAM, IMPA, Rio de Janeiro, 1978.
[3] A. A. ALBERT, *Symmetric and alternate matrices in an arbitrary field*, Trans. Amer. Math. Soc. **43**(1938), 386–436.
[4] F. BEHREND, *Über Systeme reeler algebraischer Gleichungen*, Compositio Math. **7**(1939), 1–19.
[5] F. R. GANTMACHER, The theory of Matrices I, II, Chelsea Publishing Company, New York 1960.
[6] H. HOPF, *Ein topologischer Beitrag zur reelen Algebra*, Comment. Math. Helv. **13**(1940–41), 219–239.
[7] A. HURWITZ, *Über die Komposition der quadratische Formen von beliebig vielen Variabeln*, Nachr. K. Gesellsch. Göttingen M. Ph. Kl (1898), 309–316.
[8] A. HURWITZ, *Über die Komposition der quadratischen Formen*, Math. Ann. **88**(1923), 1–25.
[9] N. JACOBSON, Basic Algebra I, M. H. Freeman and Company, San Francisco 1974.
[10] I. KAPLANSKY, Linear Algebra and Geometry, Chelsea Publishing Company, New York 1974.
[11] T. Y. LAM, The Algebraic Theory of Quadratic Forms. W. A. Benjamin, Inc. Reading, Massachusetts 1973.
[12] S. PERLIS, Theory of Matrices, Addison-Wesley Publishing Company, Inc. Reading, Massachusetts, 1958.
[13] J. RADON, *Lineare Schären orthogonalen Matrizen*, Abh. Math. Sem. Univ. Hamburg, **1**(1922), 1–14.
[14] D. B. SHAPIRO, *Spaces of Similarities*. I. *The Hurwitz Problem*, J. Algebra **46**(1977) 148–170.
[15] J. WELLSTEIN, *Über symmetrische, alternierende und orthogonale Normalformen von Matrizen*, J. Reine Angew. Math., **163**(1930) 166–182.