Boletín de la Sociedad Matemática Mexicana Vol. 26 No. 2 1981

THE CONVERGENCE OF APPROXIMATING FRACTIONS

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1. Introduction and summary

Two methods for transforming possibly divergent power series have been extensively studied. The first of these was introduced by le Roy, and is an extension of the integral transformation subsequently investigated in detail by Borel; under appropriate conditions it yields an integral expression for the suitably interpreted sum of the series being transformed. The second method involves the iterative construction of approximating fractions, of which some are convergents of an associated continued fraction, and yields a sequence of rational functions which, again under appropriate conditions, converges to the sum in question.

In this paper, two general results concerning the convergence of approximating fractions derived from power series whose coefficients are Hamburger and Stieltjes moments respectively are derived. An extension of le Roy's method of transforming power series is introduced. After conditions upon the rate of growth of the above moments have been imposed, the summability of the series by use of le Roy's method and its extension is demonstrated, the convergence of certain sequences of approximating fractions obtained from the series is established, and the consistency of these two diverse methods of defining the sum of the series considered is proved. The results derived are refinements and extensions of theorems due to F. Bernstein, Hamburger and Wall.

2. Notation and preliminaries

With α, β prescribed real numbers ($\alpha \leq \beta$), $[\alpha, \beta]^{-1}$ is the set of points $\{t: (1/t) \in [\alpha, \beta]\}$. With θ_1, θ_2 prescribed real numbers ($\theta_1 < \theta_2$), $\Delta(\theta_1, \theta_2)$ is the finite open sector containing the points of the set $\{z: \theta_1 < \arg(z) < \theta_2, 0 \leq |z| < \infty\}$; with $\theta_1 \leq \theta_2$, $\bar{\Delta}(\theta_1, \theta_2)$ is the finite closed sector (or ray) containing the points of the set $\{z: \theta_1 \leq \arg(z) \leq \theta_2, 0 \leq |z| < \infty\}$. With M a prescribed set of points in the complex plane, $\mathsf{BE}(\mathsf{M})$ is the point set $\{z: |z - t| \geq \delta$ for all $t \in \mathsf{M}, |z| < T\}$ where $\delta \in (0, \infty)$ arbitrarily small and $T \in (0, \infty)$ arbitrarily large are fixed. An expression such as z^{α}, where z is complex and $\alpha \in (-\infty, \infty)$, refers to that branch of this function which assumes positive real values for positive real z. $x!$ denotes $\Gamma(x + 1)$ for general real values of x.

The index of single summation is always ν; if the upper limit is infinity it is omitted from the summation sign; if the lower limit is also zero, it too is omitted: $\Sigma_1^n a_\nu$, $\Sigma_1 a_\nu$ and Σa_ν denote $\Sigma_{\nu=1}^n a_\nu$, $\Sigma_{\nu=1}^\infty a_\nu$ and $\Sigma_{\nu=0}^\infty a_\nu$ respectively. Order relationships (e.g. $f_\nu = O(\nu!)$) are tacitly assumed to hold for values of

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* The Director of the CIMAT very kindly placed the resources of this Institute at the disposal of the author, who wishes to record his gratitude.

the argument tending to infinity; furthermore, use of simple order relationships such as $f_\nu = O(\nu!\xi^\nu)$, $f_\nu = O(\xi'^\nu)$, $\cdots$ implies that ξ, ξ', $\cdots$ are fixed finite positive real numbers.

