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# ON THE TIGHTNESS OF A TOPOLOGICAL SPACE

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## 1. Introduction

In this note a new view of the tightness of a topological space is presented in terms of the concepts of quotient mappings and weak topologies generated by collections of subspaces. Tightness is defined by Juhasz and its interaction with various other cardinal functions are presented in [10]. The usefulness of this concept extends beyond studies dealing only with cardinal functions and it is the purpose of this study to present the fundamental but hidden connection between this cardinal function and some of its topological properties. For example, the main results in Section 3 include a characterization of spaces of tightness $m$ in terms of spaces with the weak topology generated by spaces of cardinality $\leq m$. Quotient spaces of $m$-tight spaces are $\leq m$-tight. That is, tightness is monotonic decreasing for quotient mappings. Further, tightness is preserved by countable to one closed continuous mappings.

## 2. Preliminaries

From [10], we can define the *tightness of a topological space*, $t(X)$, to be the least infinite cardinal number $m$ such that for each $A \subset X$ and for each $p \in \mathrm{cl}(A)$, there is some $B \subset A$ such that $\mathrm{card}(B) \leq m$ and $p \in \mathrm{cl}(B)$. From the definition, tightness certainly appears to be a concept which distinguishes points versus distinguishing non-closed sets. The subtle differences and implications of the "points versus non-closed set" viewpoint are the subject of many papers concerning quotient spaces whose topologies are determined by a specific collection of subspaces as: $k'$-spaces versus $k$-spaces [2], Frechet spaces versus sequential spaces [8], [9], $\Sigma'$-spaces versus $\Sigma$-spaces [7], convergence base versus convergence subbase [11] and $\Omega$-net spaces [12]. The unifying work in this area is the paper by Franklin [7] on natural covers. This study will be presented in the framework of natural covers and most of the following notions are found in or are direct variations of ideas in [7]. A *natural cover* is a function $\Sigma$ which assigns each space $X$ to a cover $\Sigma_X$ satisfying **a.)** if $S \in \Sigma_X$ and $S$ is homeomorphic to a subspace $T \subset Y$, then $T \in \Sigma_Y$ and **b.)** if $f: X \to Y$ is continuous and $S \in \Sigma_X$ there is a $T \in \Sigma_Y$ with $f(S) \subset T$. The *weak topology* on a space $X$ generated by the cover $\Sigma_X$ is the collection of all sets $U$ such that $U \cap H$ is open in $H$ for each $H \in \Sigma_X$. A space $X$ is a $\Sigma$-*space* whenever the topology of $X$ is the weak topology generated by $\Sigma_X$. In [7] many properties of spaces with the weak topology generated by natural covers are established and a vast list of consequences, which are used here at will, is included. Ordinal numbers are assigned to $\Sigma$-space as follows: for $A \subset X$, let $\mathrm{cl}_\Sigma(A) = \cup\{\mathrm{cl}_H(A \cap H): H \in \Sigma_X\}$ and let $A^0 = A$, if $\alpha = \beta + 1$ let $A^\alpha = \mathrm{cl}_\Sigma(A^\beta)$ and $A^\alpha = \cup\{A^\beta: \beta < \alpha\}$ for limit ordinals $\alpha$. The $\Sigma$-*characteristic* of $X$ is the least ordinal $\alpha$, if it exists, such that $A^\alpha = \mathrm{cl}(A)$, for each $A \subset X$. (See also Baire Order in [11], sequential and compact order in [4], [5], [6].) Franklin continues with [7, Prop. 2.14], $X$ is a $\Sigma$-space if and only if $X$ has a $\Sigma$-characteristic.

## 3. Tightness *m*

We begin by considering the weak topology generated by the collection of sets of cardinality $\leq m$. If $P_m(X) = \{B \subset X : \text{card}(B) \leq m\}$, then $P_m(X)$ forms a natural cover of $X$. Let $T_m$ be the weak topology generated by $P_m(X)$. Hence a set $U \subset X$ will be called *m-open* if and only if $U \cap H$ is open in $H$ for each $H \in P_m(X)$. Then $T_m = \{U : U \text{ is } m\text{-open}\}$ is a topology for $X$ which is finer than the original topology. We will call $X$ retopologized with this finer topology the *m-extension* of $X$ and will denote this extension by $mX$. $X$ has the weak topology determined by $P_m(X)$ provided $mX = X$ (or every $m$-open set is open). If $n \leq m$, then an $m$-open set is $n$-open and $T_m \subset T_n$. (To be in $T_m$ a set must intersect more sets in open subsets than to be in $T_n$). For every pair of infinite cardinal numbers, considered as the least ordinal of a certain cardinality, say $\omega_\alpha < \omega_\beta$, $\{\omega_\beta\}$ is $\omega_\alpha$-open in the ordinal space $[0, \omega_\beta]$ but $\{\omega_\beta\}$ is not *$\omega_\beta$-open in* $[0, \omega_\beta]$. $P_m(X)$ is a convergence subbase in the terminology of Meyer [11]. An $m$-net is a net with a directed set of cardinality $m$. The class of $m$-nets forms a natural cover and a space with the weak topology determined by $m$-nets is called $m$-sequential [11]. Every $m$-sequential space is $m$-tight. The distinction between $m$-tight as a cardinality condition on a set of points in the space and $m$-sequential as a cardinality condition on the directed set of a net in the space is illustrated in the following. Arens space $A$ [1] is $2^\omega$-Frechet and thus $2^\omega$-sequential, but it is not sequential ($\omega$-sequential). Since $A$ is countable, $A$ is $\omega$-tight. This shows that for $n < m$, the $n$-extension of an $m$-sequential ($m$-Frechet) space may fail to be $n$-sequential.

