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FUNCTIONS CONVEX IN TWO DIRECTIONS

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A plane domain is *convex in the direction of* L (a fixed line or ray) if its intersection with every line parallel to L is connected. Let $\mathfrak{F}$ be a family of normalized analytic functions $f(z) = z + a_2z^2 + \cdots$ in $E: |z| < 1$. Reade and Zlotkiewicz [1] determined the *Koebe set* of $\mathfrak{F}$, $\cap f(E): f \in \mathfrak{F}$, for a number of classes, among them that of the normalized univalent functions with image convex in a given direction. This domain was later found independently by Goodman and Saff [2]. We extend this result to functions with image convex in two directions, separated by a fixed angle $0 < \nu \leq \pi/2$. In case $\nu = 0$ the two directions coincide and our result reduces to the previous one. We conclude this note with the observation that the hypothesis of univalence is totally unnecessary.

Notation and Lemmas. Let $CD(\nu)$ denote the class of normalized univalent functions in E such that $f(E)$ is convex in the directions of $L(-\nu): \arg z = -\nu$ and the positive real axis $L(0)$. For convenience of notation we will write Q_0, Q_1, Q_2, Q_3 for the "quadrants" defined by $\arg z \in [-\nu, 0]$, $[0, \pi - \nu]$, $[\pi - \nu, \pi]$, $[\pi, 2\pi - \nu]$ respectively. Let $\beta = 2 - \nu/\pi$ throughout this paper. As the linear fractional transformation $A(z) = (e^{-\alpha(\beta+1)i}z - ie^{\alpha(\beta+1)i})/(e^{-\alpha(\beta-1)i}z - ie^{\alpha(\beta-1)i})$ carries E onto the upper half plane when $0 < \alpha < \pi/2$, the following lemma is easily verified (z^β means, as is usual, $r^\beta e^{i\beta\theta}$ where $z = re^{i\theta}$ and $0 \leq \theta < 2\pi$).

LEMMA 1. *The function* $F = F_{\nu,\alpha}$ *defined by*

$$F(z) = (A(z)^\beta - e^{2i\alpha\beta})(2\beta \sin 2\alpha)^{-1}, \; |z| < 1, \tag{1}$$

is in the class $CD(\nu)$ *for each* α, $0 < \alpha < \pi/2$. *Its image* $F(E)$ *is the complement of the translated quadrant* $Q_0 + c$ *with corner at*

$$c = c(\nu, \alpha) = -e^{2i\alpha\beta}(2\beta \sin 2\alpha)^{-1}. \tag{2}$$

Although we are mainly interested in $CD(\nu)$ for $0 < \nu \leq \pi/2$, the lemma is valid for angles in the range $0 \leq \nu < 2\pi$.

We find the polar form for the curve $c(\nu)$ which $c(\nu, \alpha)$ traces as α varies from 0 to $\pi/2$, by setting $\phi = \arg c = 2\alpha\beta - \pi$, and then

$$\rho = |c| = (2\beta \sin(\phi + \pi)/\beta)^{-1}, \; -\pi < \phi < \pi - \nu. \tag{3}$$

Let $D(\nu)$ denote the complementary domain of $c(\nu)$ which contains the origin, in other words the set $|z| < \rho$ where ρ is given by (3). In the following, $\bar{D}$ denotes the reflection of a domain D in the real axis.

LEMMA 2. *Let* $f: E \to \mathbf{C}$ *be a normalized analytic function. Let* $w \in \mathbf{C}$. *If* $f(E)$ *is disjoint from* $Q_i + w$, $i = 0, 1, 2,$ *or* 3 *respectively, then* w *lies in the complement of* $D(\nu)$, $\bar{D}(\pi - \nu)$, $-D(\nu)$, or $-\bar{D}(\pi - \nu)$ *respectively.*

Proof. We shall imitate very closely the argument given in [2] for convexity in one direction. First consider the case $i = 0$. Since $0 = f(0) \notin Q_0 + w$ we can write $w = re^{i\phi}$ with $r > 0$, $-\pi < \phi < \pi - \nu$. Setting $t = \rho/r$ where ρ is given by (3) we have $c = tw$ for suitable $0 < \alpha < \pi/2$ and $t > 0$. Now the function $tf(z)$ omits $Q_0 + c$ and F is univalent; consequently $F^{-1}(tf(z))$ is well defined from E to E and fixes the origin. By Schwarz's lemma, $t < 1$, and we conclude that w is in the complement of $D(\nu)$. In the remaining cases $i = 1, 2, 3$ we use $\overline{F_{\pi-\nu}(\bar{z})}$, $-F_\nu(-z)$, and $\overline{-F_{\pi-\nu}(-\bar{z})}$ in place of $F(z)$.

The image domain.

THEOREM 1. *The Koebe set of $CD(\nu)$ is precisely $R(\nu) = D(\nu) \cap \bar{D}(\nu) \cap (-D(\nu)) \cap (-\bar{D}(\pi - \nu))$.*

Proof. Let $f \in CD(\nu)$. If f omits the point w, it must in fact omit, by directional convexity, one of the rays $L(0) + w$, $L(\pi) + w$. Likewise it omits one of $L(-\nu) + w$, $L(\pi - \nu) + w$. As a result, f must omit some $Q_i + w$ and by Lemma 2, w cannot be in $R(\nu)$. Conversely, the functions $F(tz)/t$, $0 < t < 1$, and their reflections provide examples omitting any point not in $R(\nu)$.