Simple integral expressions and Stieltjes integral expressions denote Riemann integrals and Riemann-Stieltjes integrals respectively. All real valued functions defined over an interval of the real axis are assumed to be normalised by a condition of the form $\sigma(t) = \frac{1}{2}\{\sigma(t+) + \sigma(t-)\}$ for all $t \in (\alpha, \beta)$, $[\alpha, \beta]$ being the interval over which σ is defined. With α, β prescribed real numbers $(\alpha \le \beta)$, $\sigma \in \mathbf{BN}([\alpha, \beta])$ means that σ is a real valued function, bounded and nondecreasing over $[\alpha, \beta]$ with $\int_\alpha^\beta d\sigma(t) > 0$, and such that all moments

$$f_\nu = \int_\alpha^\beta t^\nu \, d\sigma(t) \quad (\nu = 0, 1, \cdots) \tag{1}$$

exist; conditions possibly imposed upon $[\alpha, \beta]$ are inserted in the parentheses of the above notation; $\sigma \in \mathbf{BN}([\alpha, \beta] \subseteq [0, \infty])$ means that $\sigma \in \mathbf{BN}([\alpha, \beta])$ where $[\alpha, \beta] \subseteq [0, \infty]$. With $[\alpha, \beta]$, $[\alpha', \beta']([\alpha, \beta] \subseteq [\alpha', \beta'])$ and $\sigma \in \mathbf{BN}([\alpha, \beta])$ prescribed, $\sigma' = e(\sigma; \alpha', \beta')$ is the extension of σ over $[\alpha', \beta']$, so that $\sigma'(t) = \sigma(\alpha)$ for $\alpha' \le t < \alpha$ if $\alpha' < \alpha$, $\sigma'(t) = \sigma(t)$ for $\alpha \le t \le \beta$, and $\sigma'(t) = \sigma(\beta)$ for $\beta < t \le \beta'$ if $\beta' > \beta$. With $\sigma \in \mathbf{BN}([\alpha, \beta] \subseteq [-\infty, \infty])$ and the fixed finite integer $m \ge 0$ prescribed, $\sigma^{(m)}$ is the function for which $d\sigma^{(m)}(t) = t^m \, d\sigma(t)(\alpha \le t \le \beta)$. The symbol $\{f_\nu\} = \mathbf{MS}\{\sigma; \alpha, \beta\}$ is used to indicate that the members of the moment sequence $\{f_\nu\}$ are defined by formula (1). Where convenient, the symbol $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([\alpha, \beta])\}$ is used to indicate that $\{f_\nu\} = \mathbf{MS}\{\sigma; \alpha, \beta\}$ where $\sigma \in \mathbf{BN}([\alpha, \beta])$. With $\sigma \in \mathbf{BN}([\alpha, \beta] \subseteq [-\infty, \infty])$ prescribed, $\sigma \in \mathbf{DH}$ means that the Hamburger moment problem deriving from the sequence $\{f_\nu\} = \mathbf{MS}\{\sigma; \alpha, \beta\}$ is determinate, in the sense that there is only one normalised function $\sigma' \in \mathbf{BN}([-\infty, \infty])$ for which $\{f_\nu\} = \mathbf{MS}(\sigma'; -\infty, \infty)$ (it is $\sigma' = e(\sigma; -\infty, \infty)$). $f(\sigma : \alpha, \beta; z)$ is the function

$$\int_\alpha^\beta \frac{d\sigma(t)}{1 - tz}. \tag{2}$$

$\mathscr{F}$ is the power series

$$\Sigma f_\nu z^\nu. \tag{3}$$

The approximating fraction (Näherungsbruch [10, 5]) or Padé quotient ([14], [15] Ch. 5, [21] Ch. 20) $P_{i,j}(i, j \ge 0$ being fixed finite integers) derived from the series $\mathscr{F}$, the $\{f_\nu\}$ being members of a field with $f_0 \ne 0$, is that irreducible rational function whose numerator polynomial is of degree $\le j$ and whose denominator polynomial $D_{i,j}$ is of degree $\le i$ with $D_{i,j}(0) = 1$, whose series expansion is ascending powers of z agrees with $\mathscr{F}$ for the greatest number of initial terms. The quotients $\{P_{i,j}\}$ may be placed in a two-dimensional array, the Padé table, in which i and j correspond to row and column numbers respectively. For convenience in exposition, we append the quotient $P_{0,-1}(z) = 0$ to the Padé table. For a fixed finite integer $m \ge 0$, the quotients $P_{i,i+m-1}$ $(i = 0, 1, \cdots)$ and $P_{i+m,i}$ $(i = 0, 1, \cdots)$ lie on forward diagonals in the Padé table.