It would appear that "weak topology generated by $P_m(X)$" (a condition distinguishing non-closed sets) is weaker than "$m$-tight" (a condition distinguishing points in the closure of a set). However the weak topology generated by $P_m(X)$ is much stronger than we might expect as seen in the following theorem.

THEOREM 3.1. *$X$ has the weak topology determined by the collection of subspaces of cardinality $\leq m$ if and only if $X$ has tightness $\leq m$.*

*Proof:* Let $X$ have the weak topology generated by $P_m(X)$. Let $m\text{cl}(A) = \cup\{\text{cl}_H(A \cap H) : H \in P_m(X)\}$. The key here is that $m\text{cl}(m\text{cl}(A)) = m\text{cl}(A)$. Thus $A^n = A^1$ for each $n < \omega$, $A^\omega = A^1$ and in general $A^\alpha = A^1$, for each ordinal $\alpha$. Since $\text{cl}(A) = A^\alpha$ for some $\alpha$, by the bound of the cardinality of the space, $\text{cl}(A) = A^1$ for each $A \subset X$. Thus the $P_m(X)$-characteristic is 1 and $X$ is a $P_m'(X)$-space. Hence if $p \in \text{cl}(A)$, there exists $B \in P_m(X)$ such that $B \subset A$ and $p \in \text{cl}(B)$. That is, if $X$ has the weak topology determined by $P_m(X)$, then $X$ is $m$-tight. Conversely, if $X$ is $m$-tight and $A$ is not closed with $p \in \text{cl}(A) - A$, there is a set $B \in P_m(X)$ such that $B \subseteq A$ and $p \in \text{cl}(B)$. Let $B_p = B \cup \{p\}$. Then $B_p \in P_m(X)$ and $B_p \cap A$ is not closed in $B_p$. Thus, $X$ has the weak topology determined by $P_m(X)$. This completes the proof.

From the preceding proof every $P_m(X)$-space is a $P_m'(X)$-*space* (*the convergence subbase* $P_m(X)$ is a convergence base). The Baire order of each point is 1 [11] and thus the monotonic properties of the tightness order (=1) under mappings are trivial. However, the following theorem is surprising in that the cardinal invariant tightness, $t$, is monotonic decreasing under the quotient mappings. Recall that from [5], the sequential or compact order of a space may increase under a quotient and monotonicity is guaranteed under the stronger pseudo-open mappings.

THEOREM 3.2. *if* $f{:}X \to Y$ *is a quotient mapping, then* $t(X), \geq t(Y)$.

PROOF: Let $A$ be a subset of $Y$ which is not closed. Since $f^{-1}(A)$ is not closed, there exists $B \in P_m(X)$ such that $B \cap f^{-1}(A)$ is not closed in $B$. Since $B \in P_m(X)$, $f(B) \in P_m(Y)$. If $f(B) \cap A$ is closed in $f(B)$, then $f^{-1}(f(B)) \cap f^{-1}(A)$ is closed in $f^{-1}(f(B))$. Thus since $B \subset f^{-1}(f(B))$ has a cluster point which is not in $f^{-1}(A)$, $f^{-1}(f(B)) \cap f^{-1}(A)$ is not closed in $f^{-1}(f(B))$. Thus $f(B) \cap A$ is not closed in $f(B)$. Since there exists $f(B) \in P_m(Y)$ such that $f(B) \cap A$ is not closed in $f(B)$, $Y$ has the weak topology generated by a subcollection of $P_m(Y)$. Then $Y$ has tightness $\leq m$ and this completes the proof.

COROLLARY 3.3. *$X$ has tightness $m$ if and only if $X$ is the quotient space of the disjoint topological sum of spaces of cardinality $\leq m$.*

Hence $X$ is $m$-tight if and only if $X$ is the quotient space of an $m$-tight space. In Proposition 3.6 [7], a space is a $\Sigma'$-space if and only if the natural mapping $f{:}\oplus\Sigma_X \to X$ is pseudo-open. A pseudo-open mapping [3] can be characterized as a mapping $f{:}X \to Y$ such that if $p \in \mathrm{cl}(A) \subset Y$, then $f^{-1}(p) \cap \mathrm{cl}(f^{-1}(A)) \neq \emptyset$. Thus, the natural mapping from the sum of the space in $P_m(X)$ onto $X$ must be pseudo-open as follows.

