Boletín de la Sociedad Matemática Mexicana Vol. 25 No. 2, 1980

INVARIANCE PRINCIPLE FOR BRANCHING RANDOM MOTIONS

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1. Introduction

We define a branching motion process as a supercritical age-dependent branching process with the population elements, which we call particles, moving randomly in space. The evolutions of particles that belong to different branches are independent conditional upon the initial data of their movements. To each branch of time-length T there corresponds a random process, which is interpreted as the motion of the ancestry line of the particle that lives on the branch at time T. Considering a single branch of length T (i.e. a renewal process) and introducing suitable time and space scalings, the corresponding motion process is assumed to converge weakly to a process L as $T \to \infty$. The empirical distribution of the branching motion process at time T is defined for each realization of the model by selecting at random a branch from those of length T, and taking its corresponding motion process realization. This empirical distribution is a random probability measure on an appropriate function space for each T, and our aim is to show that it converges weakly as $T \to \infty$, for almost all realizations, to a process $\tilde{L}$ which is obtained from the single branch limit process L in a certain way.

Results of this type have been obtained for Galton-Watson branching processes with general motions, and for age-dependent Markov branching processes with Brownian motions by Gorostiza and Ruiz-Moncayo [17, 18]. In both cases the limits L and $\tilde{L}$ coincide. In the former case this is due to the trivial age structure of the Galton-Watson process and in the latter it is due to the independence and stationarity of Brownian motion increments. The natural conjecture would be that in the general age-dependent case $\tilde{L}$ should be the same as L, except that the particle lifetime distribution $G\ (dt)$, on which L typically depends, is replaced by $\tilde{G}\ (dt) = me^{-\alpha t}G\ (dt)$, where m is the mean of the particle production law and α is the Malthusian parameter. For Brownian motion along the ancestry lines this change is of course not apparent. A special case where the conjecture is meaningful and has been proved is age-dependent Markov branching random walk [15]. In this paper we prove the conjecture in general, i.e. for general age-dependent branching processes with general motions (under certain technical restrictions). The proof of the general result is based upon a particular decomposition of the process. This type of approach has been used in different ways by a number of authors to solve a variety of problems. Some relevant papers are: Athreya and Kaplan [1], Kaplan and

* The first named author was partially supported by CONACYT grant PNCB 1627. The authors gratefully acknowledge the hospitality of the Instituto de Investigación en Matemáticas Aplicadas y Sistemas, Universidad Nacional Autónoma de México, and the Centre de Recherche de Mathématiques Appliquées, Université de Montreal, where parts of this research were done.

Asmussen [23], Kaplan [22], Gorostiza and Ruiz-Moncayo [17, 18]. A general discussion of the decomposition method is given by Athreya and Kaplan [2]. In the present case the application of the decomposition method is difficult due to the generality of our model; a detailed analysis of the age structure and recent results of Biggins [5] on the asymptotic shape of a random walk are employed.

A brief account of the development of the subject of empirical distributions of supercritical branching random motions is contained in [19]. Most of the existing single-time-point results (as opposed to time-interval results) deal with the asymptotics of the empirical distribution of the particle positions at a given time, as this time tends to infinity. Many of these results can be obtained from our present limit theorem (see Section 3).

Other recent works of related interest are the following: Bensoussan, Lions and Papanicolaou [4] study asymptotics of branching transport processes, using a scaling where the initial density of particles increases, and they derive as a consequence the diffusion approximation of neutron transport theory. An invariance principle proved by Fleischmann and Siegmund-Schultze [9] concerns a Galton-Watson model similar to that of [17], with a critical branching law. Branching diffusion models with infinitely many initial particles are investigated e.g. by Dawson and Ivanoff [7], and Holley and Stroock [20]. In some of the references we have cited, as well as in the present paper, due to the time scalings the branching motion process originated by each initial particle does not converge to a branching motion process. Using a different type of scaling Gorostiza and Griego [16] obtain convergence of branching transport processes to branching Brownian motion.

In Section 2 we describe the branching motion model and state the invariance principle. Section 3 gives some examples and Section 4 contains the proofs.

Proofs of results that we use without reference from the theory of branching processes can be found in Athreya and Ney [3]. Similarly, the relevant results of weak convergence can be found in Billingsley [6].

2. Branching transport processes and the invariance principle.

We will consider random processes with trajectories in the spaces $D[0, \infty)^d$ or $D[0, T]^d$ (i.e. the spaces of functions from $[0, \infty)$ or $[0, T]$ to $\mathbb{R}^d$, right-continuous with left limits), equipped with the Skorohod topology. $C[0, \infty)^d$ and $C[0, T]^d$ denote the corresponding subspaces of continuous functions. Weak convergence of probability measures is denoted $\xrightarrow{d}$, and $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$.

By a *transport process*

$$X \equiv \{X(\{\tau_i\}, t), t \geqslant 0\}$$

based on the times $\{\tau_i\}$ we mean that τ_i, $i = 1, 2, \cdots$, are independent, identically distributed (i.i.d) random variables, and if $S_0 \equiv 0$ and $S_i = \Sigma_{j=1}^{i} \tau_j$,

$i \geq 1$, then for each i, given the position and velocity (when it exists) of the process X at time S_i-, its evolution in the time interval $[S_i, S_{i+1})$ is governed by a distribution which is independent of the past. We assume $X(0) = 0$. The τ_i are the *waiting times*, the S_i are the *renewal times*, and $N(t) \equiv \max\{i : S_i \leq t\}$, $t \geq 0$, is the *renewal function* of the transport process.

The following cases exemplify different types of transport processes.

(a) *Brownian motion:* Suppose that at each renewal time a new Brownian motion starts out from the point where the previous one stopped. Then X is a Brownian motion on $C[0, \infty)^d$. In this case the τ_i are irrelevant due to the stationarity and independence of Brownian motion increments.

(b) *Random walk:* Let ξ_i, $i = 1, 2, \cdots$, be i.i.d. random vectors in $\mathbb{R}^d$, with finite, positive definite covariance matrix. Then

$$X(t) \equiv X(\{\tau_i\}, t) = \Sigma_{i=1}^{N(t)}\xi_i,\ t \geq 0,$$

is a random walk with jumps ξ_i and jump-times S_i.

(c) *Linear transport process:* Let $\{\theta_i, i = 0, 1, \cdots\}$ be a sequence of random directions (unit vectors in $\mathbb{R}^d$, $d \geq 3$) with a given distribution (as a discrete-time process), independent of the τ_i. Then

$$X(t) \equiv X(\{\tau_i\}, t) = \Sigma_{i=1}^{N(t)}\theta_{i-1}\tau_i + \theta_{N(t)}(t - S_{N(t)}),\ t \geq 0,$$

is a linear transport process with directions θ_i and direction-change-times S_i. The special case where θ_{i+1} is distributed with radial symmetry about θ_i for each i is relevant in neutron transport theory and polymer chemistry (see e.g. [10], [13], [24]).

The transport processes in the examples above converge weakly with certain time and space scalings and under certain technical conditions, and the limits are Brownian motions. There are also examples of transport processes whose limits are not Brownian motions (e.g. if space dependence of the transport motion is allowed; see e.g. [24]).

In general we will consider transport processes that converge weakly, but they and their limits need not be specified. However, we will require that the increments of the transport processes satisfy one of the following conditions for large h:

(2.1) (a) $\sup_t E[\sup_{0\leq s\leq h} \| X(t+s) - X(t) \| \mid \text{renewal times}] \leq Kh$,

(b) $\sup_t E[\sup_{0\leq s\leq h} \| X(t+s) - X(t) \| \mid \text{renewal times}] \leq K[N(t+h) - N(t)]$,

and in the case of processes having directions,

(c) $\sup_t \sup_{0\leq s\leq h} \| X(t+s) - X(t) \| \leq Kh$,

where K is a positive constant.

It is easily verified that examples (a), (b), (c) above satisfy conditions (2.1) (a), (b), (c) respectively.

Given a process $X \equiv \{X(t), 0 \leq t \leq T\}$ with trajectories in $D[0, T]^d$, the process X_T defined by the time scaling

$$X_T(t) \equiv X(Tt),\ 0 \leq t \leq 1,$$

is in $D[0, 1]^d$. We will consider processes X such that under a space scaling a_T we have

$$a_T^{-1} X_T \xrightarrow{d} L \text{ as } T \to \infty, \tag{2.2}$$

where L is a random element of $C[0, 1]^d$. Moreover, we assume a_T to be of the form

$$a_T = \eta T^{\beta}, \tag{2.3}$$

where $\beta > 0$ and η is a positive constant which may depend on moments of τ_1. A centering may be needed for convergence of X_T, and we will comment on this later on.

The linear transport process, unlike the two other examples cited, does not have independent increments on the intervals $[S_i, S_{i+1})$, $i = 0, 1, \cdots$. Indeed, the distribution of $X(t) - X(s)$, $t > s$, depends on the direction of the process at time $s-$. Therefore we make the following assumption for transport processes having directions that may depend on their past directions.

(2.4) The distribution of $X(t) - X(s)$, $t > s$, depends on the process up to time s only through its direction at $s-$, and

$$a_T^{-1} X_T \xrightarrow{d} L \quad \text{as} \quad T \to \infty$$

uniformly in the initial direction.

This condition is fulfilled in many cases. For example, it follows from Lemmas 3.4 to 3.7 in [12] that the linear transport processes in [12] and [13] satisfy (2.4)

Now we introduce the branching. We consider an age-dependent (Bellman-Harris) branching process, with particle production law $\{p_k\}_{k=0}^{\infty}$ which is supercritical, i.e. $m \equiv \sum_{k=0}^{\infty} p_k k > 1$, and such that $\sum_k p_k k^2 < \infty$, and particle lifetime distribution G with finite mean μ. In order to avoid complications that are not essential we suppose that $p_0 = 0$ (hence all the branches are infinite and we don't have to condition on non-extinction), G is non-lattice and has no atom at 0 (hence ordinary renewal theory can be used), and that there is a single initial particle, constituting generation number 0, of age 0 at time 0.

The Malthusian parameter α is the (unique) root of

$$m \int_0^{\infty} e^{-\alpha t} G\,(dt) = 1,$$

and the distribution $\tilde{G}$ is defined by

$$\tilde{G}\,(dt) = m e^{-\alpha t} G\,(dt),\ t \geq 0, \tag{2.5}$$

whose (finite) mean we denote $\tilde{\mu}$.