3. The convergence of forward diagonal sequences in the Padé table

THEOREM 1. *Subject to further conditions, let* $\{f_\nu\} = \mathbf{MS}\{\sigma \in BN([\alpha, \beta])\}$; *let* $f(z) = f(\sigma : \alpha, \beta; z)$; *let* $\{P_{i,j}\}$ *be the approximating fractions derived from the series* $\mathscr{F}$; *let* $n \geq 0$ *be a fixed finite integer.*

(i) *Let* $[\alpha, \beta] \subseteq [-\infty, \infty]$ *and let the series*

$$\Sigma_1 f_{2n+2\nu}{}^{-1/(2\nu)} \tag{4}$$

diverge. Then for increasing i the forward diagonal sequences $\{P_{i,i+2m-1}\}$ $(m = 0, 1, \cdots, n)$ *and* $\{P_{i+2m-1,i}\}$ $(m = 2, 3, \cdots, n)$ *converge uniformly to f over* $\mathbf{BE}([\alpha, \beta]^{-1})$, *and the forward diagonal sequences* $\{P_{i,i+2m}\}$ $(m = 0, 1, \cdots, n)$ *and* $\{P_{i+2m,i}\}$ $(m = 1, 2, \cdots, n)$ *converge uniformly to f over* $\mathbf{BE}([-\infty, \infty])$.

(ii) *Let* $[\alpha, \beta] \subseteq [0, \infty]$, *and let the series* $\Sigma_1 f_{n+\nu}{}^{-1/(2\nu)}$ *diverge. Then for increasing i the forward diagonal sequences* $\{P_{i,i+m}\}$ $(m = 0, 1, \cdots, n)$ *and* $\{P_{i+m,i}\}$ $(m = 1, 2, \cdots, n + 1)$ *converge uniformly to f over* $\mathbf{BE}([\alpha, \beta]^{-1})$.

Proof. If σ is a simple step function with salti $M_\nu > 0$ at the distinct points $t_\nu \in [\alpha, \beta]$ $(\nu = 1, 2, \cdots, N < \infty)$ and no other points of increase in $[\alpha, \beta]$, then $f_{2\tau} = \Sigma_1{}^N M_\nu t_\nu{}^{2\tau}$ $(\tau = 0, 1, \cdots)$; the series (4) diverges for all finite $n \geq 0$. f is the rational function

$$f(z) = \Sigma_1{}^N \frac{M_\nu}{1 - t_\nu z}$$

and [14] if none $\langle$one$\rangle$ of the $\{t_\nu\}$ is zero $P_{N+i,N+j-1}(z) \langle P_{N+i-1,N+j-1}(z) \rangle = f(z)$ for $i, j = 0, 1, \cdots$. All forward diagonal sequences of fractions $\{P_{i,j}\}$ ultimately consist of copies of f, and the results of both clauses are true. We assume henceforth that σ is not degenerate in the above sense.

For $m = 0, 1, \cdots, n$, $\sigma^{(2m)} \in \mathbf{BN}([\alpha, \beta])$. Since the series (4) diverges, $\sigma^{(2n)} \in \mathbf{DH}$ ([4] Ch. 8). If $m < n$, and the Hamburger moment problem associated with the distribution $d\sigma^{(2m)}$ were to be indeterminate, two distinct normalised solutions $d\hat{\sigma}$ and $d\tilde{\sigma}$ which differ at points other than the origin would exist ([7] §15, [13] §14). The Hamburger moment problem associated with $\sigma^{(2n)}$ would then have two distinct solutions, $\hat{\sigma}^{(2n-2m)}$ and $\tilde{\sigma}^{(2n-2m)}$. However, since $\sigma^{(2n)} \in \mathbf{DH}$, $\sigma^{(2m)} \in \mathbf{DH}$ $(m = 0, 1, \cdots, n)$ also.

