Boletín de la Sociedad Matemática Mexicana Vol. 26 No. 1 1981

SIMPLICIAL DECOMPOSITION OF Γ-STRUCTURES

BY SOLOMON M. JEKEL

Introduction

Let Γ be a topological groupoid with objects $\mathcal{O}$, and $B\Gamma$ the classifying space for Γ-structures. The loop space of $B\Gamma$, $\Omega B\Gamma$, has the following description [2]: If $\mathcal{O}$ is contractible then $\Omega B\Gamma$ is, up to weak homotopy type, the realization of a simplicial group $\mathcal{G}_*(\Gamma)$ whose group of p-cells is the free group on the p-simplexes of Γ modulo relations defined for certain composable pairs of simplexes. In this paper we give a concrete description of the complex $\mathcal{G}_*(\Gamma)$.

Each $\mathcal{G}_p(\Gamma), p \geq 0$ is isomorphic to the free product of the free group on the p-simplexes of Γ with an "isotropy group" of $S^p\Gamma$ with respect to a natural action of $S^p\Gamma$ on S^p. These isomorphisms then induce an isomorphism of $\mathcal{G}_*(\Gamma)$ with a simplicial group $\bar{\mathcal{G}}_*(\Gamma)$ which we refer to as a *simplicial group decomposition* of Γ.

The isomorphisms for each $p \geq 0$ depend on certain choices in much the same way that defining the fundamental group of a space depends on a choice of base point. In general these choices are not compatible under faces and degeneracies. When they are compatible we show that $\bar{\mathcal{G}}_*(\Gamma)$ splits into an isotropy part and a free part. Otherwise the faces and degeneracies mix the two complexes in a rather complicated way.

As an application of this construction we define a spectral sequence which converges to $H_*(B\Gamma)$ and whose E^1 term is given in terms of the homology of the isotropy groups of $S^p\Gamma$. We compute the E^1 term of this spectral sequence for $B\Gamma_1^{\omega}$, the classifying space for codimension-one real analytic Γ-structures.

I am very grateful to Francisco González-Acuña for discussing the material of this paper on numerous occasions. It is his observation that the universal groups of a discrete groupoid decompose in the described way.

§1 The Universal Simplicial Group of a Topological Groupoid

We begin by recalling the constructions and the main theorem of [2].

Let Γ be a *topological groupoid* which we view as a small topological category with all morphisms invertible. Let $\mathcal{M}(\Gamma)$ be the space of morphisms of Γ and $\mathcal{O}(\Gamma)$ the space of objects of Γ. $\mathcal{O}(\Gamma)$ is homeomorphic to the subspace of $\mathcal{M}(\Gamma)$ consisting of identity morphisms. Therefore, when there is no confusion, we shall write Γ for $\mathcal{M}(\Gamma)$ and $\mathcal{O}$ for the subspace of identity morphisms.

Consider first Γ with the *discrete* topology. The *reduced set of* Γ, $\Gamma/\mathcal{O}$, obtained by identifying the identity morphisms to a point 1, inherits a sometimes-defined multiplication "$\star$" from the groupoid composition "$\circ$".

(i) $x \star y = x \circ y$ if x, y and $x \circ y$ are not 1, and $(x, y) \to x \circ y$ is defined in Γ.

(ii) $x \star x^{-1} = x^{-1} \star x = 1$ for all $x \in \Gamma/\mathcal{O}$.

(iii) $1 \star x = x \star 1 = x$ for all $x \in \Gamma/\mathcal{O}$.

The product $\star$ makes $\Gamma/\mathcal{O}$ into a pregroup [2].

Let $\mathcal{G}_0(\Gamma)$ be the *universal group of the pregroup* $\Gamma/\mathcal{O}$. It is the free group on the elements of $\Gamma/\mathcal{O}$, $F(\Gamma/\mathcal{O})$, modulo the relations $x \cdot y = x \star y$, where $x \cdot y$ is the free product, $x \star y$ the $\star$-product and there is a relation whenever $x \star y$ is defined.

We will refer to $\mathcal{G}_0(\Gamma)$ as the *universal group of* Γ.

Now let Γ be a topological groupoid. Let $S^p\Gamma$ denote the set of singular p-simplexes on Γ, $p \geq 0$ and $S\Gamma$ the total singular complex of Γ.