Let Γ_T denote the set of branches of time-length T in the family tree, and Z_T

its cardinality, i.e. Z_T is the number of particles alive at time T. For each branch $\gamma \in \Gamma_T$, $N(\gamma, t)$ denotes the generation number of the particle alive on γ at time t, $0 \leqslant t \leqslant T$, and $\tau_i(\gamma)$, $i = 1, \cdots, N(\gamma, T) + 1$, are the ($G$-distributed) lifetimes of the successive particles on γ. Let $S_0(\gamma) \equiv 0$, $S_i(\gamma) = \Sigma_{j=1}^{i} \tau_j(\gamma)$, $i \geqslant 1$, and observe that the corresponding renewal function is precisely $N(\gamma, t)$, $0 \leqslant t \leqslant T$.

We define a *branching transport process*

$$\{X(\gamma), \gamma \in \Gamma_T\}, T > 0,$$

as an age-dependent branching process, as above, such that for each $T > 0$, to each branch $\gamma \in \Gamma_T$ there is associated a transport process $X(\gamma)$ from $D[0, T]^d$ such that

$$X(\gamma) \equiv \{X(\{\tau_i(\gamma)\}, t), 0 \leqslant t \leqslant T\},$$

i.e., $X(\gamma)$ is based on the lifetimes $\{\tau_i(\gamma)\}$ of the successive particles on γ. The increment process corresponding to the life-span of a particle on γ can be interpreted as a motion performed by that particle, and $X(\gamma)$ itself as the motion of the ancestry line of the particle alive on γ at time T. Particles that lie on different lines of descent are assumed to evolve independently of each other and of everything else in the past, conditional upon the initial data of their movements. Clearly, the motions of different particles may be dependent when their ancestry lines have a common part; this is the case for example in branching linear transport processes, because the direction of a particle affects the evolutions of all its descendants.

Branching transport processes as defined above are what we have generically called branching motion processes in the introduction.

Our main hypothesis is that the convergence condition (2.2) introduced above for the transport processes holds on the branches of the branching process, in the sense that if $\hat{\gamma}$ denotes a single branch of length T (i.e. a renewal process with G-distributed waiting times), then

$$a_T^{-1} X_T(\hat{\gamma}) \xrightarrow{d} L \text{ as } T \to \infty. \tag{2.6}$$

(Observe that the processes $X_T(\gamma)$ associated to *realizations* of different branches γ behave in general differently).

We are assuming the existence of a basic probability space $(\Omega, \mathbb{F}, P)$ where our branching transport process is defined, with Ω containing all genealogies and motions, and $\mathbb{F}$ being sufficiently large to provide all the measurability we need for our arguments.

The *empirical distribution* of the branching transport process $\{X(\gamma), \gamma \in \Gamma_T\}$, $T > 0$, is defined on $(\Omega, \mathbb{F}, P)$ for each $T > 0$ by

$$P_T(\omega, A) = Z_T(\omega)^{-1} \Sigma_{\gamma \in \Gamma_T(\omega)} \mathbf{1}[a_T^{-1} X_T(\omega, \gamma) \in A], \omega \in \Omega, \tag{2.7}$$

where A varies in the Borel field of $D[0, 1]^d$, and $\mathbf{1}[\cdot]$ is the indicator function; i.e., the empirical distribution is obtained for each realization (ω) of the model by selecting a branch γ at random (uniformly) from $\Gamma_T(\omega)$ and taking its corresponding transport process realization $X_T(\omega, \gamma)$. Thus $P_T(\omega, A)$ is a random probability measure on $D[0, 1]^d$, and we will prove that it converges weakly (i.e. as a function of A) almost-surely (a.s.) (i.e. for P-almost all ω) to the process $\tilde{L}$ described below.

The limit process $\tilde{L}$ is related to the single branch limit process L as follows: if we take a single branch (i.e. a renewal process with G-distributed waiting times), substitute G by $\tilde{G}$, defined in (2.5), and denote by $\tilde{X}_T$ the process obtained in the place of X_T, i.e.

$$\tilde{X} \equiv \{X(\{\tilde{\tau}_i\}, t), 0 \leq t \leq T\}, \quad \text{and } \tilde{X}_T(t) \equiv \tilde{X}(Tt), 0 \leq t \leq 1, \tag{2.8}$$

where the $\tilde{\tau}_i$ are i.i.d., $\tilde{G}$-distributed, then our hypothesis (2.6) implies that

$$a_T^{-1}\tilde{X}_T \xrightarrow{d} \tilde{L} \text{ as } T \to \infty. \tag{2.9}$$

This is so because the convergence (2.6) depends on G only through some of its moments, and moreover $\tilde{L}$ bears the same dependence on the moments of $\tilde{G}$ as L does on those of G. A centering may be necessary in (2.9) and a modification of a_T can be made; this is discussed in Remark 2 below. We note that $\tilde{L}$ also has continuous sample paths and $\tilde{X}$ satisfies conditions (2.1) (and 2.4) if relevant).

Summarizing, the almost-sure invariance principle for the empirical distribution (2.7) of the branching transport process $\{X_T(\gamma), \gamma \in \Gamma_T\}$ is the following

THEOREM.

$$P(\{\omega: P_T(\omega,\cdot) \xrightarrow{d} \tilde{L} \textit{ as } T \to \infty\}) = 1,$$

i.e.

$$Z_T^{-1} \Sigma_{\gamma \in \Gamma_T} \mathbf{1}[a_T^{-1}X_T(\gamma) \in A] \to P[\tilde{L} \in A] \textit{ a.s. as } T \to \infty,$$

where A is any $\tilde{L}$-continuous Borel set of $C[0, 1]^d$.

Remarks.

1. The theorem states that *the proportion of branches $\gamma \in \Gamma_T$ whose corresponding transport processes $X_T(\gamma)$ lie in the Borel set a_TA tends* a.s. *to $P[\tilde{L} \in A]$ as $T \to \infty$*. Therefore, using the continuous mapping theorem and special continuous functionals, we can obtain results on the proportion of particles alive at time T whose ancestry line trajectories satisfy given conditions (Section 3 contains examples).

2. If a centering $C_T(t)$ is required in (2.6), i.e. $a_T^{-1}(X_T - C_T) \xrightarrow{d} L$, then a centering $\tilde{C}_T(t)$ will also be needed in (2.9), i.e. $a_T^{-1}(\tilde{X}_T - \tilde{C}_T) \xrightarrow{d} L$, and in order to have convergence of the empirical distribution of the branching

transport process $\{a_T^{-1}X_T(\gamma), \gamma \in \Gamma_T\}$, the $X_T(\gamma)$ also have to be centered by $\tilde{C}_T$, not by C_T; this is clear from the proof of the theorem (convergence of $\phi_{3,T}$). Moreover, since in the normalization $a_T = \eta T^\beta$, η may depend on moments of G, we can replace a_T in (2.9) by $\tilde{a}_T = \tilde{\eta}T^\beta$, where $\tilde{\eta}$ depends on moments of $\tilde{G}$ in the same way as η on those of G. Thus $\tilde{a}_T^{-1}(\tilde{X}_T - \tilde{C}_T) \xrightarrow{d} \tilde{L}$ substitutes (2.9), and then we must also replace a_T by $\tilde{a}_T$ in the branching transport process to maintain the limit $\tilde{L}$. Summarizing, if

$$a_T^{-1}(X_T - C_T) \xrightarrow{d} L \text{ as } T \to \infty,$$

then

$$\tilde{a}_T^{-1}(\tilde{X}_T - \tilde{C}_T) \xrightarrow{d} \tilde{L} \text{ as } T \to \infty,$$

and the empirical distribution of the branching transport process

$$\{\tilde{a}_T^{-1}(X_T(\gamma, t) - \tilde{C}_T(t)), 0 \leq t \leq 1, \gamma \in \Gamma_T\}$$

converges weakly a.s. to $\tilde{L}$ as $T \to \infty$.

3. The moment condition $\Sigma p_k k^2 < \infty$ can probably be replaced by the (minimal) condition $\Sigma p_k\, k \log k < \infty$, using the methods of Athreya and Kaplan [1], and Kaplan [22], but we did not endeavor to do this.

4. This theorem is used in [27] to obtain a Gaussian limit for supercritical branching random fields.

3. Examples

A large variety of examples can be given, since from many functional central limit theorems one can obtain, using our invariance principle, respective limit theorems for the corresponding branching motion processes. A collection of (single branch) functional central limit theorems is contained in Iglehart [21], general types of transport processes are treated by Papanicolaou [24], and Papanicolaou, Stroock and Varadhan [25], random motions of $\mathbb{R}^d$ are studied in Gorostiza [14], to mention only a few sources that contain models of different kinds from which one can derive branching versions. In particular there are examples where the normalization a_T is not the typical $T^{1/2}$.

We restrict our examples to the special transport processes contained in Section 2. Standard d-dimensional Brownian motion starting at 0 will be denoted B_d.

Branching Brownian motion: If $X(\gamma)$, $\gamma \in \Gamma_T$, is B_d on $C[0, 1]^d$, then $T^{-1/2}X_T(\gamma)$ is B_d on $C[0, 1]^d$ for all T, and therefore hypothesis (2.6) is trivially satisfied with $L = B_d$. On the other hand, $\tilde{X}$ is also B_d (since the waiting times are irrelevant), and hence, from (2.9), $\tilde{L} = B_d$. It follows from the invariance principle that the empirical distribution of the branching Brownian motion $\{T^{-1/2}X_T(\gamma), \gamma \in \Gamma_T\}$ converges *a.s.* to B_d as $T \to \infty$. This is the simplest example.

Convergence of the empirical distribution of positions at time T for branching Brownian motion associated to an age-dependent branching process was first studied by S. Watanabe in the Markovian case (see [3], p. 243), and a general result was obtained by Kaplan and Asmussen [23]. The invariance principle in the Markovian case was proved by Gorostiza and Ruiz-Moncayo [18].

