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SYMMETRIC PRODUCTS OF HYPERELLIPTIC CURVES

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Let C be a complete smooth curve of genus $g > 1$ over an algebraically closed field and C_n its n-th symmetric product. Kempf proved [4] that if C is non-hyperelliptic then any small deformation of C_n is a symmetric product of a deformation of the original curve C. The non-hyperelliptic hypothesis enters in his argument when he applies Max Noether's theorem on the product of higher differentials on C. In the present paper, using a complement of Noether's theorem for hyperelliptic curves in characteristic distinct from 2 and following Kempf's argument we prove that for $g > 2$ we may remove the non-hyperelliptic hypothesis and that if $g = 2$ and $n > 1$

$$\dim H^1(C_n, \Theta_{C_n}) = \dim H^1(C, \Theta_C) + 1$$

Then working for the genus 2 case over the complex numbers we construct for $n = 2$ an effective and complete family of dimension 4, while for $n > 2$ we show that all first ordered obstructions vanish. I would like to thank George Kempf and Roy Smith for helpful conversations during the elaboration of this paper.

1. Computations of $H^1(C_n, \Theta_{C_n})$

Let C be a hyperelliptic curve over an algebraically closed field k with char $(k) \neq 2$, genus $g > 1$ and ω the sheaf of differentials. The hyperelliptic involution splits $H^0(C, \omega^m)$ into ± 1 eigenspaces. These spaces may be explicitly constructed using a plane model for C. With this decomposition the following is a simple computation [10]:

Proposition 1. *For every $m > 0$ the map*

$$H^0(C, \omega) \otimes H^0(C, \omega^m) \to H^0(C, \omega^{m+1})$$

given by multiplication is surjective except if $g = m = 2$ or $g > 2$ and $m = 1$, where the codimension of the image is 1 and $g - 2$ respectively.

We assume that the reader is familiar with [4]. Let C_n be the n-th symmetric product of C, D_n the divisor on $C \times C_n$ obtained as image of the map $C \times C_{n-1} \to C \times C_n$ given by $(p, D) \to (p, p + D)$ and denote by $\gamma_n : C \times C_{n-1} \to D_n$ the corresponding isomorphism. For any variety X denote by Θ_X the sheaf of vector fields on X. We recall from [4] the following results:

Lemma 2. $\gamma_n^*(\mathcal{O}_{C\times C_n}(D_n) \mid_{D_n}) \approx \pi_C{}^*\Theta_C \otimes \mathcal{O}_{C\times C_{n-1}}(D_{n-1})$

Proposition 3. *For $g > 1$, there exists a natural exact sequence*

$$0 \to H^1(C, \Theta_C) \to H^1(C_n, \Theta_{C_n}) \to H^0(C, \Theta_C \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_{n-1}}(D_{n-1})) \to 0$$

Our first objective will be to calculate the last term of the sequence. On

$C \times C_m$ with $m \geq 1$ we have an exact sequence

$$0 \to \pi_C{}^*\Theta_C \to \pi_C{}^*\Theta_C \otimes \mathcal{O}_{C\times C_m}(D_m) \to \pi_C{}^*\Theta_C \otimes \mathcal{O}_{C\times C_m}(D_m)\,|_{D_m} \to 0.$$

Using Lemma 2 and viewing $\mathcal{O}_{C\times C_{m-1}}$ as a module on $C \times C_m$ via γ_m we obtain the sequence

$$0 \to \pi_C{}^*\Theta_C \to \pi_C{}^*\Theta_C \otimes \mathcal{O}_{C\times C_m}(D_m) \to \pi_C{}^*\Theta_C{}^2 \otimes \mathcal{O}_{C\times C_{m-1}}(D_{m-1}) \to 0$$

We look at the exact sequence of direct image sheaves on C

$$0 \to \Theta_C \overset{\approx}{\to} \Theta_C \otimes \pi_{C^*}\Theta_{C\times C_m}(D_m) \overset{0}{\to} \Theta_C{}^2 \otimes \pi_{C^*}\mathcal{O}_{C\times C_{m-1}}(D_{m-1}) \overset{\psi m}{\to} \Theta_C$$

$$\otimes H^1(C_m, \mathcal{O}_{C_m}) \to \Theta_C \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_m}(D_m) \to \Theta_C{}^2 \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_{m-1}}(D_{m-1}).$$