COROLLRY 3.4. *If $X$ has tightness $m$, then the natural mapping $f$ from the sum of the spaces in $P_m(X)$ onto $X$ is pseudo-open.*

*Proof*: Let $p \in \mathrm{cl}(A) \subset Y$ and let $f$ be the natural mapping. Then there exists $B \in P_m(X)$ such that $B \subset A$ and $p \in \mathrm{cl}(B)$. Let $B_p = B \cup \{p\}$. Then $B_p$ is a space in the sum of the spaces in $P_m(X)$. Thus, $p \in f^{-1}(p) \cap \mathrm{cl}_{B_p}(f^{-1}(B)) \subset f^{-1}(p) \cap \mathrm{cl}(f^{-1}(A))$ and thus $f$ is pseudo-open. This completes the proof.

Since $P_m(X)$ is a collection which forms a natural cover and by Theorem 3.1 the notion of tightness is uniquely characterized by the weak topology determined by $P_m(X)$, by Lemma 2.2 [7] we have the next corollary.

COROLLARY 3.8. *The spaces of tightness $\leq m$ form a coreflective subcategory of TOP.*

Unlike most other structures with the weak topology determined by collections of subsets, tightness has very strong hereditary and productive properties as seen in the next two propositions.

PROPOSITION 3.6. *If $C$ is a subspace of $D$, then $t(C) \leq t(D)$.*

*Proof:* Let $t(D) = m$, and let $Z \subset C$ and $p \in \mathrm{cl}_C(Z)$. Then $p \in \mathrm{cl}_D(Z)$ and there exists $H \subset Z$ such that $p \in \mathrm{cl}_D(H)$ and $\mathrm{card}(H) \leq m$. Since $p \in C$, $p \in \mathrm{cl}_D(H) \cap C = \mathrm{cl}_C(H)$. Thus, $H$ is a subset of $Z$ such that $p \in \mathrm{cl}_C(H)$ and $\mathrm{card}(H) \leq m$. Thus, $t(C) \leq m$ and this completes the proof.

From the hereditary property in the previous proposition or the preservation of tightness under quotient mappings, we have the following proposition for the productive properties of tightness. This statement appears as a part of Theorem 4.2 of [10].

PROPOSITION 3.7. *If $t(\Pi X_\alpha) \leq m$, then $t(X_\alpha) \leq m$, for each $\alpha$.*

From Theorem 3.2, tightness is monotonic decreasing under quotient mappings. Tightness may increase under a one to one continuous mapping. For example, consider the identity mapping from the sequential extension of the ordinal space $[0, \omega_1]$ onto itself. Here $t(s[0, \omega_1]) = \omega_0$, but $t([0, \omega_1]) = \omega_1$. The next theorem establishes that the tightness is preserved under countable to one closed continuous mappings. The example that follows shows that the cardinality condition is required.

THEOREM 3.8. *If $f{:}X \to Y$ is a countable to one closed continuous surjection, then $t(X) = t(Y)$.*

*Proof:* Assume $t(X) = m > n = t(Y)$. Then there exists $H \subset X$ such that $H$ is $P_n(X)$-closed but not closed. Let $p \in \mathrm{cl}(H) - H$. Since $\mathrm{card}(f^{-1}(f(p))) \leq X_0$, $p \notin \mathrm{cl}(f^{-1}f(p)) \cap H)$. Then $p \in \mathrm{cl}(H - f^{-1}(f(p)))$. Let $H^* = H - f^{-1}(f(p))$. Since $f(p) \in \mathrm{cl}(f(H^*)) - f(H^*)$, there exists $C \subset f(H^*)$ such that $\mathrm{card}(C) \leq n$ and $f(p) \in \mathrm{cl}(C)$. Since $\mathrm{card}(f^{-1}(C)) \leq n$ and $H$ is $P_n(X)$- closed, $\mathrm{cl}(f^{-1}(C) \cap H^*) \subset H^*$. Accordingly, $C = f(f^{-1}(C) \cap H^*) \subset f(\mathrm{cl}(f^{-1}(C) \cap H^*)) \subset f(H^*)$. Thus since $f(\mathrm{cl}(f^{-1}(C) \cap H^*)$ is a closed set, $\mathrm{cl}(C) \subset f(H^*)$. Since $f(p) \notin f(H^*)$, $f(p) \notin \mathrm{cl}(C)$ which is a contradiction. Thus $t(X) \leq t(Y)$ and since $f$ is a quotient mapping by Theorem 3.2, $t(X) \geq t(Y)$. Hence $t(X) = t(Y)$ and this completes the proof.

EXAMPLE 3.9. *An open, perfect continuous mapping can decrease tightness from any cardinal $\alpha$ to $\omega_0$.*

Let $\alpha$ be any infinite cardinal and let $[0, \alpha]$ be the ordinal space. Let $S_1 = \{0\} \cup \{1/n{:}n \in N\}$ be the usual subspace of $R$. Then $S_1 \times [0, \alpha]$ is compact and the projection $p{:}S_1 \times [0, \alpha] \to S_1$ is open perfect and continuous. Then $t(S_1 \times [0, \alpha]) = \alpha$, but $t(p(S_1 \times [0, \alpha])) = \omega_0$.

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