Branching random walk: Let ξ_i, $i = 1, 2, \cdots$, be i.i.d. random vectors in $\mathbb{R}^d$, with mean vector 0 and finite, positive-definite covariance matrix C. Recall that the lifetime distribution G has finite mean μ. Then the random walk

$$T^{-1/2}C^{-1/2}\mu^{1/2}\Sigma_{i=1}^{N(Tt)}\xi_i,\ 0 \leqslant t \leqslant 1,$$

(the matrix $C^{-1/2}$ is the square root of C^{-1}) converges weakly to B_d as $T \to \infty$ (see [21]). The invariance principle then implies that the empirical distribution of the branching random walk

$$\{T^{-1/2}C^{-1/2}\tilde{\mu}^{1/2}\,\Sigma_{i=1}^{N(\gamma,Tt)}\xi_i(\gamma),\ 0 \leqslant t \leqslant 1,\ \gamma \in \Gamma_T\}$$

converges weakly a.s. to B_d as $T \to \infty$ (recall that $\tilde{\mu}$ is the mean of $\tilde{G}$ and see Remark 2).

We now look at random walks on $\mathbb{R}$ with general mean. Let ξ_i, $i = 1, 2, \cdots$, be i.i.d. random variables, with mean ν and finite variance ρ^2. Assume the waiting times τ_i have mean μ and finite variance σ^2. Then we have the following result, which is probably well-known.

PROPOSITION.

$$\{T^{-1/2}(\rho^2\mu^{-1} + \nu^2\sigma^2\mu^{-3})^{-1/2}(\Sigma_{i=1}^{N(Tt)}\xi_i - Tt\nu\mu^{-1}),\ 0 \leqslant t \leqslant 1\} \xrightarrow{d} B_1 \text{ as } T \to \infty.$$

Proof. The cases $\sigma^2 = 0$ and $\rho^2 = 0$ are well-known. Hence we assume $\sigma^2 > 0$ and $\rho^2 > 0$ (and obviously $\mu > 0$ and $\nu \neq 0$). We have

$$\begin{aligned} &T^{-1/2}(\rho^2\mu^{-1} + \nu^2\sigma^2\mu^{-3})^{-1/2}(\Sigma_{i=1}^{N(Tt)}\xi_i - Tt\nu\mu^{-1}) \\ &= [T^{-1/2}\rho^{-1}\mu^{1/2}(\Sigma_{i=1}^{N(Tt)}\xi_i - N(Tt)\nu)](1 + \nu^2\rho^{-2}\sigma^2\mu^{-2})^{-1/2} \\ &+ [T^{-1/2}\sigma^{-1}\mu^{3/2}(N(Tt) - Tt\mu^{-1})]\nu\sigma(\rho^2\mu^2 + \nu^2\sigma^2)^{-1/2}. \end{aligned}$$

The two terms in brackets on the right-hand side converge weakly to respective standard Brownian motions $B_1^{(1)}$ and $B_1^{(2)}$ (see [21]), and the result follows by noting that $B_1^{(1)}$ and $B_1^{(2)}$ are independent.

From this proposition and the invariance principle we have that the empirical distribution of the branching random walk

$$\{T^{-1/2}(\rho^2\tilde{\mu}^{-1} + \nu^2\tilde{\sigma}^2\tilde{\mu}^{-3})^{-1/2}(\Sigma_{i=1}^{N(\gamma,Tt)}\xi_i(\gamma) - Tt\nu\tilde{\mu}^{-1}),\ 0 \leqslant t \leqslant 1,\ \gamma \in \Gamma_T\}$$

converges weakly *a.s.* to B_1 as $T \to \infty$, where $\tilde{\mu}$ and $\tilde{\sigma}^2$ are the mean and variance of $\tilde{G}$ (see Remark 2).

Several results can be derived from this. For example:

(1) Let $Z_T(y)$ denote the number of particles alive at time T located at points

$\leqslant y$, and let

$$y_T(u) = uT^{1/2}(\rho^2\tilde{\mu}^{-1} + \nu^2\tilde{\sigma}^2\tilde{\mu}^{-3})^{1/2} + T\nu\tilde{\mu}^{-1},\ u \in \mathbb{R}.$$

Then (using the functional $x \to x(1)$, $x \in D[0, 1]$) we have

$$Z_T(y_T(u))/Z_T \to (2\pi)^{-1/2} \int_{-\infty}^{u} e^{-y^2/2}\, dy \ a.s. \text{ as } T \to \infty.$$

This result is also a special case of a theorem proved by Kaplan [22] under $\Sigma p_k k \log k < \infty$, using a different method. Previous work in the line of [22] was done by Athreya and Kaplan [1], and Kaplan and Asmussen [23]. The invariance principle for the centered random walk ($\nu = 0$) in the Markovian case was proved by Gorostiza [15], using another approach.

(2) The case $\xi_i \equiv 1$ yields results on the generation numbers along the ancestry lines. The empirical distribution of the branching generation process (branching renewal process)

$$\{T^{-1/2}\tilde{\sigma}^{-1}\tilde{\mu}^{3/2}(N(\gamma, Tt) - Tt\tilde{\mu}^{-1}),\ 0 \leqslant t \leqslant 1,\ \gamma \in \Gamma_T\}$$

converges weakly *a.s.* to B_1 as $T \to \infty$. In particular, if $R_T(y)$ denotes the number of particles alive at time T whose generation numbers are $\leqslant y$, and if

$$y_T(u) = uT^{1/2}\tilde{\sigma}\tilde{\mu}^{-3/2} + T\tilde{\mu}^{-1},\ u \in \mathbb{R},$$

then (using the functional $x \to x(1)$ on $D[0, 1]$) we have

$$R_T(y_T(u))/Z_T \to (2\pi)^{-1/2} \int_{-\infty}^{u} e^{-y^2/2}\, dy \ a.s. \text{ as } T \to \infty.$$

Samuels [26] obtained this result with convergence in probability, and it also follows from [22].

From the invariance principle for the branching generation process, or more directly from the corresponding individual branch limit theorem for renewal times, one can obtain results for the branching times along the ancestry lines. For example, using the functional $\sup_{0\leqslant t\leqslant 1} |x(t)|$, $x \in D[0, 1]$, it can be shown that the proportion of particles alive at time T such that on their ancestry lines the branching (renewal) times S_i differ from $i\tilde{\mu}$ by less than $T^{1/2}u (u > 0)$ in the time interval $[0, Ty]$ $(0 < y \leqslant 1)$ converges *a.s.* as $T \to \infty$ to

$$4\pi^{-1} \textstyle\sum_{k=0}^{\infty} (-1)^k(2k + 1)^{-1} \exp\{-\pi^2(2k + 1)^2y/8u^2\tilde{\mu}\tilde{\sigma}^{-2}\}.$$

(3) The effect of the branching mean m on the trajectories of a branching random walk can be seen in the case $\nu = 0$, $\rho^2 = 1$, and G exponential with parameter λ. In this case $\tilde{G}$ is exponential with parameter λm, and the empirical distribution of the branching random walk

$$\{T^{-1/2} \textstyle\sum_{i=1}^{N(\gamma,Tt)} \xi_i(\gamma),\ 0 \leqslant t \leqslant 1,\ \gamma \in \Gamma_T\}$$

converges weakly *a.s.* to $(\lambda m)^{1/2}B_1$ as $T \to \infty$. Using the functional $\sup_{0\leqslant t\leqslant 1} x(t)$ on $D[0, 1]$, one can show that the proportion of particles alive at time T such that the trajectories of their ancestry lines exceed the level $T^{1/2}\ a$ $(a > 0)$ in the time interval $[0, Ty]$ $(0 < y \leqslant 1)$ converges *a.s.* as $T \to \infty$ to

$$(2/\pi)^{1/2} \int_{a(\lambda m y)^{-1/2}}^{\infty} e^{-u^2/2}\, du.$$

Therefore increasing m causes an increase of this limiting proportion. This phenomenon is absent in the Galton-Watson case, where the limit is independent of m (see [17]).

Branching linear transport process: Now we look at the linear transport process in $\mathbb{R}^d$ $(d \geqslant 3)$ defined in Section 2, already scaled:

$$T^{-1/2}X_T(t) = T^{-1/2}[\Sigma_{i=1}^{N(Tt)}\, \theta_{i-1}\tau_i + \theta_{N(Tt)}(Tt - S_{N(Tt)})],\ 0 \leqslant t \leqslant 1.$$

Observe that under the scaling the particle is traveling at a speed of $T^{1/2}$. For definiteness, and due to its interest in neutron transport theory and polymer chemistry, we consider only the special case of radially symmetric direction changes, i.e., we assume that for each i,

$$\theta_{i+1} = \theta_i \cos \beta_{i+1} + \zeta_i \sin \beta_{i+1},$$

where ζ_i is uniformly distributed on the unit sphere in the hyperplane perpendicular to θ_i, and the random angles β_i are i.i.d., independent of the θ_i. In isotropic transport theory the β_i are uniformly distributed on $[0, \pi]$; in polymer theory the β_i take on a constant value called valence angle.

If $E\tau_1^{\rho} < \infty$ for some $\rho > 3$ and if the distribution of β_1 is not supported on $\{0, \pi\}$, then

$$T^{-1/2}X_T \xrightarrow{d} vB_d \text{ as } T \to \infty,$$

where

$$v^2 = \mu^{-1}[\sigma^2 + \mu^2(1 + E \cos \beta_1)(1 - E \cos \beta_1)^{-1}],$$

μ and σ^2 being the mean and variance of τ_1. This is proved in [13].

It follows from the invariance principle that the empirical distribution of the branching linear transport process

$$\{T^{-1/2}[\Sigma_{i=1}^{N(\gamma,Tt)}\, \theta_{i-1}(\gamma)\tau_i(\gamma) + \theta_{N(\gamma,Tt)}(\gamma)(Tt - S_{N(\gamma,Tt)}(\gamma))],\ 0 \leqslant t \leqslant 1,\ \gamma \in \Gamma_T\}$$

converges weakly *a.s.* as $T \to \infty$ to $\tilde{v}B_d$, where

$$\tilde{v}^2 = \tilde{\mu}^{-1}[\tilde{\sigma}^2 + \tilde{\mu}^2(1 + E \cos \beta_1)(1 - E \cos \beta_1)^{-1}],$$

$\tilde{\mu}$ and $\tilde{\sigma}^2$ being the mean and variance of $\tilde{G}$. (In the special case where G is exponential with parameter λ, as is commonly assumed in physics, then $\tilde{v}^2 = 2/\lambda m(1 - E \cos \beta_1)$).