Using the canonical isomorphisms $H^1(C_m, \mathcal{O}_{C_m}) \approx H^1(C, \mathcal{O}_C)$ and $\pi_{C^*}\mathcal{O}_{C\times C_{m-1}}(D_{m-1}) \approx \mathcal{O}_C$, we decompose this sequence into two short exact sequences on C

$$0 \to \Theta_C{}^2 \to \Theta_C \otimes H^1(C, \mathcal{O}_C) \to \operatorname{Cok} \psi_m \to 0 \qquad (1_m)$$

$$0 \to \operatorname{Cok} \psi_m \to \Theta_C \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_m}(D_m) \to \Theta_C{}^2 \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_{m-1}}(D_{m-1}) \qquad (2_m)$$

Tensoring (1_m) with $\Theta_C{}^p$ for $p \geq 0$ and writing out the long exact sequence we have

$$0 \to H^0(C, \Theta_C{}^p \otimes \operatorname{Cok} \psi_m) \to H^1(C, \Theta_C{}^{p+2}) \xrightarrow{H^1(\Theta_C{}^p \otimes \psi_m)}$$

$$\to H^1(C, \Theta_C{}^{p+1}) \otimes H^1(C, \mathcal{O}_C) \to H^1(C, \Theta_C{}^p \otimes \operatorname{Cok} \psi_m) \to 0.$$

Kempf has shown that, using Serre duality

$$H^1(\Theta_C{}^p \otimes \psi_m)^* : H^0(C, \omega) \otimes H^0(C, \omega^{p+2}) \to H^0(C, \omega^{p+3})$$

corresponds to multiplication, so using Proposition 1 we obtain:

PROPOSITION 4. *For* $m \geq 1$: **a)** *If* $g = 2$, $\dim H^0(C, \operatorname{Cok} \psi_m) = 1$. **b)** *If* $g > 2$ *and* $p \geq 0$ *or* $g = 2$ *and* $p \geq 1$, $H^0(C, \Theta_C{}^p \otimes \operatorname{Cok} \psi_m) = 0$.

LEMMA 5. $H^0(C, \Theta_C{}^p \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_m}(D_m)) = 0$ *for* $p \geq 2$ *and all* $m \geq 0$.

Proof. We prove it by induction on m. For $m = 0$ we have that $R^1\pi_{C^*}\mathcal{O}_C = 0$ so it is trivially satisfied. Assume it is true for $m \geq 0$ and all $p \geq 2$ and we prove it for $m + 1$ and arbitrary $p \geq 2$. The exact sequence of global sections of (2_{m+1}) tensored with $\Theta_C{}^{p-1}$ gives

$$0 \to H^0(C, \Theta_C{}^{p-1} \otimes \operatorname{Cok} \psi_{m+1}) \to H^0(C, \Theta_C{}^p \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_{m+1}}(D_{m+1}))$$

$$\to H^0(C, \Theta_C{}^{p+1} \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_m}(D_m)).$$

The first term is zero by Proposition 4.b and the last is zero by induction, hence the middle term is zero. ||

We arrive at our first objective:

PROPOSITION 6. *For $n > 1$, $H^0(C, \Theta_C \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_{n-1}}(D_{n-1})) = 0$, except if $g = 2$, in which case it is 1-dimensional.*

Proof: The short exact sequence of (2_{n-1}) is

$$0 \to H^0(C, \mathrm{Cok}\, \psi_{n-1}) \to H^0(C, \Theta_C \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_{n-1}}(D_{n-1})) \to H^0(C, \Theta_C{}^2 \otimes R^1\pi_{C^*}\mathcal{O}_{C\times C_{n-2}}(D_{n-2}))$$

The last term is zero by Lemma 5 and Proposition 4 now gives the desired result. ||

THEOREM 7. **a)** *If $g = 2$ and $n > 1$, then the natural injection $H^1(C, \Theta_C) \to H^1(C_n, \Theta_{C_n})$ has codimension 1.*
b) *Let C be a curve with $g > 2$, then there exists a natural isomorphism $H^1(C_n, \Theta_{C_n}) \approx H^1(C, \Theta_C)$*
c) *Let $S_n: \mathscr{V}_n \to \mathscr{T}$ be the deformation of C_n constructed by symmetrising the universal deformation $s: \mathscr{V} \to \mathscr{T}$ of C. Then for $g > 2$ s_n is the universal deformation of C_n.*

Proof: **a)** and **b)** follow from Proposition 3 and 6. **b)** is the only ingredient lacking for hyperelliptic curves in Kempf's theorem 4.2 in [4]. So **b)** and his argument give **c)**. ||

2. The complete family of C_2 for $g = 2$

We can conclude that deformation theory of higher symmetric products of curves of genus 2 differs from those of $g > 2$, at least up to first order and we are interested in seeing if this difference extends to actual deformations. For the rest of this paper we assume that k is the complex number field $\mathbf{C}$ and that C is a curve of genus 2.