$S^p\Gamma$ is, in a natural way, the set of morphisms of a discrete groupoid with objects $S^p\mathcal{O}$. Let $\mathcal{G}_p(\Gamma)$ be the universal group of (the discrete groupoid) $S^p\Gamma$.

Consider the disjoint union $\mathcal{G}_*(\Gamma) = \cup_{p \geq 0} \mathcal{G}_p(\Gamma)$. $\mathcal{G}_*(\Gamma)$ is a simplicial group with faces and degeneracies induced by those of $S\Gamma$ which we call the *universal simplicial group of* Γ.

The following theorem is proved in [2].

THEOREM 1: *Let Γ be a topological groupoid with its space of objects $\mathcal{O}$ contractible. Let $\mathcal{G}_*(\Gamma)$ be the universal simplicial group of Γ. Let $B\Gamma$ be the classifying space for Γ-structures. Then there is a weak homotopy equivalence*

$$|\mathcal{G}_*(\Gamma)| \to \Omega B\Gamma.$$

§2. Elements of the Decomposition

In this section Γ will be a discrete groupoid with objects $\mathcal{O}$ and source and target maps s and t. To Γ we associate an *isotropy group* $\mathcal{I}(\Gamma)$ and a *base path group* $\mathcal{F}(\mathcal{O})$ which will be the factors of the decomposition of $\mathcal{G}_0(\Gamma)$. The definition of these groups will depend, up to isomorphism, on a choice of a *base for* Γ.

(1) Γ-*components and paths*:

Define an equivalence relation $\sim$ on the objects of Γ by $x \sim y$ if there exists a morphism $\gamma \in \Gamma$ such that $s(\gamma) = x$, $t(\gamma) = y$.

A Γ-*component* $\mathcal{O}_\alpha \subset \mathcal{O}$ is an equivalence class of objects of Γ under the above equivalence.

A path of a component $\mathcal{O}_\alpha \subset \mathcal{O}$ is a morphism $p \in \Gamma$ such that $s(p)$ and $t(p)$ are in $\mathcal{O}_\alpha$.

Let Γ_α be the set of paths of $\mathcal{O}_\alpha$. Each Γ_α is a groupoid with objects $\mathcal{O}_\alpha$, and the groupoid Γ is the disjoint union of the groupoids Γ_α. $\{\Gamma_\alpha\}$ is a partition of Γ into its "groupoid" components.

(2) *A base for* Γ:

A base for Γ, $B = \{\beta, \rho\}$ is given by the following data.

a) A collection of *base points* $\beta = \{b_\alpha\}$. This is a choice of a single element $b_\alpha \in \mathcal{O}\alpha$ for each component $\mathcal{O}_\alpha$.

b) A collection of *base paths* $\rho = \{\rho(x)\}$. This is a choice of a single morphism

$\rho(x) \in \Gamma$ for each $x \in \mathcal{O}$ such that

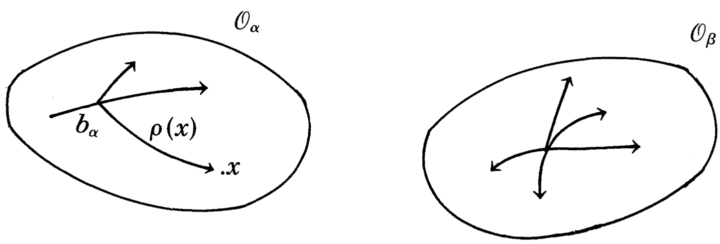
(i) If $x \in \mathcal{O}_\alpha$ then $s(\rho(x)) = b_\alpha$ and $t(\rho(x)) = x$
(ii) $\rho(b_\alpha) = \mathrm{id}_{b\alpha}$ = identity morphism of b_α.

A base for Γ, $B = \{\beta, \rho\}$ is given by the following data.

a) A collection of *base points* $\beta = \{b_\alpha\}$. This is a choice of a single element $b_\alpha \in \mathcal{O}\alpha$ for each component $\mathcal{O}_\alpha$.

b) A collection of *base paths* $\rho = \{\rho(x)\}$. This is a choice of a single morphism $\rho(x) \in \Gamma$ for each $x \in \mathcal{O}$ such that

(i) If $x \in \mathcal{O}_\alpha$ then $s(\rho(x)) = b_\alpha$ and $t(\rho(x)) = x$
(ii) $\rho(b_\alpha) = \mathrm{id}_{b\alpha}$ = identity morphism of b_α.