Then, for example, the proportion of particles alive at time T whose ancestry line trajectories remain during the time interval $[0, Ty]$ $(0 < y \leqslant 1)$ inside the ball centered at the origin with radius $aT^{1/2}$ $(a > 0)$, converges *a.s.* as $T \to \infty$ to

$$P(Q_{a/\tilde{v}} > y),$$

where Q_u is the first passage time through $u > 0$ of a Bessel process of index $\nu = (d-2)/2$ starting at 0, and Q_u has Laplace transform

$$Ee^{-sQ_u} = (u(2s)^{1/2})^{\nu}\Gamma(\nu+1)I_\nu(u(2s)^{1/2})$$

(see Getoor and Sharpe [11]).

4. Proofs

We list for easy reference some results from branching processes and renewal theory that are well-known or not hard to derive, and which are valid under the conditions we are assuming.

Branching processes (see [3]).

Recall $\{p_k\}$, m, G, α, $\tilde{G}$, Z_t.

Definitions:

$$A(z) = \left(\int_0^z e^{-\alpha u}[1-G(u)]\,du\right)\left(\int_0^\infty e^{-\alpha u}[1-G(u)]\,du\right)^{-1}, \qquad z \geqslant 0$$

(stationary age distribution).

$$G_z(x) = [G(z+x) - G(z)][1-G(z)]^{-1}, \qquad x \geqslant 0, \quad z \geqslant 0,$$

with G taken as distribution function ($G_z(\cdot)$ is the lifetime distribution function of a particle initially of age z). Similarly (see (2.5)),

$$\tilde{G}_z(x) = [\tilde{G}(z+x) - \tilde{G}(z)][1-\tilde{G}(z)]^{-1}, \qquad x \geqslant 0, \quad z \geqslant 0.$$

$$V(z) = m\int_0^\infty e^{-\alpha u}G_z(du), \qquad z \geqslant 0$$

(reproductive age value).

We denote expectation when the initial particle has lifetime distribution G_z by E_z.

Results:

A1. $e^{-\alpha t}Z_t \to W$ *a.s.* as $t \to \infty$, where W is an *a.s.* positive random variable.

A2. $EZ_t \leqslant Ke^{\alpha t}$ and $EZ_t^2 \leqslant Ke^{2\alpha t}$ for all t, where K is a positive constant.

A3. Let $A(z, t)$ = proportion of particles alive at time t of age $\leqslant z$. Then

$$A(z,t) \to A(z) \quad a.s. \quad \text{as} \quad t \to \infty,$$

and

$$\int_0^\infty V(z)A(dz,t) \to \int_0^\infty V(z)A(dz) = m\tilde{\mu}\alpha(m-1)^{-1} \quad \text{a.s.} \quad \text{as} \quad t \to \infty.$$

A4.
$$mG_z(dx) = e^{\alpha x}V(z)\tilde{G}_z(dx).$$

Renewal theory (see [8]).

Let $\tau_1, \tau_2, \cdots,$ be i.i.d. random variables with non-lattice distribution function G, with finite mean μ; $S_0 \equiv 0$, $S_n = \sum_{i=1}^n \tau_i$, $n \geqslant 1$; and $N(t) = \max\{n : S_n \leqslant t\}$, $t \geqslant 0$.

Result:

B1.
$$S_{N(t)+1} - t \xrightarrow{d} H \qquad \text{as} \quad t \to \infty,$$

where

$$P[H \leq x] = \mu^{-1} \int_0^x [1 - G(y)]\, dy, \qquad x \geq 0.$$

Consequence:

Let $\tilde{\tau}_1, \tilde{\tau}_2, \cdots,$ be i.i.d. random variables with distribution $\tilde{G}(dx) = me^{-\alpha x}G(dx)$, mean $\tilde{\mu}$; $\tilde{S}_0 \equiv 0$, $\tilde{S}_n = \Sigma_{i=1}^n \tilde{\tau}_i$, $n \geq 1$, $\tilde{N}(t) = \max\{n : \tilde{S}_n \leq t\}$, $t \geq 0$. Then B1 (and uniform integrability) implies

B2.
$$Ee^{\alpha(\tilde{S}_{\tilde{N}(t)+1}-t)} \to (m-1)(\tilde{\mu}\alpha)^{-1} \qquad \text{as} \quad t \to \infty.$$

Weak convergence will be proved using the following lemma, which can be obtained from standard theory. Before stating the lemma we need to define a class of complex-valued functions on $D[0, 1]^d$. For $x = (x^1, \cdots, x^d) \in D[0, 1]^d$, let

$$F_0 = \{f : f(x) = \exp(i\, \Sigma_{j,k} u_{jk} x^j(t_k)),\ t_k \in [0, 1],$$
$$u_{jk} \in \mathbb{R},\ j = 1, \cdots, d;\ k = 1, \cdots, m;\ m = 1, 2, \cdots\},$$

and

$$F_1 = \{f : f(x) = \exp(iuw'(x, \delta)),\ u \in \mathbb{R},\ 0 < \delta < 1\},$$

where

$$w'(x, \delta) = \max_{1\leq i\leq d} \max\{\sup_{t-\delta\leq t'\leq t\leq t''\leq t+\delta} \min[\,|x^i(t) - x^i(t')|,\ |x^i(t) - x^i(t'')|\,],$$
$$\sup_{0\leq t\leq \delta} |x^i(t) - x^i(0)|,\ \sup_{1-\delta\leq t\leq 1} |x^i(t) - x^i(1)|\},$$

and let $F = F_0 \cup F_1$.

It is easy to show that for each $f \in F$ there exists a constant $M > 0$ such that

(4.1)
$$|f(x) - f(y)| \leq M \sup_{0\leq t\leq 1} \|x(t) - y(t)\|, \qquad x, y \in D[0, 1]^d.$$

LEMMA. *Let X_T, $T > 0$, and X be random elements of $D[0, 1]^d$ defined respectively on probability spaces $(\Omega_T, \mathfrak{F}_T, P_T)$, $T > 0$, and $(\Omega, \mathfrak{F}, P)$. If*

$$\int_{\Omega_T} f(X_T)\, dP_T \to \int_\Omega f(X)\, dP \qquad \textit{as} \quad T \to \infty$$

for each $f \in F$, and X is left-continuous at $t = 1$ P-a.s., then

$$X_T \xrightarrow{d} X \qquad \textit{as} \quad T \to \infty.$$

Proof of the Theorem.

Take $f \in F$ and let

$$\phi_T = Z_T^{-1} \Sigma_{\gamma\in\Gamma_T} f(a_T^{-1} X_T(\gamma)).$$

Due to the lemma, the theorem will be proved if we show that

$$\phi_T \to Ef(\tilde{L}) \quad a.s. \qquad \text{as} \quad T \to \infty. \tag{4.2}$$

(The space $(\Omega_T, \mathfrak{F}_T, P_T)$ of the lemma corresponds to the random choice of $\gamma \in \Gamma_T$).

Let s_T be a time such that $0 < s_T < T$, and consider the decomposition

$$\phi_T = \phi_{1,T} + \phi_{2,T} + \phi_{3,T},$$

where

$$\phi_{1,T} = Z_T^{-1} \textstyle\sum_{\gamma \in \Gamma_T} [f(a_T^{-1} X_T(\gamma)) - f(a_T^{-1} X_T^{s_T}(\gamma))],$$

$$\phi_{2,T} = Z_T^{-1} \textstyle\sum_{\gamma \in \Gamma_{s_T}} \{\sum_{\gamma' \in \Gamma_T^\gamma} f(a_T^{-1} X_T^{s_T}(\gamma')) - E^{s_T} \{\sum_{\gamma' \in \Gamma_T^\gamma} f(a_T^{-1} X_T^{s_T}(\gamma'))\},$$

$$\phi_{3,T} = Z_T^{-1} \textstyle\sum_{\gamma \in \Gamma_{s_T}} E^{s_T} \sum_{\gamma' \in \Gamma_T^\gamma} f(a_T^{-1} X_T^{s_T}(\gamma')),$$

with E^{s_T} meaning conditional expectation given everything up to time s_T,

$$X_T^{s_T}(\gamma, t) = \begin{cases} 0, & 0 \leq t < p_T(\gamma)/T, \\ X_T(\gamma, t) - X_T(\gamma, p_T(\gamma)/T), & p_T(\gamma)/T \leq t \leq 1, \end{cases}$$

where

$$p_T(\gamma) = S_{N(\gamma, s_T)+1}(\gamma) \wedge T,$$

and Γ_T^γ denoting the elements of Γ_T that are continuations of $\gamma \in \Gamma_s$, $s < T$.

We will prove (4.2) by showing that with an appropriate choice of s_T one obtains

$$\phi_{1,T} \to 0, \ \phi_{2,T} \to 0 \quad \text{and} \quad \phi_{3,T} \to Ef(\tilde{L}) \quad a.s. \qquad \text{as} \quad T \to \infty.$$

At this point we require that $s_T = o(t)$ (hence $T - s_T \to \infty$) and $s_T = o(a_T)$ as $T \to \infty$. A more precise choice of s_T will be made later on.

