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EMBEDDING THE DEFORMATION SPACE OF A FUCHSIAN GROUP OF THE FIRST KIND

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1. Introduction

A coordinate covering of a Riemann surface S such that the transition functions are the restrictions of elements of the group of all Möbius transformations Aut $\hat{\mathbf{C}} = PSL(2, \mathbf{C})$ is said to determine a *projective structure* (or *Möbius structure*) on S.

Let $\psi_0: U \to S$, $U \subset \hat{\mathbf{C}}$, be a coordinate chart and let ψ be a lift of ψ_0 to the universal cover Δ of S. Extending ψ^{-1} analytically to all of Δ yields a meromorphic local homeomorphism f of Δ into $\hat{\mathbf{C}}$. Identifying $\pi_1(S)$ with the covering group G of S, we obtain a homomorphism $\chi: G \to$ Aut $\hat{\mathbf{C}}$ satisfying $\chi(\gamma)\circ f = f\circ\gamma$, $\gamma \in G$. The pair (f, χ) is called a *deformation* of G; an *equivalent* deformation (one corresponding to the same projective structure) is obtained by replacing f with $A\circ f$ for any $A \in$ Aut $\hat{\mathbf{C}}$.

Now assume Δ is the unit disk and G is a Fuchsian group acting on Δ. We are interested in the case that S is a surface of type (p, n), that is, a compact surface of genus p from which n points have been removed. Given a quadratic differential ϕ on S one obtains in a natural way (by solving a Schwarzian differential equation) a "normalized" mapping f and a deformation (χ, f). Taking an ordered set of r generators for G, the images of these generators under χ define a point in (Aut $\hat{\mathbf{C}})^r$; this point determines χ completely. The image of the mapping Φ: {quadratic differentials} $\to$ (Aut $\hat{\mathbf{C}})^r$ defined in this manner consists of points representing all possible conjugacy classes of homomorphisms of G arising from projective structures.

For the compact case ($p \geq 2$, $n = 0$), Gunning [5] and Kra [9] have shown that Φ is a holomorphic injective mapping; various authors (see Earle [1], Hejhal [7], Hubbard [8]) have shown that Φ is nonsingular. Injectivity for the case of punctures was proven by Kra [12], provided of course that one restricts the domain of Φ to the finite dimensional space $B_2(G)$ of bounded quadratic differentials. In this article we show that Φ is nonsingular for surfaces with punctures. The idea of the proof presented here is the same as that in [1].

2. Notation and statement of main result

G will always denote a finitely generated, torsion free Fuchsian group acting on $\Delta = \{z : |z| < 1\}$ such that Δ/G is a surface of type (p, n). Aut $\hat{\mathbf{C}}$ is the group of mappings γ of the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ *where* $\gamma(z) = (az + b)(cz + d)^{-1}$, a b, c, $d \in \mathbf{C}$ and $ad - bc = 1$.

Let $B_2(\Delta, G) = B_2(G)$ denote the vector space of quadratic differentials for G; that is, the space of functions ϕ, holomorphic in Δ, such that

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z),\ z \in \Delta, \gamma \in G:$$

$$\sup\{|\phi(z)|(1 - |z|^2),\ z \in \Delta\} < \infty.$$

If g is a meromorphic function on Δ we will write

$$[g] = (g''/g')' - \tfrac{1}{2}(g''/g')^2, \tag{3}$$

the *Schwarzian derivative* of g, which is holomorphic in Δ whenever g is a meromorphic local homeomorphism.

Let $[f_\phi] = \phi \in B_2(G)$ where $f = f_\phi$ is the particular solution satisfying the normalization

$$f(0) = 0, \qquad f'(0) = 1 \quad \text{and} \quad f''(0) = 0. \tag{4}$$

One may obtain f in the following manner: Let y_1 and y_2 be solutions of the differential equation

$$2y'' + \phi y = 0 \tag{5}$$

normalized by the initial conditions

$$y_1(0) = y_2'(0) = 0 \quad \text{and} \quad y_1'(0) = y_2(0) = 1.$$

Then $f = y_1/y_2$. It is well known that $f' = y_2^{-2}$.

The complete set of solutions of $[g] = \phi$ is comprised of the functions $g = A \circ f$ where $A \in \text{Aut}\, \hat{\mathbf{C}}$.