(3) *The isotropy group of* Γ:

Let $x \in \mathcal{O}$. The *isotropy group of x*, $\mathcal{I}(x)$, is defined by $\mathcal{I}(\mathrm{x}) = \{\gamma \in \Gamma \mid \mathrm{s}(\gamma) = \mathrm{t}(\gamma) = \mathrm{x}\}$. $\mathcal{I}(\mathrm{x})$ is a group under the multiplication induced by the groupoid structure of Γ.

Now fix a collection $\beta = \{b_\alpha\}$ of base points for Γ. The *isotropy group of* Γ (relative to the base points β) is given by

$$\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma, \beta) = \star_\alpha \mathcal{I}(b_\alpha)$$

the free product over all the base points $b_\alpha \in \beta$.

When we write $\mathcal{I}(\Gamma)$ it is understood to be defined relative to a fixed collection β of base points for Γ.

(4) *The base path group of* Γ:

Let $\beta = \{b_\alpha\}$ be a given collection of base points for Γ. The *base path group* of Γ (relative to the base points β) is given by

$$\mathcal{F}(\mathcal{O}) = \mathcal{F}(\mathcal{O}, \beta) = F(\mathcal{O})/\{b_\alpha = 1\}$$

the free group on the objects of Γ modulo the relations $b_\alpha = 1$ for each base point b_α. $\mathcal{F}(\mathcal{O})$ is a free group.

When we write $\mathcal{F}(\mathcal{O})$ it is understood to be defined relative to a fixed collection of base points for Γ.

(5) *The base point map and the isotropy map*:

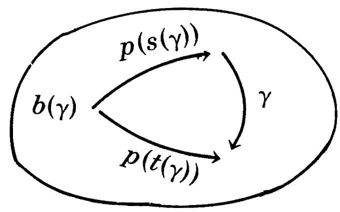
The base point map is a function $b:\Gamma \to \mathcal{O}$ defined by letting $b(\gamma)$ be the base point of the component of which $\gamma \in \Gamma$ is a path.

Let $\beta = \{b_\alpha\}$, $\rho = \{\rho(x)\}$ be a base for Γ, and let $\mathcal{I}(\Gamma)$ be the isotropy group of Γ (relative to β). The *isotropy map* $\iota: T \to \mathcal{I}(\Gamma)$ is given by

$$\iota(\gamma) = \rho(s(\gamma)) \circ \gamma \circ \rho(t(\gamma))^{-1}.$$

As in §1, "$\circ$" denotes composition in Γ.

When we write ι it is understood to be defined relative to a fixed base B for Γ.



Remarks: (i) If $\gamma \in \Gamma$ is a base path $\rho(x)$, then $\iota(\gamma) = \mathrm{id}_{b(\gamma)} \circ \gamma \circ \gamma^{-1} = \mathrm{id}_{b(\gamma)}$ which is the identity element of the group $\mathcal{I}(\Gamma)$.

(ii) ι restricted to the isotropy group of an object $x \in \mathcal{O}$ takes $\mathcal{I}(x)$ to $\mathcal{I}(b(x))$ and is conjugation by $\rho(x)$ as a map between these two groups.

(iii) If Γ is the fundamental groupoid of a connected space X, then there is only one component. The base point map chooses a base point $x \in X$. $\mathcal{I}(\Gamma)$ is the fundamental group of X. The map described in (ii) above gives an isomorphism of the fundamental groups at two different base points via a particular path joining them.

§3. Decomposition of the Universal Group of Γ.

Consider Γ with the discrete topology. Let $\mathcal{G}_0(\Gamma)$ be its universal group. Suppose we are given a base $B = \{\beta = \{b_\alpha\}, \rho = \{\rho_x\}\}$ for Γ. We then obtain the following description of $\mathcal{G}_0(\Gamma)$. Compare [1].

THEOREM 2: *The base B defines an isomorphism $B_{\#}: \mathcal{G}_0(\Gamma) \to \mathcal{I}(\Gamma) * \mathcal{F}(\mathcal{O})$.*

Proof: It is enough to consider the case when there is only one component. So we assume that $\mathcal{O}$ is itself a component and $b \in \mathcal{O}$ its base point. $\{p_x\}$ is its family of base paths.