Proof that $\phi_{1,T} \to 0$. We have, using (4.1),

$$\begin{aligned} |\phi_{1,T}| &\leq Z_T^{-1} \textstyle\sum_{\gamma \in \Gamma_T} |f(a_T^{-1} X_T(\gamma)) - f(a_T^{-1} X_T^{s_T}(\gamma))| \\ &\leq Z_T^{-1} M a_T^{-1} \textstyle\sum_{\gamma \in \Gamma_T} \sup_{0 \leq t \leq 1} \| X_T(\gamma, t) - X_T^{s_T}(\gamma, t) \| \\ &= Z_T^{-1} M a_T^{-1} \textstyle\sum_{\gamma \in \Gamma_T} \sup_{0 \leq t \leq p_T(\gamma)/T} \| X_T(\gamma, t) \| \\ &= Z_T^{-1} M a_T^{-1} \textstyle\sum_{\gamma \in \Gamma_T} \sup_{0 \leq t \leq p_T(\gamma)} \| X(\gamma, t) \| \\ &\leq Z_T^{-1} M a_T^{-1} \textstyle\sum_{\gamma \in \Gamma_T} \sum_{i=0}^{N(\gamma, s_T)} D_i(\gamma), \end{aligned}$$

where

$$D_i(\gamma) = \sup_{S_i(\gamma) \leq t \leq S_{i+1}(\gamma)} \| X(\gamma, t) - X(\gamma, S_i(\gamma)) \|, \qquad i = 0, \cdots, N(\gamma, s_T).$$

Hence

$$|\phi_{1,T}| \leq M(a_T^{-1} s_T)(s_T^{-1} \max_{\gamma \in \Gamma_{s_T}} \textstyle\sum_{i=0}^{N(\gamma, s_T)} D_i(\gamma)),$$

and since $a_T^{-1}s_T \to 0$ *a.s.*, it suffices to show that

$$s_T^{-1} \max_{\gamma \in \Gamma_{s_T}} \Sigma_{i=0}^{N(\gamma, s_T)} D_i(\gamma)$$

remains bounded *a.s.* as $T \to \infty$, but this holds due to results on the asymptotic shape of a branching random walk. Indeed, Biggins' Corollary to Theorem B [5] can be applied to show that there is a finite random variable K such that

$$\max_{\gamma \in \Gamma_{s_T}} N(\gamma, s_T) \leq [K s_T] \text{ a.s.} \quad \text{for large } s_T,$$

where $[\cdot]$ denotes integer part. Then for each $\gamma \in \Gamma_{s_T}$,

$$\Sigma_{i=0}^{N(\gamma, s_T)} D_i(\gamma) \leq \Sigma_{i=1}^{[K s_T]+1} D_i(\theta) \text{ a.s.} \quad \text{for large } s_T,$$

where θ is the terminal node of an arbitrary continuation of γ up to generation number $[K s_T] + 1$, and the $D_i(\theta)$ denote the D_i along the corresponding branch. Therefore

$$\max_{\gamma \in \Gamma_{s_T}} \Sigma_{i=0}^{N(\gamma, s_T)} D_i(\gamma) \leq \max_{\theta \in H_T} \Sigma_{i=1}^{[K s_T]+1} D_i(\theta) \text{ a.s.} \quad \text{for large } s_T,$$

where H_T is the set of nodes (particles) of generation $[K s_T] + 1$. Now, by Theorem A [5],

$$([K s_T] + 1)^{-1} \max_{\theta \in H_T} \Sigma_{i=1}^{[K s_T]+1} D_i(\theta)$$

is *a.s.* bounded as $s_T \to \infty$. This yields the desired result.

Proof that $\phi_{2,T} \to 0$. Let

$$R(\gamma) = \Sigma_{\gamma' \in \Gamma_T^\gamma} f(a_T^{-1} X_T^{s_T}(\gamma')) - E^{s_T} \Sigma_{\gamma' \in \Gamma_T^\gamma} f(a_T^{-1} X_T^{s_T}(\gamma')), \qquad \gamma \in \Gamma_{s_T}.$$

Due to A1, it suffices to prove that

$$e^{-\alpha T} \Sigma_{\gamma \in \Gamma_{s_T}} R(\gamma) \to 0 \text{ a.s.}$$

Using Chebyshev's inequality, the fact that the $R(\gamma)$ are (conditionally) independent and $E^{s_T} R(\gamma) = 0$, and $|f| = 1$, we have, for $\epsilon > 0$,

$$P[|e^{-\alpha T} \Sigma_{\gamma \in \Gamma_{s_T}} R(\gamma)| > \epsilon] \leq \epsilon^{-2} e^{-2\alpha T} E E^{s_T} |\Sigma_{\gamma \in \Gamma_{s_T}} R(\gamma)|^2$$

$$= \epsilon^{-2} e^{-2\alpha T} E\, \Sigma_{\gamma \in \Gamma_{s_T}} E^{s_T} |R(\gamma)|^2$$

$$\leq \epsilon^{-2} e^{-2\alpha T} E\, \Sigma_{\gamma \in \Gamma_{s_T}} E^{s_T} |\Sigma_{\gamma' \in \Gamma_T^\gamma} f(a_T^{-1} X_T^{s_T}(\gamma'))|^2$$

$$\leq \epsilon^{-2} e^{-2\alpha T} E\, \Sigma_{\gamma \in \Gamma_{s_T}} E^{s_T} (Z_T^\gamma)^2,$$

where Z_T^γ denotes the cardinality of Γ_T^γ (i.e. the number of descendants of $\gamma \in \Gamma_{s_T}$ at time T). Conditioning upon the remaining lifetime of a particle at time s_T and using A2 it is easy to show that

$$E^{s_T}(Z_T^\gamma)^2 \leq K_1 e^{2\alpha(T - s_T)}$$

for all $\gamma \in \Gamma_{s_T}$, where K_1 is a constant independent of T, s_T. Hence, by A2 again, we obtain

(4.3) $$P[|e^{-\alpha T}\Sigma_{\gamma\in\Gamma_{s_T}} R(\gamma)| > \epsilon] \leqslant K_2\, e^{-\alpha s_T},$$

where K_2 is a constant.

Let $T_n \to \infty$ on a lattice $\{n\delta\}_n (\delta > 0)$ and choose s_{T_n} so that $\Sigma_n \exp(-\alpha s_{T_n}) < \infty$ (e.g. $s_{T_n} = 2\alpha^{-1} \log n$). Then by the Borel-Cantelli lemma and (4.3),

$$\phi_{2,T_n} \to 0 \; a.s. \qquad \text{as } n \to \infty.$$

Further arguments are needed to obtain the result for $T \to \infty$ continuously, but this will be taken care of at the end of the proof.

Proof that $\phi_{3,T} \to Ef(\tilde{L})$.

There are two cases to consider depending on whether the increments $\{X(u + S_i) - X(S_i), 0 \leqslant u \leqslant \tau_{i+1}\}$, $i = 0, 1, \cdots$, of the underlying transport process are independent or not. We will carry out the arguments for the non independent case (satisfying conditions (2.1)(c) and (2.4)) and indicate the modifications needed to handle the independent case.

Observe that

$$\begin{aligned}\phi_{3,T} &= Z_T^{-1}\, \Sigma_{\gamma\in\Gamma_{s_T}} E^{s_T}\, \Sigma_{\gamma'\in\Gamma_T^{\gamma}} f(a_T^{-1}X_T^{s_T}(\gamma')) \\ &= (Z_{s_T}Z_T^{-1})Z_{s_T}^{-1}\, \Sigma_{\gamma\in\Gamma_{s_T}} E^{s_T}\, \Sigma_{\gamma'\in\Gamma_T^{\gamma}} f(a_T^{-1}X_T^{s_T}(\gamma')) \\ &= (Z_{s_T}Z_T^{-1}) \textstyle\iint (E_{z,\theta}\, \Sigma_{\gamma'\in\Gamma_T^{\gamma_0}} f(a_T^{-1}X_T^{s_T}(\gamma')))\, A(dz, d\theta, s_T),\end{aligned}$$

where γ_0 is a fixed (arbitrary) branch in Γ_{s_T}, and under the conditional expectation $E_{z,\theta}$ the particle living on γ_0 at time s_T has age z and is traveling (at time s_T-) in the direction θ (θ is a unit vector in $\mathbb{R}^d$). $A(dz, d\theta, s_T)$ is the proportion of particles alive at time s_T with ages in $(z, z + dz)$ and directions in $(\theta, \theta + d\theta)$.

In the last expression we used the fact that for transport processes under consideration the conditional distribution of $X(t) - X(s)$, $t > s$, given the process up to time s depends only on the direction of the process at time $s-$.

In the independent case, θ is not a relevant quantity and so $A(dz, d\theta, s_T)$ becomes $A(dz, s_T)$, which represents the proportion of particles alive at time s_T with ages in $(z, z + dz)$.

Hence

$$\phi_{3,T} = (e^{-\alpha s_T}Z_{s_T})(e^{-\alpha T}Z_T)^{-1}\psi_T,$$

where

(4.4) $$\psi_T = e^{-\alpha(T-s_T)} \textstyle\iint (E_{z,\theta}\, \Sigma_{\gamma'\in\Gamma_T^{\gamma_0}})\, f(a_T^{-1}X_T^{s_T}(\gamma')))\, A(dz, d\theta, s_T),$$

and since $(e^{-\alpha s_T}Z_{s_T})(e^{-\alpha T}Z_T)^{-1} \to$ *a.s.*, by A1, we must show that

(4.5) $$\psi_T \to Ef(\tilde{L}) \; a.s. \qquad \text{as } T \to \infty.$$

We have

$$(4.6)\quad E_{z,\theta}\, \Sigma_{\gamma' \in \Gamma_T^{\gamma_0}} f(a_T^{-1} X_T^{s_T}(\gamma'))$$
$$= \int_0^{T-s_T} (E_\theta\, \Sigma_{\gamma' \in \Gamma_T^{\gamma(x)}} f(a_T^{-1} X_T^{s_T}(\gamma'))) G_z\,(dx) + o(1) \quad \text{as } T \to \infty,$$

where $\gamma(x)$ is a fixed branch in Γ_{s_T} such that the particle at s_T has remaining life x, and the $o(1)$ term accounts for the possibility that this particle does not die by time T.

In the following calculation, $\gamma_0(x)$ is a fixed branch in Γ_{s_T+x}, $N(\gamma, (s_T + x, T])$ denotes the number of splits on γ in the interval $(s_T + x, T]$, ξ_k is the number of kth generation descendants of $\gamma_0(x)$, and $\gamma_1, \cdots, \gamma_{\xi_k}$ are the corresponding branches.

$$E_\theta\, \Sigma_{\gamma' \in \Gamma_T^{\gamma(x)}} f(a_T^{-1} X_T^{s_T}(\gamma')) = m E_\theta\, \Sigma_{\gamma_T' \in \Gamma_T^{\gamma_0(x)}} f(a_T^{-1} X_T^{s_T}(\gamma'))$$
$$= m\, \Sigma_{k=0}^{\infty} E_\theta\, \Sigma^* f(a_T^{-1} X_T^{s_T}(\gamma'))$$

(where Σ^* denotes summation over γ' such that $\gamma' \in \Gamma_T{}^{\gamma_0(x)}$ and $N(\gamma', (s_T + x, T]) = k$)

$$= m\, \Sigma_{k=0}^{\infty} E_\theta\, \Sigma_{j=1}^{\xi_k} f(a_T^{-1} X_T^{s_T}(\gamma_j)) 1[N(\gamma_j, (s_T + x, T]) = k]$$
$$= m\, \Sigma_{k=0}^{\infty} E_\theta \xi_k E f(a_T^{-1} X_T^{s_T}(\gamma_1)) 1[N(\gamma_1, (s_T + x, T]) = k]$$
$$= m\, \Sigma_{k=0}^{\infty} m^k E_\theta f(a_T^{-1} X_T^{s_T}(\gamma_1)) 1[N(\gamma_1, (s_T + x, T]) = k];$$

in the last three lines we conditioned upon ξ_k, and used the facts that the terms with the different γ_j have the same distribution and $E\xi_k = m^k$.