We fix a set $\gamma_1, \gamma_2, \cdots, \gamma_r$ of generators for G. For each $\gamma \in G$, $\chi(\gamma)$ is defined by the relation $\chi(\gamma) \circ f = f \circ \gamma$. Then we define

$$\Phi : B_2(G) \to (\text{Aut}\, \hat{\mathbf{C}})^r$$

by

$$\Phi(\phi) = (\chi(\gamma_1), \cdots, \chi(\gamma_r)).$$

(Here Aut $\hat{\mathbf{C}}$ is viewed as the Lie group $SL(2, \mathbf{C})$ modulo its center.) It follows from the theory of ordinary differential equations that Φ is holomorphic. It is injective by [5] and [9].

THEOREM. *The holomorphic function* $\Phi : B_2(G) \to (\text{Aut}\, \hat{\mathbf{C}})^r$ *defined by* $\Phi(\phi) = (\chi(\gamma_1), \cdots, \chi(\gamma_r))$ *is nonsingular.*

The proof of the Theorem for the compact case will be found in section 4, while the case of punctures will be treated separately in Section 5.

3. Variational formulas

In this section we derive an integral formula for the variation of the mapping function with respect to a complex parameter λ.

Let $\phi_0, \phi \in B_2(G)$ be two fixed quadratic differentials. For $\lambda \in \mathbf{C}$ and $z \in \Delta$ let $y(z, \lambda)$ be the solution of

$$2y'' + (\phi_0 + \lambda\phi)y = 0 \tag{6}$$

where

$$y(0, \lambda) = 1 \quad \text{and} \quad y'(0, \lambda) = 0.$$

Throughout y' means $\partial y/\partial z$; we will write $\dot{y}$ for $\partial y/\partial \lambda$.

Define

(7) $$f(z, \lambda) = \int_0^z y(t, \lambda)^{-2}\, dt.$$

Since $y(z, \lambda)$ has at most simple poles, it follows that (7) is well defined and independent of the path of integration. We will write $f_\lambda = f(z, \lambda)$. Hence $[f_\lambda] = \phi_0 + \lambda\phi$ and

(8) $$f_\lambda(0) = 0, \qquad f_\lambda'(0) = 1 \quad \text{and} \quad f_\lambda''(0) = 0, \qquad \lambda \in \mathbf{C}.$$

Differentiation of (6) with respect to λ gives an inhomogeneous linear equation in $\dot{y}$

$$2\dot{y}'' + (\phi_0 + \lambda\phi)\dot{y} = -\phi y,$$

which can be solved by the elementary method of variation of parameters in terms of $\dot{y}$. Specifically, substitution of $\dot{y} = uy$ yields

$$\begin{aligned} -2u(z) &= \int_0^z y(s)^{-2} \int_0^s \phi(t)y(t)^2\, dt\, ds \\ &= \int_0^z \phi(t)y(t)^2 \int_t^z y(s)^{-2}\, ds\, dt \\ &= \int_0^z \phi(t)y(t)^2(f(z) - f(t))\, dt \end{aligned}$$

at $\lambda = 0$. (The interchange of integrals may be justified by defining all the line integrals along radial segments.) Thus we have

(9) $$\dot{y}(z) = -(y(z)/2) \int_0^z \phi(t)y(t)^2(f(z) - f(t))\, dt$$

at $\lambda = 0$, except for the possible addition of a solution of the homogeneous equation (6). However, (9) is seen to hold as written because it satisfies the initial condition $\dot{y}(0) = \dot{y}'(0) = 0$.

Now from (3) we have $\dot{f}' = -2y^{-3}\dot{y}$, so

$$\begin{aligned} \dot{f}(z) &= \int_0^z y(s)^{-2} \int_0^s \phi(t)y(t)^2(f(s) - f(t))\, dt\, ds \\ &= \int_0^z \phi(t)y(t)^2 \int_t^z (f(s) - f(t))y(s)^{-2}\, ds\, dt, \end{aligned}$$

giving

(10) $$\dot{f} = \tfrac{1}{2} \int_0^z \phi(t)y(t)^2(f(z) - f(t))^2\, dt.$$

at $\lambda = 0$.

These formulas simplify considerably if one assumes that $\phi_0 = 0$, for then we have $y(z) = 1$ and $f(z) = z$ at $\lambda = 0$. Formulas (9) and (10) for this particular case are used implicitly in [2]. A formula similar to (9) appears in [6].

4. Proof of the theorem for the compact case

We fix two elements $\phi_0, \phi \in B_2(G)$ and suppose that

(11) $$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \Phi(\phi_0 + \lambda\phi) = 0, \qquad \lambda \in \mathbf{C}\ .$$

Then it suffices to show that $\phi = 0$.