Let $\iota:\Gamma \to \mathcal{I}(b)$ be the isotropy map relative to the given base. Then each $\gamma \in \Gamma$ can be writen uniquely in Γ as $\rho(s(\gamma))^{-1} \circ \iota(\gamma) \circ \rho(t(\gamma))$. We define a function $B_0: \Gamma \to \mathcal{I}(b) * \mathcal{F}(\mathcal{O})$ by

$$B_0(\gamma) = \langle s(\gamma) \rangle^{-1} \cdot \iota(\gamma) \cdot \langle t(\gamma) \rangle$$

where $\langle x \rangle$ denotes the generator of $\mathcal{F}(\mathcal{O})$ determined by $x \in \mathcal{O}$, and "$\cdot$" indicates multiplication in the free product.

Note if $\gamma \in \mathcal{O} \subset \Gamma$ then $B_0(\gamma) = 1$, so that B_0 induces a function $B_1 : \Gamma/\mathcal{O} \to \mathcal{I}(b) * \mathcal{F}(\mathcal{O})$ and hence a homomorphism $B_2 : F(\Gamma/\mathcal{O}) \to \mathcal{I}(b) * \mathcal{F}(\mathcal{O})$. B_2 respects the relations in the presentation of $\mathcal{G}_0(\Gamma)$ so induces a homomorphism $B_{\#} : \mathcal{G}_0(\Gamma) \to \mathcal{I}(b) * \mathcal{F}(\mathcal{O})$.

A homomorphism $A_{\#} : \mathcal{I}(b) * \mathcal{F}(\mathcal{O}) \to \mathcal{G}_0(\Gamma)$ is completely determined by the following rule: $A_{\#}(\alpha) = [\alpha]$ for $\alpha \in I(b)$; $A_{\#}(\langle x \rangle) = [\rho(x)]$ for $x \in \mathcal{O}$. Here $[\alpha]$ and $[\rho(x)]$ indicate the elements in $\mathcal{G}_0(\Gamma)$ represented by the morphisms α and $\rho(x)$.

$A_{\#}$ is a two sided inverse for $B_{\#}$, so that $B_{\#}$ is an isomorphism.

§4. Decomposition of the Universal Simplicial Group of Γ.

The decomposition of $\mathcal{G}_0(\Gamma)$ described in Theorem 2 induces a decomposition of $\mathcal{G}_*(\Gamma)$.

Let Γ be a topological groupoid with space of objects $\mathcal{O}$. Then $S^k\Gamma$ is a discrete groupoid with objects $S^k\mathcal{O}$, $k \geq 0$. For each $k \geq 0$ we fix a base $B^k = \{\beta^k, \rho^k\}$ for $S^k\Gamma$, and all the constructions which follow will be made with respect to the given bases.

Let $\bar{\mathcal{G}}_k(\Gamma) = \mathcal{I}(S^k\Gamma) * \mathcal{F}(S^k\mathcal{O})$. Then $\bar{\mathcal{G}}_*(\Gamma) = \cup_{k \geq 0} \bar{\mathcal{G}}_k(\Gamma)$ is a simplicial group which is isomorphic to $\mathcal{G}(\Gamma)$, the universal simplicial group of Γ. The isomorphism is determined for each $k \geq 0$ by the choice of base B^k as in §3. The faces $\{\bar{\partial}_i\}$ and degeneracies $\{\bar{\sigma}_j\}$ of $\bar{\mathcal{G}}_*(\Gamma)$ are determined via these isomorphisms:

$$\begin{array}{ccc}
\mathcal{G}_k(\Gamma) & \xrightarrow{\;B_{\#}^{\,k}\;} & \bar{\mathcal{G}}_k(\Gamma) \\
\partial \downarrow \uparrow \sigma & & \bar{\partial} \downarrow \uparrow \bar{\sigma} \\
\mathcal{G}_{k-1}(\Gamma) & \xrightarrow[\;B_{\#}^{\,k-1}\;]{} & \bar{\mathcal{G}}_{k-1}(\Gamma)
\end{array}$$

$$\bar{\partial}_i = \mathrm{B}_{\#}^{\,k-1} \circ \partial_i \circ (\mathrm{B}_{\#}^{\,k})^{-1} \tag{1}$$

$$\bar{\sigma}_j = \mathrm{B}_{\#}^{\,k} \circ \sigma_j \circ (\mathrm{B}_{\#}^{\,k-1})^{-1} \tag{2}$$

We call $\bar{\mathcal{G}}_*(\Gamma) = \cup_{k \geq 0} [\mathcal{I}(S^k\Gamma) * \mathcal{F}(S^k\mathcal{O})]$ with face maps $\{\bar{\partial}_i\}$ and degeneracy maps $\{\bar{\sigma}_j\}$ *a simplicial group decomposition of the topological groupoid* Γ. Again, it depends on a choice of base B^k for $S^k\Gamma$ for each $k \geq 0$.

