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THE STRONG LAW OF LARGE NUMBERS IN LOCALLY CONVEX SUSLIN SPACES

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Introduction

G.E.F. Thomas raised the question, whether the strong law of large numbers is valid for i.i.d. totally summable sequences $\{X_n\}_{n=1}^{\infty}$ of random variables with values in a quasicomplete locally convex Suslin-space E. We show by a simple truncation-argument that the answer is yes and—using an idea of Chatterji—that for Saksspaces E the assumption of total summability is also necessary for the strong law of large numbers to hold. A final example shows that the assumption that E is Suslin, is essential. The theorem generalizes the result of Mourier [7], Padgett and Taylor [8] who considered the case of separable Banach and Fréchet spaces.

1. Denote by (Ω, Σ, μ) an (abstract) probability space, by (E, τ) a vector space E with a locally convex Hausdorff topology τ. Throughout this paper—except in the example 10)—we shall assume that E is a Suslin space. The letter X denotes a random variable defined on Ω with values in E (measurable with respect to the Borel-σ-algebra of E).

2. *Definition* (c.f. [6], §5): X is called *totally summable*, if there is a closed, absolutely convex, bounded subset B of E, such that if $\|\cdot\|_B$ denotes the gauge-function of B,

$$\int_\Omega \| X(\omega) \|_B \, d\mu(\omega) < \infty.$$

If E is assumed to be quasicomplete, then one may define for $A \in \Sigma$ the Pettis-integral

$$\int_A X(\omega) \, d\mu(\omega),$$

which is an element of E([6]).

3. Let $\{X_n\}_{n=1}$ be a sequence of independent identically distributed (i.i.d.) E-valued random-variables and denote by S_n the partial sums $X_1 + \cdots + X_n$.

4. LEMMA. *Assume there is a compact, convex, metrisable subset K of E such that X_1 takes its values almost surely in K. Then the strong law of large numbers holds,* i.e.

$$\lim_{n\to\infty} n^{-1} S_n(\omega) = E(X_1) \qquad \mu - \text{a.s.}$$

Proof. K being metrisable, there is a sequence $\{f_k\}_{k=1}^{\infty}$ in E' which induces the τ-topology on K(c.f. [2] for example). For each of the sequences of i.i.d. realvalued, bounded random variables $\{f_k \circ X_n\}_{n=1}^{\infty}$ we may apply the strong law of large numbers, i.e. for each $k \in \mathbf{N}$

$$\lim_{n\to\infty} n^{-1} f_k \circ S_n(\omega) = E(f_k \circ X_1) \qquad \mu - \text{a.s.}$$

Whence, as $n^{-1}S_n$ lies almost surely in the convex set K,

$$\lim_{n\to\infty} n^{-1} S_n(\omega) = E(X_1) \qquad \mu - \text{a.s.},$$

where $E(X_1)$ denotes the expectation of X_1, which is an element of K. q.e.d.

5. THEOREM. *Let $\{X_n\}_{n=1}^{\infty}$ be an* i.i.d. *sequence of totally summable random variables with values in a quasicomplete locally convex Suslin space E. Then the strong law of large numbers holds*, i.e.

$$\lim_{n\to\infty} n^{-1}S_n(\omega) = E(X_1) \qquad \mu - \text{a.s.}$$

Proof. Replacing if necessary Ω by $E^{\mathbf{N}}$ and μ by its image under the map $\omega \to \{X_n(\omega)\}_{n\in\mathbf{N}}$, we may assume without loss of generality that the underlying probability space is a product space $(\Omega^{\mathbf{N}}, \Sigma^{\mathbf{N}}, \mu^{\mathbf{N}})$ and there is one random-variable $X:\Omega \to E$ such that $X_n = X \circ p_n$, p_n denoting the projection onto the n-th coordinate in $\Omega^{\mathbf{N}}$.

Let B be a closed, absolutely convex, bounded subset of E such that

$$\int \| X(\omega) \|_B \, d\mu(\omega) < \infty.$$

Fix $\epsilon > 0$. There is $\delta > 0$ such that $A \subseteq \Omega$, $\mu(A) < \delta$ implies $\int_A \| X(\omega) \|_B \, d\mu(\omega) \leqq \epsilon$. E being a Suslin space, every probability measure on the Borel sets of E is tight ([4], p. 122 th. 10). Applying this to the image of μ under X we can find a compact set K_1 in E such that $\mu\{\omega : X(\omega) \in K_1\} > 1 - \delta$. By the quasicompleteness of E, the closed convex hull K of $K_1 \cup \{0\}$ is still compact and—using again the fact that E is Suslin-metrisable (c.f. [4], p. 106, cor. 2).