We note that for each $\gamma \in G$ there is a unique element $\gamma_\lambda \in \operatorname{Aut} \hat{\mathbf{C}}$ such that

$$\gamma_\lambda \circ f_\lambda = f_\lambda \circ \gamma, \qquad \lambda \in \mathbf{C} . \tag{12}$$

We regard γ_λ as a matrix in $SL(2, \mathbf{C})$, determined up to sign. It follows from (11) that

$$\dot{\gamma} = \frac{d}{d\lambda}\bigg|_{\lambda=0} \gamma_\lambda = 0 \tag{13}$$

for each of the generators $\gamma = \gamma_i$, hence for all $\gamma \in G$. Consequently, if we now regard γ_λ as a Möbius transformation, an easy computation shows

$$\dot{\gamma}(z) = \frac{\partial}{\partial\lambda}\bigg|_{\lambda=0} \gamma_\lambda(z) = 0, \qquad z \in \hat{\mathbf{C}} \quad \text{and} \quad \gamma \in G. \tag{14}$$

Differentiating (12) with respect to z and λ yields

$$(\gamma_\lambda' \circ f_\lambda) f_\lambda' = (f_\lambda' \circ \gamma)\gamma' \quad \text{and} \tag{15}$$

$$(\gamma_\lambda' \circ f_\lambda)\dot{f}_\lambda + \dot{\gamma}_\lambda \circ f_\lambda = \dot{f}_\lambda \circ \gamma \tag{16}$$

respectively. Setting $\lambda = 0$ and combining these relations with (14) yields

$$\frac{\dot{f} \circ \gamma}{f' \circ \gamma}(\gamma')^{-1} = \frac{\dot{f}}{f'} \quad \text{for all} \quad \gamma \in G.$$

Thus $(\dot{f}/f')|_{\lambda=0}$ is an automorphic form for G of degree (-1). Since $\dot{f}/f' = (\dot{y}_1 y_2 - \dot{y}_2 y_1)/(y_1' y_2 - y_2' y_1)$ and the Wronskian $y_1'y_2 - y_2'y_1$ is identically equal to 1, one sees that $\dot{f}/f'$ is holomorphic. Since Δ/G is a compact Riemann surface and the dimension of the vector space of holomorphic automorphic forms of degree (-1) is zero (see for instance [9]), we conclude that $\dot{f}_{\lambda|=0} = 0$.

Hence by (10)

$$\textstyle\int_0^z \phi(t) y(t)^2 (f(z) - f(t))^2 \, dt = 0. \tag{17}$$

By calculating the third derivative of (17) with respect to z, we obtain $\phi y^2 \equiv 0$. Hence $\phi \equiv 0$. This concludes the proof of the theorem.

Recalling that $\Phi(\phi)$, $\phi \in B_2(G)$, represents a projective structure on Δ/G, we now consider the effect of taking different representatives by allowing normalizations other than (8).

Let $\phi \mapsto A_\phi$ be a holomorphic function from $B_2(G)$ into $\operatorname{Aut} \hat{\mathbf{C}}$. For each $\phi \in B_2(G)$ and $\gamma \in G$ let $\chi_A(\gamma) = A_\phi \circ \chi(\gamma) \circ A_\phi^{-1}$ (where χ is defined in section 2). We define $\Phi_A : \mathrm{B}_2(\mathrm{G}) \to (\operatorname{Aut} \hat{\mathbf{C}})^r$ by

$$\Phi_A(\phi) = (\chi_A(\gamma_1), \cdots, \chi_A(\gamma_r)).$$

By the results cited earlier Φ_A is holomorphic and injective.

COROLLARY. *Φ_A is nonsingular.*

Proof. Let ϕ_0, $\phi \in B_2(G)$. The homomorphism χ_A corresponding to $\phi_0 + \lambda\phi$, $\lambda \in \mathbf{C}$, is characterized by $\chi_A(\gamma) \circ g_\lambda = g_\lambda \circ \gamma$ where $\gamma \in G$ and

$$g_\lambda = A_\lambda \circ f_\lambda. \tag{18}$$

A_λ is of course an abbreviation for $A_{\phi_0+\lambda\phi}$.