We are now concerned with a single branch, $\gamma_0(x)$ followed by γ_1. We omit writing the branch but we write X' in place of X in order to remember the role of x in X. So,

$$E_\theta\, \Sigma_{\gamma' \in \Gamma_T^{\gamma(x)}} f(a_T^{-1} X_T^{s_T}(\gamma')) = \Sigma_{k=0}^{\infty} m^{k+1} E_\theta f(a_T^{-1} X_T'^{s_T}) 1[N((s_T + x, T]) = k].$$

To proceed with the calculation we denote τ_i' the waiting times *after* time $s_T + x$, and correspondingly S_i' the renewal times and N' the renewal function, and we bring in the factor $\exp\{-\alpha(T - s_T)\}$ from (4.4).

$$e^{-\alpha(T-s_T)} E_\theta\, \Sigma_{\gamma' \in \Gamma_T^{\gamma(x)}} f(a_T^{-1} X_T^{s_T}(\gamma'))$$
$$= e^{-\alpha(T-s_T)}\, \Sigma_{k=0}^{\infty} m^{k+1} E_\theta f(a_T^{-1} X_T'^{s_T}) 1[N'(T - s_T - x) = k]$$
$$= e^{-\alpha(T-s_T)}\, \Sigma_{k=0}^{\infty} m^{k+1} E_\theta f(a_T^{-1} X_T'^{s_T}) 1[S_k' \leqslant T - s_T - x < S_{k+1}']$$
$$= e^{-\alpha(T-s_T)}\, \Sigma_{k=0}^{\infty} m^{k+1} \int^* E_\theta[f(a_T^{-1} X_T'^{s_T})) \mid \tau_1' = t_1, \cdots, \tau_{k+1}' = t_{k+1}]$$
$$\cdot G(dt_1) \cdots G(dt_{k+1})$$

(where $\int^*$ denotes integration over $t_1, \cdots, t_{k+1}$ such that $t_1 + \cdots + t_k \leqslant T - s_T - x < t_1 + \cdots t_{k+1}$)

$$= \Sigma_{k=0}^{\infty} \int^* E_\theta[f(a_T^{-1}X_T'^{s_T})e^{\alpha\{t_1+\cdots+t_{k+1}-(T-s_T)\}} \mid \tau_1' = t_1, \cdots, \tau_{k+1}' = t_{k+1}]$$

$$\cdot me^{-\alpha t_1} G(dt_1) \cdots me^{-\alpha t_{k+1}} G(dt_{k+1})$$

$$= E_\theta f(a_T^{-1}X_T'^{s_T})\exp[\alpha\{\tilde{S}_{\tilde{N}'(T-s_T-x)+1}' - (T - s_T)\}],$$

because the distribution of the τ_i', which was G, has been replaced by $\tilde{G}$, defined by (2.5); the tilde over $\tilde{S}'$ and $\tilde{N}'$ refers to this fact and the prime has the same meaning as above; $\tilde{X}_T^{s_T}$ is defined by

$$\tilde{X}_T^{s_T}(t) = \begin{cases} 0, & 0 \leq t \leq \tilde{p}_T/T, \\ \tilde{X}_T(t) - \tilde{X}_T(\tilde{p}_T/T), & \tilde{p}_T/T \leq t \leq 1, \end{cases} \tag{4.7}$$

with

$$\tilde{p}_T = \tilde{S}_{\tilde{N}(s_T)+1} \wedge T,$$

where $\tilde{X}_T$ is given by (2.8), and the prime on $\tilde{X}_T'^{s_T}$ keeps reminding us that in the calculation the remaining life at time s_T is x.

Now we go back to (4.6) and change the distribution of the particle at time s_T, which has remaining life x from G to $\tilde{G}$ by means of A4.

$$\begin{aligned} &e^{-\alpha(T-s_T)} \int_0^{T-s_T} (E_\theta \Sigma_{\gamma' \in \Gamma_T^{\gamma(x)}} f(a_T^{-1}X_T^{s_T}(\gamma')))G_z(dx) \\ &= \int_0^{T-s_T} E_\theta(f(a_T^{-1}X_T'^{s_T}) \\ &\quad \cdot \exp[\alpha\{\tilde{S}_{\tilde{N}'(T-s_T-x)+1}' - (T - s_T)\}])m^{-1}e^{\alpha x}V(z)\tilde{G}_z(dx) \\ &= m^{-1}V(z) \int_0^{T-s_T} E_\theta(f(a_T^{-1}X_T^{s_T}) \\ &\quad \cdot \exp[\alpha\{\tilde{S}_{\tilde{N}'(T-s_T-x)+1}' - (T - s_T - x)\}])\tilde{G}_z(dx). \end{aligned} \tag{4.8}$$

From (4.4), (4.6), (4.8) and (4.11) below we have

$$\begin{aligned} \psi_T = m^{-1} \iint (&\textstyle\int_0^{T-s_T} E_\theta\{f(a_T^{-1}\tilde{X}_T'^{s_T}) \\ &\cdot \exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1}' - (T - s_T - x))]\}\tilde{G}_z(dx)) \\ &\cdot V(z)\, A(dz, d\theta, s_T) + o(1) \qquad \text{as } T \to \infty. \end{aligned} \tag{4.9}$$

Let, for $B > 0$,

$$\begin{aligned} \psi_T^B = m^{-1} \iint (&\textstyle\int_0^{(T-s_T)\wedge b} E_\theta\{f(a_T^{-1}\tilde{X}_T'^{s_T}) \\ &\cdot \exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1}' - (T - s_T - x))]\}\tilde{G}_z(dx)) \\ &\cdot V(z)\, (dz, d\theta, s_T) \end{aligned} \tag{4.10}$$

Then, since $E_\theta\{\ \}$ is bounded (see B2) and (see A3)

$$\int_0^\infty \int_0^\infty \tilde{G}_z(dx)V(z)\, A(dz) < \infty \ a.s., \tag{4.11}$$

it follows from (4.9) and (4.10) that (4.5) will be proved if we show that

$$\lim_{B\to\infty} \lim_{T\to\infty} \psi_T^B = Ef(\tilde{L}) \ a.s. \tag{4.12}$$

Let $q_T(x)$ be such that $s_T + x < q_T(x) < T$,

$$\tilde{q}_T(x) = (s_T + x + \tilde{S}'_{\tilde{N}'(q_T(x)-s_T-x)+1'}) \wedge T,$$

and

$$\hat{X}_T(t) = \begin{cases} \tilde{X}_T{}'^{s_T}(t), & 0 \leqslant t \leqslant \tilde{q}_T(x)/T, \\ \tilde{X}_T{}'^{s_T}(\tilde{q}_T(x)/T), & \tilde{q}_T(x)/T \leqslant t \leqslant 1 \end{cases} \tag{4.13}$$

Replacing $\tilde{X}_T{}'^{s_T}$ by $\hat{X}_T$ in (4.10), we define

(4.14)

$$\hat{\psi}_T{}^B = m^{-1} \int\int\left(\int_0^{(T-s_T)\wedge B} E_\theta\{f(a_T{}^{-1}\hat{X}_T\right) \exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1'}$$
$$- (T - s_T - x))]\} \tilde{G}_z(dx)) V(z)A(dz, d\theta, s_T).$$

We will show that with an appropriate choice of $q_T(x)$,

$$\psi_T{}^B - \hat{\psi}_T{}^B \to 0 \ a.s. \text{ as } T \to \infty \tag{4.15}$$

and

$$\lim_{B\to\infty}\lim_{T\to\infty} \hat{\psi}_T{}^B = Ef(\tilde{L}) \ a.s., \tag{4.16}$$

which proves (4.12).

We now prove (4.15). First, using (4.1) we have

$$|\psi_T{}^B - \hat{\psi}_T{}^B| \leqslant m^{-1} \int\int\int_0^{(T-s_T)\wedge B} E_\theta|\{f(a_T{}^{-1}\tilde{X}_T{}'^{s_T}) - f(a_T{}^{-1}\hat{X}_T)|\exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1'}$$
$$- (T - s_T - x))]\} \tilde{G}_z(dx) V(z)A(dz, d\theta, s_T)$$
$$\leqslant m^{-1}Ma_T{}^{-1} \int\int\int_0^{(T-s_T)\wedge B} E_\theta\{\sup_{\tilde{q}_T(x)\leqslant t\leqslant T}\|\tilde{X}(t) - \tilde{X}(\tilde{q}_T(x))\| \exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1'}$$
$$- (T - s_T - x))]\} \tilde{G}_z(dx) V(z)A(dz, d\theta, s_T).$$

Now, denoting $e_T(x) = \exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1'} - (T - s_T - x))]$ and using condition (2.1) (c) for $\tilde{X}$,

$$E_\theta\{\sup_{\tilde{q}_T(x)\leqslant t\leqslant T}\|\tilde{X}(t) - \tilde{X}(\tilde{q}_T(x))\| e_T(x)\} \leqslant K(T - q_T(x))Ee_T(x)$$

for large $T - q_T(x)$. Note that the dependence on θ is no longer present. It follows from B2 that

$$\lim_{T\to\infty}\sup_{x\leqslant B}Ee_T(x) = N < \infty.$$

Thus

$$|\psi_T{}^B - \hat{\psi}_T{}^B| \leqslant m^{-1}MKNa_T{}^{-1} \int_0^{(T-s_T)\wedge B} (T - q_T(x)) \int_0^\infty \tilde{G}_z(dx) V(z)A(dz, s_T),$$

and since $x \leqslant B$ and $T - s_T \to \infty$ we can choose $q_T = q_T(x)$ independently of

x so that $s_T + x < q_T < T$ for all $x \leqslant B$. So, using A3 and (4.11),

$$|\psi_T{}^B - \hat{\psi}_T{}^B| \leqslant m^{-1}MKN \int_0^{(T-s_T)\wedge B} \int_0^\infty \tilde{G}_z(dx)V(z)A(dz, s_T)a_T{}^{-1}(T - q_T)$$
$$\leqslant Qa_T{}^{-1}(T - q_T),$$

where $Q > 0$ is a constant. Taking q_T such that $T - q_T \to \infty$ and $T - q_T = o(a_T)$ as $T \to \infty$ obtains (4.15).