Let $\Omega^1 = \{\omega \in \Omega : X(\omega) \in K\}$ and $\Omega^2 = \Omega \setminus \Omega^1$. Let $X^1 = X \cdot \chi_{\Omega^1}$ and $X^2 = X \cdot \chi_{\Omega^2}$. Note that $\{X^1 \circ p_n\}_{n=1}^{\infty}$ and $\{X^2 \circ p_n\}_{n=1}^{\infty}$ are both sequences of i.i.d. random variables, the former satisfying the hypothesis of the preceding lemma. Whence

$$\lim_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^{n} X^1 \circ p_i((\omega_m)_{m=1}^{\infty}) = E(X^1)$$

for $\mu^{\mathbf{N}}$ – almost all $(\omega_m)_{m=1}^{\infty}$ in $\Omega^{\mathbf{N}}$. Note that $\| E(X) - E(X^1) \|_B = \| E(X^2) \|_B \leqq E(\| X^2 \|_B) \leqq \epsilon$. Of the remaining part $\{X^2 \circ p_n\}_{n=1}^{\infty}$ the following estimate takes care:

$$\begin{aligned} &\limsup_{n\to\infty} \| n^{-1} \textstyle\sum_{i=1}^{n} X^2 \circ p_i((\omega_m)_{m=1}^{\infty}) \|_B \\ &\leqq \limsup_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^{n} \| X^2 \circ p_i((\omega_m)_{m=1}) \| \qquad \mu - \text{a.s.} \\ &\leqq \epsilon. \end{aligned}$$

The last inequality holds almost surely, as $\{ \| X^2 \circ p_n \|_B \}_{n=1}^{\infty}$ is a sequence of positive i.i.d. random variables. By the scalar strong law of large numbers we know that their means converge a.s. to $E(\| X^2 \|_B)$ which is less than ϵ.

Noting that

$$n^{-1} \textstyle\sum_{i=1}^{n} X \circ p_n = n^{-1} \sum_{i=1}^{n} X^1 \circ p_n + n^{-1} \sum_{i=1}^{n} X^2 \circ p_n$$

we see that for each $\epsilon > 0$ the sequence $\{n^{-1}S_n\}_{n=1}^{\infty}$ may almost surely be written as a sum of a sequence, τ-converging to a value which is ϵ-close to $E(X)$ in the $\|\cdot\|_B$-gauge and a sequence with the lim sup of the $\|\cdot\|_B$-gauge bounded by ϵ. Letting $\epsilon = k^{-1}$, $k = 1, 2, \cdots$, one concludes that on a set of measure 1

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} X \circ p_i((\omega_m)_{m=1}{}^{\infty}) = \lim_{n\to\infty} n^{-1} S_n((\omega_m)_{m=1}{}^{\infty}) = E(X). \quad \text{q.e.d.}$$

6. If on a locally convex space E there is one closed bounded absolutely convex set such that its scalar multiples form a fundamental system for the bounded sets in E, E is called a Saks space. For definitions and notations we refer to [2].

The following result was proved by Chatterji for the case of Banach-valued Pettis-integrable functions ([1]). His argument carries over to the following more general case, establishing a converse to proposition 5 for the case of Saks-space.

7. PROPOSITION. *Let $(E, \|\cdot\|, \tau)$ be a Saks-space and $\{X_n\}_{n=1}{}^{\infty}$ an* i.i.d. *sequence of τ-measurable E-valued random variables. If*

$$\lim_{n\to\infty} n^{-1}S_n(\omega)$$

converges almost surely with respect to the mixed topology $\gamma(\|\cdot\|, \tau)$, then

$$\int_\Omega \|X_1\| \, d\mu < \infty,$$

i.e., *X_1 is totally summable.*

Proof: If $\{n^{-1}S_n(\omega)\}_{n=1}{}^{\infty}$ γ-converges, it is γ-bounded and therefore norm-bounded ([2]). As

$$\begin{aligned} n^{-1}X_n &= n^{-1}(S_n - S_{n-1}) \\ &= n^{-1}S_n - (1 - 1/n)\cdot(n-1)^{-1}S_{n-1}, \end{aligned}$$

we infer that $\{n^{-1}X_n(\omega)\}_{n=1}{}^{\infty}$ is almost surely bounded. Hence there is $M > 0$ such that

$$\mu\{\omega : \limsup \|n^{-1}X_n(\omega)\| \leqq M\} > 0.$$

By Kolmogoroff's 0-1-law the probability of the above event is actually 1.