We reformulate Theorem 1.

THEOREM 3: *Let Γ be a topological group with its space of objects $\mathcal{O}$ contractible. Let $B\Gamma$ be the classifying space for Γ-structures. Let $\bar{\mathcal{G}}_*(\Gamma)$ be a simplicial group decomposition of Γ. Then there is a weak homotopy equivalence*

$$\bar{\mathcal{G}}_*(\Gamma) \to \Omega B\Gamma.$$

The faces $\bar{\partial}_i: \bar{\mathscr{G}}_k(\Gamma) \to \bar{\mathscr{G}}_{k-1}(\Gamma)$ *and the degeneracies* $\bar{\sigma}_j: \bar{\mathscr{G}}_k(\Gamma) \to \bar{\mathscr{G}}_{k+1}(\Gamma)$ *are completely determined by the following rules*, where $\{\partial_i\}$ and $(\sigma_j\}$ denote the faces and degeneracies of $S\Gamma$.

$$\text{a) } \bar{\partial}_i(x) = \iota(\partial_i(x)) \quad \text{for} \quad x \in \mathscr{I}(b_\alpha).$$

$$\text{b) } \bar{\partial}_i(\langle y \rangle) = \langle \partial_i(b(y)) \rangle^{-1} \cdot \iota(\partial_i(p(y))) \cdot \langle \partial_i(y) \rangle \quad \text{for} \quad y \in S^k \mathscr{O}.$$

$$\text{c) } \bar{\sigma}_j(x) = \iota(\sigma_j(x)) \quad \text{for} \quad x \in \mathscr{I}(b_\alpha)$$

$$\text{d) } \bar{\sigma}_j(\langle y \rangle) = \langle \sigma_j(b(y)) \rangle^{-1} \cdot \iota(\sigma_j(p(y))) \cdot \langle \sigma_j(y) \rangle \quad \text{for} \quad y \in S^k \mathscr{O}.$$

Remarks: Computing the operators $\{\bar{\partial}_i\}$ and $\{\bar{\sigma}_j\}$ explicitly is a matter of evaluating the compositions on the right hand side of equations (1) and (2).

Recall, for each $k \geq 0$ we have the isotropy map $\iota^k: S^k\Gamma \to \mathscr{I}(S^k\Gamma)$ and the base point map $b^k: S^k\Gamma \to S^k \mathscr{O}$. We have suppressed the k in the notation when it is clear in which dimension the functions are acting.

We have $\mathscr{I}(S^k\Gamma) = \star_{b\alpha}\mathscr{I}(b_\alpha)$ where $\beta^k = \{b_\alpha\}$. So to define $\bar{\partial}_i: \bar{\mathscr{G}}_k(\Gamma) \to \bar{\mathscr{G}}_{k-1}(\Gamma)$ we had to define $\bar{\partial}_i(x)$ for each $x \in \mathscr{I}(b_\alpha)$ and for each b_α, and $\bar{\partial}_i(\langle y \rangle)$ for each $y \in S^k \mathscr{O}$. Then $\bar{\partial}_i$ extends to all of $\bar{\mathscr{G}}_k(\Gamma)$. Similarly to define $\bar{\sigma}_j: \bar{\mathscr{G}}_k(\Gamma) \to \bar{\mathscr{G}}_{k+1}(\Gamma)$ we had to define $\bar{\sigma}_j(x)$ for each $x \in \mathscr{I}(b_\alpha)$ and for each b_α, and $\bar{\sigma}_j(\langle y \rangle)$ for each $y \in S^k \mathscr{O}$.