It will be important to look at the Abelian integral map $u_n: C_n \to P_n$ where P_n is the Picard variety of C of degree n. From the point of view of C_n, u_n is the Albanese map obtained by integrating the global 1-forms on C_n modulo periods. The basic idea is that for any deformation of C_n there exists a natural deformation of u_n and this will allow us to compare the deformation of C_n with the deformation of P_n.

To begin, we briefly recall the definition of the Albanese mapping of a local family. Let $p: \mathscr{V} \to U$ be a family of compact Kähler manifolds, where U is a neighborhood of 0 which may be shrinked at will. In particular we may assume that $\mathscr{V}$ is topologically $V_0 \times U$. Let $\Omega^1_{\mathscr{V}|U}$ be the sheaf of relative differentials. Then $p_*\Omega^1_{\mathscr{V}|U}$ is a vector bundle on U. After shrinking U we may choose a basis $\sigma_1, \cdots, \sigma_q$ which may be thought as a family of basis for the 1-forms on $V_t = p^{-1}(t)$. After choosing a local section s of p we can integrate $\sigma = (\sigma_1, \cdots, \sigma_q)$ over paths starting in $s(t)$ and contained in V_t defined modulo periods; i.e. if $\alpha_1, \cdots, \alpha_{2q}$ is a basis for $H_1(V_0, Z)$ and

$$\Phi(t) = (\textstyle\int_{\alpha_j} \sigma_i)$$

we obtain the Albanese map

$$\begin{array}{ccc} \mathscr{V} & & \mathrm{Alb}(\mathscr{V}) = U \times \mathbf{C}^q/\Phi \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Φ is called the period map and may be thought as a holomorphic map of U into D, where D is the period matrix space [1].

Denote by S the space of 2×2 matrices A with $\det(ImA) > 0$. Then over S we may construct a family of complex tori $\mathscr{W} \xrightarrow{q} S$ $\mathscr{W} = \mathbf{C}^2 \times S/\sim$, where two points are equivalent $(z_1, z_2, A) \sim (w_1, w_2, A)$ if and only if

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} + A \begin{pmatrix} p \\ q \end{pmatrix}$$

where m, n, p, q are integers. $q^{-1}(A)$ is the complex tori with period matrix A. In [5] it is shown that this family of normalized complex tori is complete and effectively parametrized.

It will be easier to understand first the case $n = 2$. We can view $u_2 : C_2 \to P_2$ as the blow up of the canonical point in P_2. Now P_2 corresponds to a point $s \in S$ and there exists a unique isomorphism between P_2 and $q^{-1}(s)$ which sends the canonical point to 0. Let $\tilde{\mathscr{W}}$ be the variety obtained by blowing up $\mathscr{W}$ along $0 \times S$:

$$\begin{array}{ccc} \tilde{\mathscr{W}} & \xrightarrow{\varphi} & \mathscr{W} \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{3}$$

$\varphi_t : \tilde{\mathscr{W}}_t \to \mathscr{W}_t$ is the blow up at 0, hence $\tilde{\mathscr{W}}$ is a deformation space of $\tilde{\mathscr{W}}_s = C_2$.

THEOREM 8. *The family $\tilde{\mathscr{W}}$ is complete and effectively parametrized at s. In particular, for curves of genus 2 there are deformations of C_2 which are not symmetric products of curves, moreover which are not even algebraic.*

Proof: The Kodaira-Spencer map of (3) at s gives a commutative diagram

$$\begin{array}{ccc} H^1(C_2, \Theta_{C_2}) & \xrightarrow{d\varphi} & H^1(P_2, \Theta_{P_2}) \\ \tilde{\rho} \nwarrow & & \nearrow \rho \\ & T_sS & \end{array}$$

By construction ρ is an isomorphism. By the behaviour of $H^q(\ \ , \Theta)$ under blowing up [2, p. 798] $d\varphi$ is an isomorphism, hence $\tilde{\rho}$ is an isomorphism. So the first part follows from Kodaira-Spencer's theorem of completeness [6]. The second part follows since a general torus of dimension 2 is not algebraic and the Albanese variety of an algebraic variety is algebraic. ||