In the independent case the preceding argument to prove (4.15) is somewhat more involved because (2.1)(c) cannot be used. We will give the details using (2.1)(b), which is satisfied by the random walk (the case (2.1)(a) is easier). Note that θ does not appear here.

$$E\{\sup_{\tilde{q}_T(x)\leqslant t\leqslant T}\|\tilde{X}(t) - \tilde{X}(\tilde{q}_T(x))\|e_T(x)\}$$
$$= E\{E[\sup_{\tilde{q}(x)\leqslant t\leqslant T}\|\tilde{X}(t) - \tilde{X}(\tilde{q}_T(x))\|\,|\,\text{renewal times}]e_T(x)\}$$
$$\leqslant KE\{[\tilde{N}'(T - s_T - x) - \tilde{N}'(\tilde{q}_T(x) - s_T - x)]e_T(x)\}$$
$$\leqslant KE\{[\tilde{N}'(T - s_T - x) - \tilde{N}'(\tilde{q}_T(x) - s_T - x)] \exp(\alpha\tilde{\tau}_{\tilde{N}'(T-s_T-x)+1}{}')\}.$$

By standard arguments one can show that

$$E[(\tilde{N}'(T - s_T - x) - \tilde{N}'(\tilde{q}_T(x) - s_T - x))$$
$$\cdot\exp(\alpha\tilde{\tau}_{\tilde{N}'(T-s_T-x)+1}{}')\,|\,\tilde{N}'(\tilde{q}_T(x) - s_T - x)]$$
$$= \Sigma^*(k - \tilde{N}'(\tilde{q}_T(x) - s_T - x))E[\mathbf{1}[\tilde{S}_k{}' \leqslant T - s_T - x]$$
$$\cdot E[\mathbf{1}[\tilde{\tau}_{\tilde{N}'(T-s_T-x)+1}{}' > T - s_T - x - \tilde{S}_k{}']$$
$$\cdot\exp(\alpha\tilde{\tau}_{\tilde{N}'(T-s_T-x)+1}{}')\,|\,\tilde{S}_k{}', \tilde{N}'(\tilde{q}_T(x) - s_T - x)]\,|\,\tilde{N}'(\tilde{q}_T(x) - s_T - x)],$$

(where Σ^* denotes summation over k such that $k \geqslant \tilde{N}'(\tilde{q}_T(x) - s_T - x)$), and, recalling (2.5),

$$E[\mathbf{1}[\tilde{\tau}_{\tilde{N}'(T-s_T-x)+1}{}' > T - s_T - x - \tilde{S}_k{}']$$
$$\cdot\exp(\alpha\tilde{\tau}_{\tilde{N}'(T-s_T-x)+1}{}')\,|\,\tilde{S}_k{}', \tilde{N}'(\tilde{q}_T(x) - s_T - x)]$$
$$= \int_{T-s_T-x-\tilde{S}_k{}'}^\infty e^{\alpha u}\tilde{G}(du) = m\int_{T-s_T-x-\tilde{S}_k{}'}^\infty G(du) \leqslant m.$$

Hence

$$E\{\sup_{\tilde{q}_T(x)\leqslant t\leqslant T}\|\tilde{X}(t) - \tilde{X}(\tilde{q}_T(x))\|e_T(x)\}$$
$$\leqslant KmE\,\Sigma^*(k - \tilde{N}'(\tilde{q}_T(x) - s_T - x))P[\tilde{N}'(T - s_T - x)$$
$$\geqslant k\,|\,\tilde{N}'(\tilde{q}_T(x) - s_T - x)]$$
$$= KmE(\tilde{N}'(T - s_T - x) - \tilde{N}'(\tilde{q}_T(x) - s_T - x))^2,$$

and conditioning upon $\tilde{q}_T(x)$,

$$\leq H\,E(T - s_T - x - (\tilde{q}_T(x) - s_T - x))^2$$
$$= H\,E(T - \tilde{q}_T(x))^2 \leq H(T - q_T(x))^2$$

for large $T - q_T(x)$, where $H > 0$ is a constant. Therefore

$$|\psi_T{}^B - \hat{\psi}_T{}^B| \leq m^{-1}MHa_T{}^{-1} \int_0^{(T-s_T)\wedge B} (T - q_T(x))^2 \int_0^\infty \tilde{G}_z(dx)V(z)A(dz, s_T),$$

and as before we can choose $q_T = q_T(x)$ independently of x so that $s_T + x < q_T < T$ for all $x \leq B$. So

$$|\psi_T{}^B - \hat{\psi}_T{}^B| \leq m^{-1}MH \int_0^{(T-s_T)\wedge B} \int_0^\infty \tilde{G}_z(dx)V(z)A(dz, s_T)a_T{}^{-1}(T - q_T)^2$$
$$\leq Qa_T{}^{-1}(T - q_T)^2,$$

where $Q > 0$ is a constant. Taking q_T such that $T - q_T \to \infty$ and $(T - q_T)^2 = o(a_T)$ as $T \to \infty$ obtains (4.15).

Now (4.16). We will show first that for all $x \leq B$,

(4.17) $$E_\theta\{f(a_T{}^{-1}\hat{X}_T) \exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1}{}' - (T - s_T - x))]\}$$
$$\to Ef(\tilde{L})(m - 1)(\tilde{\mu}\alpha)^{-1} \text{ uniformly in } \theta \text{ as } T \to \infty.$$

$\hat{X}_T$ and $s_T + x + \tilde{S}_{\tilde{N}'(T-s_T-x)+1}{}' - \tilde{q}_T(x)$ are independent conditional upon $\tilde{q}_T(x)$, provided that $\tilde{q}_T(x) < T$. Since $T - q_T \to \infty$ implies that $T - \tilde{q}_T(x) \to \infty$ *a.s.* for all $x \leq B$, then

$$E_\theta\{f(a_T{}^{-1}\hat{X}_T) \exp[\alpha(\tilde{S}_{\tilde{N}'(T-s_T-x)+1}{}' - (T - s_T - x))]\}$$
$$= E_\theta\{\mathbf{1}[\tilde{q}_T(x) < T]E_\theta[f(a_T{}^{-1}\hat{X}) \mid \tilde{q}_T(x)]E[\exp(\alpha(s_T + x + \tilde{S}_{\tilde{N}'(T-s_T-x)+1}{}' - \tilde{q}_T(x)$$
$$- (T - \tilde{q}_T(x)))) \mid \tilde{q}_T(x)]\} + o(1) \text{ as } T \to \infty;$$

moreover, $T - q_T(x) \to \infty$ a.s. implies, by B2, that

$$E[\exp(\alpha(s_T + x + \tilde{S}_{\tilde{N}'(T-s_T-x)+1}{}' - \tilde{q}_T(x) - (T - \tilde{q}_T(x)))) \mid \tilde{q}_T(x)]$$
$$\to (m - 1)(\tilde{\mu}\alpha)^{-1} \text{ a.s. as } T \to \infty.$$

Then it is easy to see that (4.17) follows if we show that

(4.18) $$E_\theta f(a_T{}^{-1}\hat{X}_T) \to Ef(\tilde{L}) \text{ uniformly in } \theta \text{ as } T \to \infty,$$

which we will do by proving that

(4.19) $$E_\theta \sup_{0\leq t\leq 1} \| a_T{}^{-1}\hat{X}_T(x) - a_T{}^{-1}\tilde{X}_T(t) \| \to 0 \text{ uniformly in } \theta \text{ as } T \to \infty,$$

for this, with (2.9) and condition (2.4) for $\tilde{X}$ can be shown to imply (4.18).

From (2.8), (4.7) and (4.13) we have, denoting $t_T = q_T(p_T - s_T)(= q_T(x))$,

$$\sup_{0\leq t\leq 1} \| a_T{}^{-1}\hat{X}_T(t) - a_T{}^{-1}\tilde{X}_T(t) \|$$
$$= a_T{}^{-1} \sup_{0\leq t\leq T} \| \hat{X}(t) - \tilde{X}(t) \|$$

$$\leq a_T^{-1}(\sup_{0\leq t\leq \tilde{p}_T} \|\tilde{X}(t)\| + \sup_{t_T\leq t\leq T}\|\tilde{X}(t) - \tilde{X}^{s_T}(t_T)\|)$$

$$\leq a_T^{-1}(2\sup_{0\leq t\leq \tilde{p}_T} \|\tilde{X}(t)\| + \sup_{t_T\leq t\leq T}\|\tilde{X}(t) - \tilde{X}(t_T)\|),$$

and the result follows using condition (2.1)(*c*)for $\tilde{X}$ similarly as below.

Under condition (2.1)(*b*) we have

$$E \sup_{0\leq t\leq 1} \| a_T^{-1}\hat{X}_T(t) - a_T^{-1}\tilde{X}_T(t)\|$$

$$\leq a_T^{-1}K[2(E(\tilde{N}(s_T) + 1) + E(\tilde{N}(T) - \tilde{N}(q_T))]$$

$$\leq H(a_T^{-1}s_T + a_T^{-1}(T - q_T)) + o(1) \text{ as } T\to\infty,$$

where $H > 0$ is a constant, and then (4.19) follows because $s_T = o(a_T)$ and $T - q_T = o(a_T)$. Finally, from (4.14), (4.17) and A3 we have

$$\lim_{T\to\infty} \hat{\psi}_T^B = m^{-1}Ef(\tilde{L})(m-1)(\tilde{\mu}\alpha)^{-1}\int_0^\infty\int_0^B \tilde{G}_z(dx)V(z)A(dz),$$

and

$$\int_0^\infty\int_0^\infty \tilde{G}_z(dx)V(z)A(dz) = m\tilde{\mu}\alpha(m-1)^{-1},$$

and therefore (4.16) is proved.