§5. Compatibility of Bases

Suppose the bases B^k for $S^k\Gamma$ can be chosen *compatibly* for all $k \geq 0$, that is any face and any degeneracy of each base point and base path is again a base point and base path. Then by remark (i) in §2 $\iota(\partial_i(\rho(y)))$ and $\iota(\sigma_j(\rho(y)))$ are the identity for all $y \in \mathscr{O}$. In this case, as formulas b) and d) show $\{\bar{\partial}_i\}$ and $\{\bar{\sigma}_j\}$ take elements of $\mathscr{F}_\star(\mathscr{O}) = \cup_{k \geq 0} \mathscr{F}(S^k \mathscr{O})$ into itself. On the other hand, as formulas a) and c) show $\{\bar{\partial}_i\}$ and $\{\bar{\sigma}_j\}$ always map $\mathscr{I}_\star(\Gamma) = \cup_{k\geq 0} \mathscr{I}(S^k\Gamma)$ into itself.

We therefore obtain the following theorem.

THEOREM 4: *Let* Γ *be3 a topological groupoid and suppose there exists a compatible choice of bases for* $S^k\Gamma$, $k \geq 0$. *Then* $\bar{\mathscr{G}}_\star(\Gamma) = \mathscr{I}_\star(\Gamma) \star \mathscr{F}_\star(\mathscr{O})$. *The faces and degeneracies are given by*

$$\bar{\partial}_i(x) = \partial_i(x),$$

$$\bar{\sigma}_j(x) = \sigma_j(x), \quad \textit{for} \quad x \in \mathscr{I}_\star(\Gamma).$$

$$\bar{\partial}_i(\langle y \rangle) = \langle \partial_i(b(y)) \rangle^{-1} \cdot \langle \partial_i(y) \rangle,$$

$$\bar{\sigma}_j(\langle y \rangle) = \langle \sigma_j(b(y)) \rangle^{-1} \cdot \langle \sigma_j(y) \rangle, \quad \textit{for} \quad y \in \mathscr{F}_\star(\mathscr{O}).$$

Example: If $\Gamma = G$ is a topological group then $\mathscr{I}_\star(G) = SG$ the singular complex of G, $\mathscr{F}_\star(\mathscr{O}) = 1$ in every dimension, and $\bar{\mathscr{G}}_\star(\Gamma) = SG$. Of course, as is well known there is a weak homotopy equivalence $|SG| \to \Omega BG$.

§6. A Spectral Sequence Converging to $H_*(B\Gamma)$.

Let Γ be a topological groupoid with contractible object space $\mathcal{O}$. Let $H_*(\cdot)$ denote integral homology. We define a spectral sequence which converges to $H_*(B\Gamma)$ and whose E^1 term is given in terms of the isotropy and base path groups of Γ.

Consider the bigraded group $E_{p,q} = \tilde{H}_q(\bar{\mathscr{G}}_p(\Gamma)), p, q \geq 0.$

Then

$$E_{p,q} = \begin{cases} 0 & q = 0 \\ H_1(\mathscr{I}(S^p\Gamma)) \oplus H_1(\mathscr{F}(S^p\,\mathcal{O})) & q = 1 \\ H_q(\mathscr{I}(S^p\Gamma)) & q \geq 2 \end{cases}$$

and we have a differential $d: E_{p,q} \to E_{p-1,q}$ induced by the face maps $\{\bar{\partial}_i\}$ of $\bar{\mathscr{G}}_*(\Gamma)$.

PROPOSITION 1: *$\{E_{p,q},\ d\}$ is the E^1 term of a spectral sequence which converges to $\tilde{H}_{p+q}(B\Gamma)$.*

Proof: This is a consequence of the proof of the main theorem of [2].

Let $N\Gamma^p$ be the space of p-cells of the simplicial nerve of Γ. Let $N\mathcal{O}^O = \mathcal{O}$, $N\mathcal{O}^p = \text{diagonal}\,(\mathcal{O}^P), p \geq 1$. $N\mathcal{O}^p$ is a contractible subspace of $N\Gamma^p$ and $N\mathcal{O} = \cup_{p\geq 0} N\mathcal{O}^p$. Consider the bisimplicial set $S(N\Gamma, N\mathcal{O}) = \cup_{p,q\geq 0} \dfrac{S^pN\Gamma^q}{S^pN\mathcal{O}^q}$ with horizontal (q-fixed) faces and degeneracies induced by those of S and vertical (p-fixed) faces and degeneracies induced by those of $N\Gamma$. In [2] we show that there is a weak homotopy equivalence of the realization of the diagonal complex of $S(N\Gamma, N\mathcal{O})$ to $B\Gamma$. If then follows that $H_p{}^hH_q{}^v(|S(N\Gamma, N\mathcal{O})|)$ is the E^2 term of a spectral sequence converging to $H_{p+q}(B\Gamma)$. (Here $H_p{}^hH_q{}^v(\cdot)$ is the p-th homology of the horizontal simplicial abelian group obtained by taking the q-th homology of each of the vertical simplicial groups of a given bisimplicial set). But the p-th vertical simplicial set of $S(N\Gamma, N\mathcal{O})$ is a $K(\mathscr{G}_p(\Gamma), 1) = K(\bar{\mathscr{G}}_p(\Gamma), 1)$, [2]. Hence $H_q{}^v(|S(N\Gamma, N\mathcal{O})|)$ is the E^1 term described in the proposition.