The Borel-Cantelli-lemma implies that

$$\sum_{n=1}^{\infty} \mu\{\|X_n\| \leqq n\cdot M\} < \infty.$$

As the sequence $\{X_n\}_{n=1}{}^{\infty}$ is identically distributed

$$\sum_{n=1}^{\infty} \mu\{\|X_1\| \leqq n\cdot M\} < \infty$$

or equivalently

$$\int_\Omega \|X_1(\omega)\| \, d\mu(\omega) < \infty. \quad \text{q.e.d.}$$

8. COROLLARY. *Let F be a separable Banach space and let $E = F'$ with $\tau = \sigma(F', F)$. (Then E is quasicomplete and Suslin.) If $(X_n)_{n\geqq 1}$ is a sequence of* i.i.d. *Pettis summable random variables with values in* (E, τ) *and* $\frac{1}{n}S_n$ *converges almost surely in* (E, τ) *the* X_n *are totally summable,* i.e. $\int \| X_n \| \, d\mu < +\infty$.

Proof. By the theorem of Banach-Steinhaus $\{\frac{1}{n}S_n(\omega)\}$ is bounded almost surely, hence converges a.s. with respect to γ([2], p. 9, proposition 1–10).

9. In particular we can construct the following: *Example.* Of a case where E is Suslin, quasi complete, but where for Pettis summable i.i.d. random variables the strong law of large numbers fails.

It suffices to take $E = l^2$, τ the weak topology, $\Omega = [0, 1]^{\mathbf{N}}$ μ product Lebesgue measure, and $X_n(\omega) = X(\omega_n)$ where $X: [0, 1] \to l^2$ is Pettis integrable but not Bochner integrable (e.g. if $[0, 1] = \sum_{n=1}^{\infty} A_n$ with $| A_n | = c/n^2)X(t) = ne_n$ for $t \in A_n$, $(e_n)_{n \geqq 1}$ being the canonical basis of l^2).

10. *Example.* We now give an example of a locally convex space E that fails to be Suslin and an i.i.d. sequence $\{X_n\}_{n=1}^{\infty}$ of Borel-measurable, uniformly bounded Pettis-integrable E-valued random variables such that

$$\lim_{n\to\infty} n^{-1} S_n(\omega)$$

does not exist almost surely.

Denote by $[0, \omega_1]$ (resp. $[0, \omega_1[$) the compact (resp. locally compact) space of ordinals less than or equal to (resp. less than) ω_1, the first uncountable one. Let $C([0, \omega_1])$ be the Banach space of continuous functions on $[0, \omega_1]$ and $(M([0, \omega_1]), \sigma^*)$ the dual space, the Radon-measures on $[0, \omega_1]$, equipped with the weak*-topology.

Let $(\Omega, \Sigma, \mu) = ([0, \omega_1[^{\mathbf{N}}, \text{Borel}\,([0, \omega_1[)^{\mathbf{N}}, \nu^{\mathbf{N}})$, where ν denotes the σ-additive Borel measure on $[0, \omega_1[$ that gives measure 1 or 0 to each Borel set in $[0, \omega_1[$, according to whether it contains an uncountable closed set or not. (This famous example, due to J. Diendonné, may be found in [3] for example). Let $\delta: [0, \omega_1[\to M([0, \omega_1])$ denote the Dirac transform, i.e., the map associating to each $\alpha \in [0, \omega_1[$ the Dirac measure δ_α. Define a sequence $\{X_n\}_{n=1}^{\infty}$ of i.i.d. $M([0, \omega_1])$-valued Borel-measurable (w.r. to the σ^*-topology) random variables on Ω by putting $X_n = \delta \circ p_n$, p_n denoting the projection onto the n-th coordinate of $[0, \omega_1[^{\mathbf{N}}$.

It is easily seen that a Σ-measurable subset of Ω has measure 1 iff it contains a set of the form $F^{\mathbf{N}}$ for some uncountable closed subset F of $[0, \omega_1[$.

But as it is evidently absurd that for some closed uncountable F of $[0, \omega_1[$ we have, that for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ in F the limit

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \delta_{\alpha_i}$$

converges in the weak-$*$-topology of $M[0, \omega_1]$, we arrive at a contradiction, showing that the strong law of large numbers does not hold for $\{X_n\}_{n=1}^{\infty}$.

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