3. First order obstructions for $g = 2$ and $n > 2$

Now, we investigate the deformations of C_n with $n > 2$. The Abelian integral map $u_n : C_n \to P_n$ gives a representation of C_n as a bundle of projective spaces P^{n-2}. Let τ be the sub-bundle of Θ_{C_n} consisting of those vector fields which are everywhere tangent to the fibers of u_n. We can fit it into an exact sequence

$$0 \to \tau \to \Theta_{C_n} \to u_n{}^*\Theta_{P_n} \to 0 \tag{4}$$

Proposition 9. **a)** $R^q u_{n*}\tau = R^q u_{n*}\mathcal{O}_{C_n} = R^q u_{n*}\Theta_{C_n} = 0$ *for* $q > 0$
b) *For every* q: $H^q(C_n, \tau) \approx H^q(P_n, u_{n*}\tau)$ *and* $H^q(C_n, \Theta_{C_n}) \approx H^q(P_n, u_{n*}\Theta_{C_n})$
c) $H^q(C_n, \Theta_{C_n}) = H^q(C_n, \tau) = 0$ *for* $q > 2$.
d) *We have an exact sequence*

$$0 \to H^0(P_n, \Theta_{P_n}) \to H^1(C_n, \tau) \to H^1(C_n, \Theta_{C_n}) \to H^1(P_n, \Theta_{P_n})$$
$$\to H^2(C_n, \tau) \to H^2(C_n, \Theta_{C_n}) \to H^2(P_n, \Theta_{P_n}) \to 0 \tag{5}$$

Proof: For U an affine open set in P_n we have $H^q(U \times P^{n-2}, \tau) = H^0(U, \mathcal{O}) \otimes H^q(P^{n-2}, \Theta_{P^{n-2}}) = 0$ for $q > 0$ since we know that $H^q(P^{n-2}, \Theta_{P^{n-2}}) = 0$ for $q > 0$. Hence $R^q u_{n*}\tau = 0$. Repeating the same argument for $\mathcal{O}_{C_n}$ we obtain that $R^q u_{n*}\mathcal{O}_{C_n} = 0$ for $q > 0$. Now using that $u_n{}^*\Theta_{P_n} \approx \mathcal{O}_{C_n} \oplus \mathcal{O}_{C_n}$ we can conclude that the only non-zero part of the long exact sequence of higher cohomololgy groups of (4) is

$$0 \to u_{n^*}\tau \to u_{n^*}\Theta_{C_n} \to \Theta_{P_n} \to 0 \tag{6}$$

In particular $R^q u_{n*}\Theta_{C_n} = 0$ for $q > 0$. This proves **a**). **b**) follows from **a**) since the spectral sequence for $u_{n*}\tau$ $u_{n*}\Theta_{C_n}$ degenerates. **c**) follows from **b**) since P_n has dimension 2. Writing the long exact sequence of (6) together with **b**) and the fact that C_n has no vector fields gives **d**). ‖

Let $p \in C$ and consider the sheaf $\mathcal{O}_{C_n}(p + C_{n-1})$. The dual of its direct image sheaf $W = (u_{n*}\mathcal{O}_{C_n}(p + C_{n-1}))^*$ is a Picard sheaf on P_n and $C_n = \mathrm{Proj}(W)$. The Euler sequence for the tangent bundle of a projective space globalizes to

$$0 \to \mathcal{O}_{C_n} \to u_n{}^*W \otimes \mathcal{O}_{C_n}(p + C_{n-1}) \to \tau \to 0$$

The only non-trivial part of the exact sequence of higher direct image sheaves is by Proposition 9.a

$$0 \to \mathcal{O}_{P_n} \xrightarrow{\alpha} W \otimes W^* \to u_{n^*}\tau \to 0$$

α corresponds to multiplying by the identity automorphism of W, but this map splits naturally via $(1/n)$Trace, hence we obtain a family of short exact sequences for $q \geq 0$:

$$0 \to H^q(P_n, \mathcal{O}_{P_n}) \to H^q(P_n, W \otimes W^*) \to H^q(P_n, u_{n^*}\tau) \to 0 \tag{7}$$

Theorem 10: *Let C be a curve of genus* 2 *over the complex numbers and n* ≥ 3. *Then*:
a) $\dim H^1(C_n, \tau) = 2$ *and* $H^2(C_n, \tau) = 0$

b) *The sequence* (5) *gives rise to canonical isomorphisms*

$$H^q(C_n, \Theta_{C_n}) \approx H^q(P_n, \Theta_{P_n}) \quad \text{for} \quad q = 1, 2.$$

c) *The graded Lie algebgra $H^*(C_n, \Theta_{C_n})$ is non-zero only in degrees* 1 *and* 2, *and it is Abelian.*