§7. A Simplicial Group Decomposition of Γ_1^ω

The E^2 term of the above spectral sequence simplifies considerably when $B\Gamma_1^\omega$ is the classifying space for codimension-1 real analytic foliations.

Let $\Gamma = \Gamma_1^\omega$ be the groupoid of germs of local, real analytic homeomorphisms of $\mathbf{R}$ with the sheaf topology. We will compute the groups $\bar{\mathscr{G}}_p(\Gamma)$.

First consider $\bar{\mathscr{G}}_0(\Gamma) = \mathscr{I}(S^0\Gamma) \star \mathscr{F}(\mathbf{R})$. There is only one $S^0\Gamma$-component since any two points of $\mathbf{R}$ are joined by some germ. Suppose we choose $0 \in \mathbf{R}$ as a base point. *Let $T = \mathscr{I}(0)$ be the group of germs of* Γ *keeping* 0 *fixed.* (T is the group of convergent Taylor series expansions at the origin under composition). Then $\bar{\mathscr{G}}_0(\Gamma) \simeq \mathrm{T} \star \mathscr{F}(\mathbf{R}) \simeq T \star \{F(\mathbf{R})/\langle\text{origin}\rangle = 1\}$.

Now consider $\bar{\mathscr{G}}_p(\Gamma), p \geq 0$. There are uncountably many $S^p\Gamma$-components. The set of all $\sigma \in S^pR$ which are constant belong to the same $S^p\Gamma$-component.

Let us choose O^p, the p-simplex which is constant and equal to 0, as a base point. Then $\mathscr{I}(O^p) \simeq T$.

Let $\sigma = S^p\mathbf{R}$ be any simplex of any other component. Then σ is not constant, hence its image contains some open set in $\mathbf{R}$. By analytic continuation it follows that $\mathscr{I}(\sigma) = 1$.

Then we obtain

Proposition 2: $\bar{\mathscr{G}}_p(\Gamma) = T \star \mathscr{F}(S^p\mathbf{R})$ *for all* $p \geq 0$. *The faces and degeneracies of* $\bar{\mathscr{G}}_*(\Gamma)$ *can be computed by Proposition* 1.

Note that the proposition implies that the E^2 term of the spectral sequence of proposition 1 in this case reduces to

$$E_{p,q}{}^2 = \begin{cases} 0 & q = 0;\quad p \geq 1 \quad \text{and} \quad q \geq 2 \\ H_q(T) & p = 0 \\ H_p(H_1(\bar{\mathscr{G}}_*(\Gamma)) & q = 1 \end{cases}$$

since for fixed $q \geq 2$ we are computing the homology of the constant simplicial group which is $H_q(T)$ for all $p \geq 0$ and this is a $K(H_q(T), 0)$.

For the spaces $B\Gamma = B\Gamma_1{}^r$, $0 \leq r \leq \infty$ we can also show that $E_{p,q}{}^2 = 0$ for $p \geq 1$ and $q \geq 2$, but this requires a more detailed analysis of $\mathscr{G}_*(\Gamma)$ in the E^1 term.

CENTRO DE INVESTIGACION DEL IPN., MEXICO 14, D.F.
NORTHEASTERN UNIVERSITY, BOSTON, MASSACHUSETTS

REFERENCES

[1] P. H. HIGGENS, *Presentations of Groupoids with Applications to Groups*, Proc. Cambridge Philos. Soc., **60**(1964), 7–20.

[2] S. JEKEL, *Loops on the Classifying Space for Foliations*, Amer. J. of Math., **102**(1980), 13–23.