As in the proof of the theorem, the assumption that $\left.\frac{d}{d\lambda}\right|_{\lambda=0} \Phi_A(\phi_0 + \lambda\phi) = 0$ implies that $(\dot{g}/g')|_{\lambda=0}$ is holomorphic (-1) differential and hence $\dot{g}|_{\lambda=0} = 0$. Assuming this then, differentiation of (18) with respect to λ yields

$$-(\dot{A} \circ f)/(A' \circ f) = \dot{f}.$$

Observe, on the other hand, that

$$-\dot{A}(z)/A'(z) = a_2 z^2 + a_1 z + a_0 \quad \text{at} \quad \lambda = 0,$$

where a_0, a_1, $a_2 \in \mathbf{C}$. We now have

$$a_2 f(z)^2 + a_1 f(z) + a_0 = \dot{f}(z),\ z \in \Delta.$$

It follows from (8) that

$$\dot{f}(0) = \dot{f}'(0) = \dot{f}''(0) = 0$$

and one deduces that $a_0 = a_1 = a_2 = 0$. Hence $\dot{f}|_{\lambda=0} = 0$. The proof now proceeds as before

5. Surfaces with punctures

We now permit S to have finitely many punctures, $S = \hat{S} - \{s_1, s_2, \cdots, s_n\}$ where $\hat{S}$ is compact. We consider the variation of the deformation $(f_\lambda, \chi_\lambda)$ determined by $\phi_0 + \lambda\phi$ with ϕ_0, $\phi \in B_2(G)$ as in the previous section. Clearly the proof of the theorem and the corollary will carry through exactly as before if only we show that $(\dot{f}/f')|_{\lambda=0}$ projects to the restriction of a (-1) differential holomorphic on $\hat{S}$.

Let $z_0 \in \partial\Delta$ be the fixed point of a parabolic element $P \in G$. Thus a cusped region R in Δ at z_0 projects to a (punctured) neighborhood in S of one to the s_j. Let $B \in \operatorname{Aut} \hat{\mathbf{C}}$ take R to the horizontal strip $\{\operatorname{Re} w < 0, |\operatorname{Im} w| < \pi\}$. Then the variable $\zeta = e^{B(z)}$, $|\zeta| < 1$, is a local coordinate on $\hat{S}$ at s_j.

By a lemma of Kra [11], the transformed element $P_\lambda = \chi_\lambda(P)$ is always parabolic. Let $p(\lambda) \in \hat{\mathbf{C}}$ be its fixed point, and define

$$C_\lambda(z) = (z - p(\lambda))^{-1}. \tag{19}$$

(We can assume that $p(0) \neq \infty$ by replacing f by $A \circ f$ for some $A \in \operatorname{Aut} \hat{\mathbf{C}}$.) Thus

$$C_\lambda \circ P_\lambda \circ C_\lambda^{-1}(z) = z + b(\lambda),\ b(\lambda) \neq 0. \tag{20}$$

Clearly, C_λ, $b(\lambda)$ are holomorphic.

Recall that we are assuming $(d/d\lambda)\Phi(\phi_0 + \lambda\phi)|_{\lambda=0} = 0$. Hence by (14), $\dot{P}$

= 0, and by (19) and then (20) we have

$$\dot{C}_0 = 0, \qquad \dot{b}(0) = 0. \tag{21}$$

Now define

$$g_\lambda = C_\lambda \circ f_\lambda. \tag{22}$$

Writing $w = B(z)$, we see that $g \circ B^{-1}(w)$ increases by the constant $b(\lambda)$ as w increases by $2\pi i$. Thus $(g \circ B^{-1})'$ is periodic in w and may be regarded as a single-valued function of the variable ζ in $0 < |\zeta| < 1$. Kra ([11], page 503) has shown that we may write

$$g_\lambda(z) = g_\lambda \circ B^{-1}(w) = \textstyle\sum_{k=0}^{\infty} b_k(\lambda)\zeta^k + b(\lambda)B(z). \tag{23}$$

Since $g_\lambda'(z)$ is holomorphic in λ and meromorphic as a function of ζ, one sees immediately that all the coefficients $b_k(\lambda)$ are holomorphic functions of λ.