Proof: In [3, p. 272], Kempf has computed the dimension of $H^q(P_n, W \otimes W^*)$, which is 4 and 1 for $q = 1, 2$ respectively. Since P_n is isomorphic to the Jacobian of C, $H^q(P_n, \mathcal{O}_{P_n})$ has dimension 2 and 1 for $q = 1, 2$. So **a)** follows from (7) and these calculations. Now **b)** follows from **a)** and the long exact sequence (5).

To check **c)** we only have to prove that the map

$$[\ ,\ ]: H^1(C_n, \Theta_{C_n}) \otimes H^1(C_n, \Theta_{C_n}) \to H^2(C_n, \Theta_{C_n})$$

given by the Poisson bracket of (0, 1)-vector forms vanishes, since all the other cohomology groups vanish. To do this, consider the $\bar{\partial}$-resolution of (4)

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tau & \to & E(\tau) & \xrightarrow{\bar{\partial}} & E^{0,1}\tau & \xrightarrow{\bar{\partial}} & E^{0,2}\tau & \xrightarrow{\bar{\partial}} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Theta_{C_n} & \to & E\Theta_{C_n} & \xrightarrow{\bar{\partial}} & E^{0,1}\Theta_{C_n} & \xrightarrow{\bar{\partial}} & E^{0,2}\Theta_{C_n} & \xrightarrow{\bar{\partial}} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & u_n^*\Theta_{P_n} & \to & Eu_n^*\Theta_{P_n} & \xrightarrow{\bar{\partial}} & E^{0,1}u_n^*\Theta_{P_n} & \xrightarrow{\bar{\partial}} & E^{0,2}u_n^*\Theta_{P_n} & \xrightarrow{\bar{\partial}} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where $E^{p,q}\#$ means the C^∞-sections of the sheaf $\#$. The complex is vertically exact and the complex of global sections computes horizontally the cohomology of the first sheaves.

We may choose a harmonic basis for $H^1(P_n, \Theta_{P_n})$ of the form $\varphi_{ij} = (\partial/\partial z_i)\, d\bar{z}_j$ where (z_1, z_2) are affine coordinates of the universal cover of P_n. Since the spectral sequence of $u_{n*}(u_n^*\Theta_{P_n}) = \Theta_{P_n} \otimes u_{n*}\mathcal{O}_{C_n} = \Theta_{P_n}$ degenerates by Proposition 9, we may consider φ_{ij} as elements of $H^0(C_n, E^{0,1}\Theta_{P_n})$ and they form a basis for $H^1(C_n, u_n^*\Theta_{P_n})$. Since $H^1(C_n, \Theta_{C_n}) \to H^1(C_n, u_n^*\Theta_{P_n})$ is an isomorphism, we may choose $\bar{\partial}$-closed liftings $\tilde{\varphi}_{ij} \in H^0(C_n, E^{0,1}\Theta_{C_n})$. In local coordinates $(y_1, \cdots, y_{n-2}, z_1, z_2)$

$$\tilde{\varphi}_{ij} = (\partial/\partial z_i + A_{ij}(y, z))\, d\bar{z}_j + \textstyle\sum_{m=1}^{n-2} B_{ij}{}^m(y, z)\, d\bar{y}_m$$

where A and B are sections of τ, that is vector fields in $\partial/\partial y_m$. Since $[\varphi_{ij}, \varphi_{rs}] = 0$, and by the local definition of the Poisson bracket of (0, 1)-vector forms we have $[\tilde{\varphi}_{ij}, \tilde{\varphi}_{rs}] \in H^0(C_n, E^{0,2}\tau)$, but since $H^2(C_n, \mathcal{T}) = 0$ by part a) we have that $[\tilde{\varphi}_{ij}, \tilde{\varphi}_{rs}] = \bar{\partial}\psi$ with $\psi \in E^{0,1}\mathcal{T}$. But this implies that on the level of cohomology the Poisson bracket is trivial, hence $H^*(P_n, \Theta_{P_n})$ is an Abelian Lie Algebra. ||

The fact that the Lie algebra is abelian means that all first order obstructions vanish. Whether higher order obstructions vanish for $n > 2$ is unsettled at this point.

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