Denote the Hankel determinant of order $r + 1$ whose $(i, j)^{\text{th}}$ element is $f_{m+i+j-2}$ $(i, j = 1, 2, \cdots, r + 1)$ by $H_{m,r}$ $(m, r = 0, 1, \cdots)$ and set $H_{m,-1} = 1$ $(m = 0, 1, \cdots)$. Let m be fixed in the range $0 \leq m \leq n$. $\sigma^{(2m)} \in \mathbf{BN}([\alpha, \beta] \subseteq [-\infty, \infty])$ is nondegenerate; hence $H_{2m,r} > 0$ $(r = 0, 1, \cdots)$ ([7] §5, [13] §6). The series $\Sigma f_{2m+\nu} z^\nu$ generates an associated continued fraction whose extended convergents are

$$C_i^{(2m)}(z, T) = \frac{a_1{}^{(2m)}}{1 - b_1{}^{(2m)}z-} \frac{a_2{}^{(2m)}z^2}{1 - b_2{}^{(2m)}z-} \cdots \frac{a_i{}^{(2m)}z^2}{1 - (b_i{}^{(2m)} + T)z} \tag{5}$$

$$(i = 2, 3, \cdots)$$

(with an obvious interpretation of $C_1^{(2m)}(z, T)$). For a fixed finite $z \notin [-\infty, \infty]$ the $\{C_i^{(2m)}(z, T)\}$ describe, as T varies in the range $-\infty \leq T \leq \infty$, a system of circles $\{\mathscr{C}_i^{(2m)}(z)\}$ in the complex plane. With $C_0^{(2m)}(z, T) = 0$ and $\mathscr{C}_0^{(2m)}(z)$ taken to be the line forming an angle $-\arg(z)$ with the real axis at the origin, $\mathscr{C}_{i+1}^{(2m)}(z)$ touches $\mathscr{C}_i^{(2m)}(z)$ at $C_i^{(2m)}(z, 0)$ $(i = 0, 1, \cdots)$; $\mathscr{C}_{i+1}^{(2m)}(z)$ lies inside $\mathscr{C}_i^{(2m)}(z)$ $(i = 1, 2, \cdots)$. The value of $f(\sigma^{(2m)}:\alpha,\beta;z)$ lies inside $\mathscr{C}_i^{(2m)}(z)$ $(i = 1, 2, \cdots)$. When $\sigma^{(2m)} \in \mathbf{DH}$, the radii of the $\mathscr{C}_i^{(2m)}(z)$ tend to zero; the associated continued fraction is then said to be completely convergent (vollständig konvergent [8, 7], [15] §38). In particular, the sequence $\{C_r^{(2m)}(z, 0)$ converges to $f(\sigma^{(2m)} : \alpha, \beta; z)$ at an infinite sequence of distinct points in $\mathbf{BE}([-\infty, \infty])$. The $\{C_i^{(2m)}(z, 0)\}$ also have a decomposition

$$C_i^{(2m)}(z, 0) = \Sigma_1{}^i \frac{M_{i,\nu}{}^{(2m)}}{1 - t_{i,\nu}{}^{(2m)} z},$$

where $\alpha < t_{i,\nu}{}^{(2m)} < \beta$, $M_{i,\nu}{}^{(2m)} > 0$ $(\nu = 1, 2, \cdots, i)$ and $\Sigma_1{}^i M_{i,\nu}{}^{(2m)} = f_{2m}$ $(i = 1, 2, \cdots)$. The $\{C_i^{(2m)}(z, 0)\}$ are thus uniformly bounded over $\mathbf{BE}([\alpha, \beta]^{-1})$. Uniform convergence of the $\{C_i^{(2m)}(z, 0\}$ to $f(\sigma^{(2m)} : \alpha, \beta; z)$ therefore holds, by the Stieltjes-Vitali theorem, over $\mathbf{BE}([\alpha, \beta]^{-1})$. Furthermore

$$P_{i,i+2m-1}(z) = \Sigma_0{}^{2m-1} f_\nu z^\nu + z^{2m} C_i^{(2m)}(z, 0) \quad (i = 0, 1, \cdots)$$

and

$$f(z) = \Sigma_0{}^{2m-1} f_\nu z^\nu + z^{2m} f(\sigma^{(2m)} : \alpha, \beta; z). \tag{6}$$

The result concerning the sequences $\{P_{i,i+2m-1}\}$ stated in the first part of the theorem follows.