By (21), (22) and the chain rule, we have that $\dot{g}/g' = \dot{f}/f'$ at $\lambda = 0$. Now by (21) and (23),

$$\dot{g}(z) = \textstyle\sum_0^\infty \dot{b}_k(0)\zeta^k, \tag{24}$$

$$g'(z) = (\textstyle\sum_{k=1}^{\infty} kb_k(0)\zeta^k + b(0))B'(z). \tag{24'}$$

Consequently the differential $(\dot{f}/f')\,dz^{-1}$, written in terms of the local coordinate ζ on $\hat{S}$, becomes

$$\begin{aligned}(\dot{f}(z)/f'(z))\,dz^{-1} &= (\dot{g}(z)/g'(z))(d\zeta/dz)\,d\zeta^{-1}\\ &= (\textstyle\sum_0^\infty \dot{b}_k(0)\zeta^{k+1})(\sum_1^\infty kb_k(0)\zeta^k + b(0))^{-1}\,d\zeta^{-1}.\end{aligned} \tag{25}$$

Since $b(0) \neq 0$, this differential is holomorphic in the neighborhood of $\zeta = 0$, and the proof is concluded.

Comments. Following Gunning [3], let $R_0 \subset SL(2, \mathbf{C})^r$ be the subset of r-tuples of matrices whose underlying Möbius transformations have no common fixed point. Let V_0 denote the quotient of R_0 modulo conjugation by elements of Aut $\hat{\mathbf{C}}$. Since $B_2(G)$ is simply connected, we may life Φ to a holomorphic function $\Phi_0 : B_2(G) \to SL(2, \mathbf{C})^r$. It is nonsingular by our results. An argument given by Kra ([11], Lemma 4) can be used to show that for $\phi \in B_2(G)$, $\chi(G)$ is never an affine group. Hence the image of Φ_0 lies in R_0. Following Φ_0 by the natural projection onto V_0, we obtain a map $\Phi_0^* : B_2(G) \to V_0$, where G is an arbitrary Fuchsian group representing a surface of type (p, n).

Gunning [3], [4] describes a natural holomorphic structure on V_0 in the following manner: each $x \in R_0$ lies in a codimension-3 submanifold U of R_0 such that the function $(u, A) \to AuA^{-1}$ is a biholomorphic mapping of $U \times$ Aut $\hat{\mathbf{C}}$ onto a neighborhood of x. A neighborhood U_0 of the projection $x_0 \in V_0$ of x is therefore homeomorphic to $U \times$ {identity}. Thus we may construct a holomorphic section $s : U_0 \to U$. It is easy to verify that $s \circ \Phi_0^*$ is of the form Φ_A as in the Corollary. It follows that Φ_0^* is nonsingular.

Replacing all matrices in the construction above by their underlying Möbius transformations we may form R, V, Φ^* analogous to R_0, V_0, $\Phi_0{}^*$ (note that $R \subseteq (\text{Aut } \mathbf{C})^r$) where $\Phi^*: B_2(G) \to V$ is holomorphic. V_0 is a branched cover of V; however, by Kra ([11], Theorem 1) $\chi(G)$ cannot be a finite group, so $\Phi_0{}^*(\phi) \in V_0$ never lies over the branched set of V for any $\phi \in B_2(G)$. Hence Φ^* is nonsingular.

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References

[1] C. J. Earle, *On variation of projective structures*, Proceedings of the 1978 Stony Brook conference, Ann. of Math. Studies, Princeton University Press, **97**(1981), 87–100.

[2] F. Gardiner and I. Kra, *Stability of Kleinian groups*, Indiana Univ. Math. J., **21**(1972), 1037–1059.

[3] R. C. Gunning, *Analytic structures on the space of flat vector bundles over a compact Riemann surface*, Several Complex Variables II, Maryland, 1979. Springer Lecture Notes, **185**(1971), 47–62.

[4] ———, *Lectures on Riemann surfaces*, Princeton University Press, (Mathematical Notes 2), 1966.

[5] ———, *Special coordinate coverings of Riemann surfaces*, Math. Ann., **170**(1967), 67–86.

[6] D. Hejhal, *The variational theory of linearly polymorphic functions*, J. d'Analyse Math., **30**(1976), 215–264.

[7] ———, *Monodromy groups and Poincaré series*, Bull. Amer. Math. Soc., (1978), 339–376.

[8] J. Hubbard, *The monodromy of projective structures*, Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Studies, Princeton University Press, **97**(1981), 257–276.

[9] I. Kra, *On affine and projective structures on Riemann surfaces*, J. d'Analyse Math., **22**(1969), 285–298.

[10] ———, *Deformations of Fuchsian groups*, Duke Math. J., **36**(1969), 537–546.

[11] ———, *Deformations of Fuschian groups II*, Duke Math. J., **38**(1971), 499–508.

[12] ———, *A generalization of a theorem of Poincaré*, Proc. Amer. Math. Soc., **27**(1971), 299–302.