This finishes the proof that $\phi_{3,T}\to Ef(\tilde{L})$ a.s.

We have now proved the theorem in the special case when $T\to\infty$ on lattices $\{n\delta\}$, i.e.

(4.20) $\qquad Z_{n\delta}^{-1}\sum_{\gamma\in\Gamma_{n\delta}}\mathbf{1}[a_{n\delta}^{-1}X_{n\delta}(\gamma)\in A]\to P[\tilde{L}\in A]$ *a.s.* as $n\to\infty$,

for all $\delta > 0$, where A is any $\tilde{L}$-continuous Borel set of $D[0, 1]^d$.

The final step is to pass from the lattice to the continuum, i.e. to prove that (4.20) implies

(4.21) $\quad Z_T^{-1}\sum_{\gamma\in\Gamma_T}\mathbf{1}[a_T^{-1}X_T(\gamma)\in A]\to P[\tilde{L}\in A]\quad$ *a.s.* $\quad$ as $\quad T\to\infty$.

For $T > 0$, and $\delta > 0$ small, let the integer n be such that $(n-1)\delta\leq T < n\delta$, and set $\delta_T = n\delta$. Then $T/\delta_T\to 1$ as $T\to\infty$, and therefore $a_T/a_{\delta_T}\to 1$ as $T\to\infty$, due to the form of a_T (2.3). Thus we can replace a_T by a_{δ_T} in (4.21). Hence, to prove (4.21) it is sufficient to show

(4.22) $\qquad \limsup_{T\to\infty} Z_T^{-1}\sum_{\gamma\in\Gamma_T}\mathbf{1}[a_{\delta_T}^{-1}X_T(\gamma)\in F]\leq P[\tilde{L}\in F]\quad$ *a.s.*

for closed sets F in $D[0, 1]^d$.

Without loss of generality we can assume that the sum in (4.22) is over the branches in Γ_T which do not split before time δ_T. This follows because

$$\limsup_{T\to\infty} Z_T^{-1}\sum_{\gamma\in\Gamma_T}\mathbf{1}[\text{particle on } \gamma \text{ dies in next } \delta \text{ time unit})$$

is a.s. arbitrarily small as $\delta\to 0$, as can be shown using the arguments of

Athreya and Kaplan [1] (p. 47). Therefore

$$(4.23)\quad \begin{aligned} Z_T^{-1}\,\Sigma_{\gamma\in\Gamma_T}\,\mathbf{1}[a_{\delta_T}^{-1}X_T(\gamma)\in F] &= Z_T^{-1}\,\Sigma_{\gamma\in\Gamma_{\delta_T}}\,\mathbf{1}[a_{\delta_T}^{-1}X_T(\gamma)\in F] \\ &= C_T(\delta)Z_{\delta_T}^{-1}\,\Sigma_{\gamma\in\Gamma_{\delta_T}}\,\mathbf{1}[a_{\delta_T}^{-1}X_T(\gamma)\in F], \end{aligned}$$

where

$$(4.24)\quad C_T(\delta)\leqslant e^{\alpha\delta}(Z_{\delta_T}e^{-\alpha\delta_T})/(Z_Te^{-\alpha T}).$$

Let $\epsilon>0$, and $F_\epsilon=F+S_\epsilon$, where S_ϵ is an open Skorohod ball of center 0 and radius ϵ. Then

$$(4.25)\quad \begin{aligned} Z_{\delta_T}^{-1}\,\Sigma_{\gamma\in\Gamma_{\delta_T}}\,\mathbf{1}[a_{\delta_T}^{-1}X_T(\gamma)\in F] &\leqslant Z_{\delta_T}^{-1}\,\Sigma_{\gamma\in\Gamma_{\delta_T}}\,\mathbf{1}[a_{\delta_T}^{-1}X_{\delta_T}(\gamma)\in F_\epsilon] \\ &\quad + Z_{\delta_T}^{-1}\,\Sigma_{\gamma\in\Gamma_{\delta_T}}\,\mathbf{1}[\rho(a_{\delta_T}^{-1}X_T(\gamma),\,a_{\delta_T}^{-1}X_{\delta_T}(\gamma))\geqslant\epsilon], \end{aligned}$$

where ρ denotes the Skorohod metric

$$(4.26)\quad \begin{aligned} &\rho(x,y)=\max_{1\leqslant i\leqslant d}\inf_{\lambda\in\Lambda}\{\sup_{0\leqslant t\leqslant 1}|x^i(t)-y^i(\lambda(t))|+\sup_{0\leqslant t\leqslant 1}|t-\lambda(t)|\}, \\ &\qquad x=(x^1,\cdots,x^d)\quad\text{and}\quad y=(y^1,\cdots,y^d)\quad\text{in } D[0,1]^d, \end{aligned}$$

with Λ being the set of all strictly increasing and continuous mappings of [0, 1] onto itself.

If we show that

$$(4.27)\quad Z_{\delta_T}^{-1}\,\Sigma_{\gamma\in\Gamma_{\delta_T}}\,\mathbf{1}[\rho(a_{\delta_T}^{-1}X_T(\gamma),\,a_{\delta_T}^{-1}X_{\delta_T}(\gamma))\geqslant\epsilon]\to 0\quad a.s.\qquad\text{as } T\to\infty,$$

then it will follow from (4.23), (4.24), (4.25), A1 and the invariance theorem for the lattice (4.20), that

$$\limsup_{T\to\infty} Z_T^{-1}\,\Sigma_{\gamma\in\Gamma_T}\,\mathbf{1}[a_{\delta_T}^{-1}X_T(\gamma)\in F]\leqslant e^{\alpha\delta}P[\tilde{L}\in F_\epsilon]\quad a.s.,$$

where ϵ is taken so that F_ϵ is $\tilde{L}$-continuous, whence (4.22) results by letting $\delta\to 0$ and $\epsilon\to 0$.

We now prove (4.27).

Setting $d=1$ and deleting γ to simplify notation, we have from (4.26),

$$\rho(a_{\delta_T}^{-1}X_T,\,a_{\delta_T}^{-1}X_{\delta_T})\leqslant a_{\delta_T}^{-1}\sup_{0\leqslant t\leqslant 1}|X_T(\lambda_T(t))-X_{\delta_T}(t)|+\sup_{0\leqslant t\leqslant 1}|t-\lambda_T(t)|,$$

where

$$\lambda_T(t)=\begin{cases}\delta_T T^{-1}t, & 0\leqslant t\leqslant r_T,\\ \text{linear in } r_T\leqslant t\leqslant 1,\end{cases}$$

with $0<r_T<T\delta_T^{-1}<1$. Hence

$$\sup_{0\leqslant t\leqslant 1}|t-\lambda_T(t)|=(\delta_T T^{-1}-1)r_T\leqslant(\delta_T(\delta_T-\delta)^{-1}-1)r_T,$$

and

$$\begin{aligned}\sup_{0\le t\le 1}|X_T(\lambda_T(t)) - X_{\delta_T}(t)| &= \sup_{r_T\le t\le 1}|X_T(\lambda(t)) - X_{\delta_T}(t)|\\ &\le \sup_{r_T\le t\le 1}|X_T(\lambda_T(t)) - X_T(\lambda_T(r_T))|\\ &\quad + \sup_{r_T\le t\le 1}|X_{\delta_T}(t) - X_{\delta_T}(r_T)|\\ &\le 2\sup_{\delta_T r_T\le t\le \delta_T}|X(t) - X(\delta_T r_T)|\\ &= 2\sup_{0\le s\le \delta_T(1-r_T)}|X(\delta_T r_T + s) - X(\delta_T r_T)|.\end{aligned}$$

Choosing $r_T = 1 - s_{\delta_T}/\delta_T$ (remember s_T?), which is clearly possible, we have, for each $\gamma \in \Gamma_{\delta_T}$,

$$\rho(a_{\delta_T}{}^{-1}X_T(\gamma),\, a_{\delta_T}{}^{-1}X_{\delta_T}(\gamma)) \le \nu_{\delta_T}(\gamma) + (\delta_T(\delta_T - \delta)^{-1} - 1)r_T,$$

where

$$\nu_{\delta_T}(\gamma) = a_{\delta_T}{}^{-1}2\sup_{0\le s\le s_{\delta_T}}|X(\gamma, \delta_T r_T + s) - X(\gamma, \delta_T r_T)|.$$

Since $(\delta_T(\delta_T - \delta)^{-1} - 1)r_T \to 0$, (4.27) will be proved if we show that

$$Z_{\delta_T}{}^{-1}\,\Sigma_{\gamma\in\Gamma_{\delta_T}} \mathbf{1}[\nu_{\delta_T}(\gamma) \ge \epsilon] \to 0 \quad a.s. \qquad \text{as} \quad T\to\infty. \tag{4.28}$$

By any of the conditions (2.1) and $s_T = o(a_T)$ we have that $\nu_{\delta_T}(\gamma) \to 0$ in probability as $T\to\infty$ for each γ. The random functions defined by

$$Y_{\delta_T}(\gamma, t) = \begin{cases} 0, & 0\le t < r_T,\\ 2\sup_{0\le s\le t-r_T}|X_{\delta_T}(\gamma, r_T + s) - X_{\delta_T}(\gamma, r_T)|, & r_T \le t \le 1,\end{cases}$$

for $\gamma \in \Gamma_{\delta_T}$, satisfy the conditions of the invariance principle by construction, and $a_{\delta_T}{}^{-1}Y_{\delta_T}(\gamma)$ converges in distribution to the function 0 for all γ because $\sup_{0\le t\le 1} a_{\delta_T}{}^{-1}|Y_{\delta_T}(\gamma, t)| = \nu_{\delta_T}(\gamma)$. Since the $Y_{\delta_T}(\gamma)$ are indexed on the lattice $\{n\delta\}$, the invariance principle in the lattice case (4.20) and the functional $x \to x(1)$, $x \in D[0, 1]$, can be applied to obtain (4.28) because $\nu_{\delta_T}(\gamma) = a_{\delta_T}{}^{-1}Y_{\delta_T}(\gamma, 1)$.

The theorem is proved.

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