Although $H_{2m,i} > 0$ $(i = 0, 1, \cdots)$, it can occur that for a fixed $i \geq 1$, $H_{2m+1,i-1} = 0$, when $H_{2m+1,i-2} \neq 0$, $H_{2m+1,i} \neq 0$ ([17] §22). The numbers

$$T_i^{(2m)} = -\frac{H_{2m+1,i-1} H_{2m,i-2}}{H_{2m,i-1} H_{2m+1,i-2}} \quad (i = 1, 2, \cdots)$$

are well determined. From the theory given above, the sequence $C_i^{(2m)}(z, T_i^{(2m)})$ converges to $f(\sigma^{(2m)} : \alpha, \beta; z)$ at an infinite sequence of distinct points in $\mathbf{BE}([-\infty, \infty])$. The diameter of the circle $\mathscr{C}_1^{(2m)}(z)$ described above is $f_{2m} | z/\mathrm{Im}(z) |$, and is thus bounded for all z lying in a prescribed domain $\mathbf{BE}([-\infty, \infty])$. Hence the sequence $\{C_i^{(2m)}(z, T_i^{(2m)})\}$ is uniformly bounded over this domain, and converges uniformly to $f(\sigma^{(2m)} : \alpha, \beta; z)$ over it. As is easily deduced from theory given in [16] §18, [7] §13

$$P_{i-1,2m+i-1}(z) = \Sigma_0{}^{2m-1} f_\nu z^\nu + z^{2m} C_i^{(2m)}(z; T_i^{(2m)}) \quad (i = 1, 2, \cdots).$$

Joint use of this result and formula (6) establishes the result concerning the sequences $\{P_{i,i+2m}(z)\}$ stated in the first part of the theorem.

The function g defined by the relationship

$$f(z) = \frac{f_0}{1 - b_1 z - z^2 g(z)}, \tag{7}$$

where $b_1 = f_1/f_0$, has a representation $g(z) = f(\tilde{\sigma}: \alpha', \beta'; z)$ where $\tilde{\sigma} \in \mathbf{BN}([\alpha', \beta'] \subseteq [\alpha, \beta])$ ([13] §1). Set $\{g_\nu\} = \mathbf{MS}(\sigma'; \alpha', \beta')$. Then in conjunction

(8) $$f(z) \sim \Sigma f_\nu z^\nu, \qquad g(z) \sim \Sigma g_\nu z^\nu$$

as z tends to zero along, in particular, the imaginary axis ([7] §5, [13] §1). The $\{f_\nu\}$ and $\{g_\nu\}$ are thus connected by the relationship

(9) $$f_0^2 g_r = f_0 f_{r+2} - f_1 f_{r+1} - f_0 \Sigma_1^r f_\nu g_{r-\nu} \quad (r = 0, 1, \cdots).$$

Denote the approximating fractions derived from the series $\Sigma g_\nu z^\nu$ by $\tilde{P}_{i,j}$ $(i, j = 0, 1, \cdots)$. Then

(10) $$P_{j+2,i}(z) = \frac{f_0}{1 - b_1 z - z^2 \tilde{P}_{i,j}(z)} \quad (i, j = 0, 1, \cdots).$$

If, in the above, $|\alpha|, \beta < \gamma < \infty$, then $|g_{2\nu}| < \{\tilde{\sigma}(\beta') - \tilde{\sigma}(\alpha')\}\gamma^{2\nu}$. The series