The domain $R(\nu)$ is symmetric in the directions of $L(-\nu/2)$, $L((\pi - \nu)/2)$, as would be expected from the fact $e^{-i\nu}\overline{f(e^{i\nu}\bar{z})}$ is in $CD(\nu)$ when f is. $R(0)$ is the domain which the results cited earlier specify for functions convex in the horizontal direction. For $f \in CD(\nu)$, both f and $e^{i\nu}f(e^{-i\nu}z)$ are in $CD(0)$, so we have trivially $R(0) \cup e^{-i\nu}R(0) \subseteq R(\nu)$; in fact, this inclusion is strict for $0 < \nu \leq \pi/2$.

The curves $c(\nu)$, $-c(\nu)$ meet at $z_0 = (2\beta \cos \pi/2\beta)^{-1}e^{i(\pi-\nu)/2}$ and at $-z_0$. For this value of ϕ, $\bar{c}(\pi - \nu)$ passes through $(2(3 - \beta))^{-1}e^{i(\pi-\nu)/2}$. Therefore $c(\nu)$, $-c(\nu)$, and $\bar{c}(\pi - \nu)$ (likewise $c(\nu)$, $-c(\nu)$ and $-\bar{c}(\pi - \nu)$) are concurrent when $\nu = \nu_0 = \pi(2 - \beta_0)$, where β_0 satisfies the transcendental equation

$$\beta \cos \pi/2\beta = 3 - \beta. \tag{4}$$

The approximate value is $\nu_0 \approx .18138\pi$. For $0 \leq \nu \leq \nu_0$, $D(\nu) \cap (-D(\nu))$ is contained in $\bar{D}(\pi - \nu) \cap (-\bar{D}(\pi - \nu))$ and hence $R(\nu)$ reduces to the former set (see Figure 1). When $0 \leq \nu_1 < \nu_2 \leq \nu_0$, $R(\nu_1)$ is properly contained in $R(\nu_2)$, but on the other hand when $\nu_0 \leq \nu_1 < \nu_2$ or $\nu_1 < \nu_0 < \nu_2$ neither of $R(\nu_1)$, $R(\nu_2)$ contains the other. All of the above statements may be verified by diligent application of trigonometry.

We may also consider the class $CD^*(\nu)$ of normalized univalent functions convex in any two directions separated by the angle ν. Its Koebe domain is clearly the largest disk contained in $R(\nu)$, centered at the origin. The radius of the largest such disk in $D(\nu) \cap (-D(\nu))$ is $1/2\beta$, for $\bar{D}(\pi - \nu) \cap (-\bar{D}(\pi - \nu))$ it is $1/2(3 - \beta)$. Therefore we have

THEOREM 2. *The Koebe domain of $CD^*(\nu)$ is the disk $|z| < 1/2\beta$.*

Nonunivalent functions. A review of the proofs in the preceding section reveals that the univalence of the functions in $CD(\nu)$ is never used. This is an

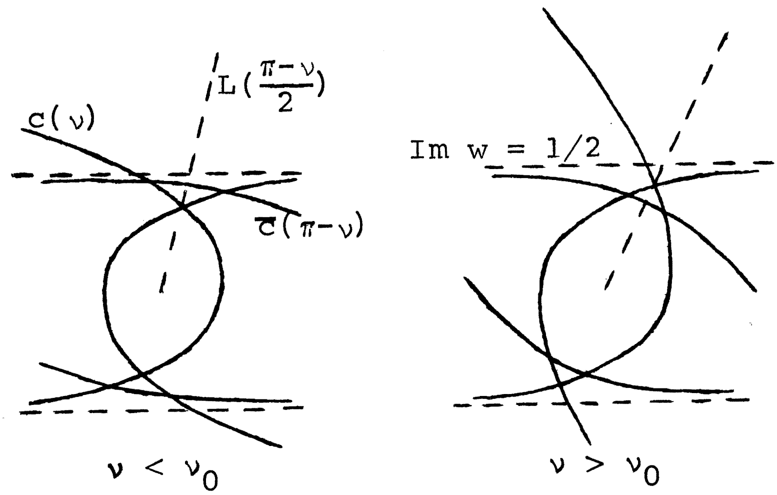

Fig. 1

instance of a more general phenomenon. In loose terms, for a class $\mathfrak{F}$ of normalized analytic functions defined by a suitable geometric condition on the image, the Koebe set of $\mathfrak{F}$ is the same as that of the family of univalent functions in $\mathfrak{F}$.

THEOREM 3. *Let* $f, g: E \to \mathbf{C}$ *be normalized analytic functions. Suppose* g *is univalent and* $g(E) = tf(E)$ *for some* $t > 0$. *If* $R \subseteq g(E)$ *where* R *is starshaped with respect to the origin, then also* $R \subseteq f(E)$.

Proof. The function $g^{-1}(tf(z))$ is well-defined from E to E and by Schwarz's lemma, $t \leq 1$. By hypothesis $f(E)$ contains $t^{-1}R$, and since R is starshaped, $t^{-1}R$ contains R.

For a simple illustration we apply this to the classical Koebe-Bieberbach theorem and obtain the following: if f is normalized and if its (not necessarily one-to-one) image is simply connected, then that image must include the disk of radius $\frac{1}{4}$ about the origin.

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