(11) $$\Sigma_1 g_{2n+2\nu-2}{}^{-1/(2\nu)},$$

in particular, diverges. The results derived above may be used to establish uniform convergence of the sequences $\{\tilde{P}_{i,i+2m-1}\}(m = 0, 1, \cdots, n-1)$ over $\mathbf{BE}([\alpha', \beta']^{-1}) \supseteq \mathbf{BE}([\alpha, \beta]^{-1})$ and $\{\tilde{P}_{i,i+2m}\}(m = 0, 1, \cdots n-1)$ over $\mathbf{BE}([-\infty, \infty])$, to g in both cases. Comparison of formulae (7, 10) leads to the remaining results of the first part of the theorem.

If no $\gamma \in (0, \infty)$ exists for which $|\alpha|, \beta < \gamma$, then for any $u_0 \in (0, \infty)$ an interval $[\alpha'', \beta'']$ for which $|\alpha''|, |\beta''| > u_0$ exists such that $\tilde{\sigma}$ contains points of increase over it. Extracting the contribution to the integral expression for f_{2r+2} over the interval $[\alpha'', \beta'']$, integrating this component by parts, and using the mean value theorem, it is easily shown that for any B, $u_0 \in (0, \infty)$, a finite integer r' exists such that $f_{2r+2} > Bu_0^{2r}(r = r', r'+1, \cdots)$. Formula (9) yields the relationship

$$f_0^2 g_{2r} = f_0 f_{2r+2} - f_1 f_{2r+1} - f_0 f_1 g_{2r-1} - f_0 \int_\alpha^\beta \int_{\alpha'}^{\beta'} \frac{t^2(u^{2r-1} - t^{2r-1})}{u - t} d\sigma(t)\, d\tilde{\sigma}(u)$$

(in this and the following formulae $r = 1, 2, \cdots$). The integrand in the above double integral is positive. Hence

$$f_0^2 g_{2r} < f_0 f_{2r+2} - f_1 f_{2r+1} - f_0 f_1 g_{2r-1}$$

or, since g_{2r} and f_{2r+2} are positive,

(12) $$g_{2r} < f_0^{-2}\{f_0 f_{2r+2} + |f_1|(|f_{2r+1}| + f_0|g_{2r-1}|)\}.$$

For any $u_0 \in (0, \infty)$ and all $u \in (-\infty, \infty)$

$$|u^{2r-1}| \leq \frac{u_0^{2r} + (2r-1)u^{2r}}{2ru_0}$$

and hence

$$|g_{2r-1}| < \frac{u_0^{2r} g_0 + (2r-1) g_{2r}}{2ru_0}.$$

A similar inequality may be derived for $| f_{2r+1} |$. Formula (12) now yields an inequality of the form

$$g_{2r} < Af_{2r+2} + Bu_0^{2r} + Cg_{2r}$$

where $A, B \in (0, \infty)$ and $C = | f_1 | / (f_0^2 u_0)$. u_0 can be so chosen that $C < 1$, and using the inequality derived above for f_{2r+2}, we finally derive an inequality of the form $g_{2r} < Df_{2r+2} (r = r', r' + 1, \cdots)$ where $D \in (0, \infty)$ is independent of r. Since the series (4) diverges, the series (11) does likewise. The remaining results of the first part of the theorem are again established.

To prove the second part of the theorem, we set $\alpha' = -\beta^{1/2}$, $\beta' = \beta^{1/2}$ and $\gamma = \alpha^{1/2}$, and define the function σ' over $[\alpha', \beta']$ by setting $\sigma'(t) = \frac{1}{2} \{\sigma(\gamma) - \sigma(t^2)\} (\alpha' \leq t \leq -\gamma)$, $\sigma'(t) = 0$ if $\gamma > 0$ $(-\gamma < t < \gamma)$, and $\sigma'(t) = -\sigma'(-t) (\gamma \leq t \leq \beta')$. Now $\sigma' \in \mathbf{BN}([\alpha', \beta'] \subseteq [-\infty, \infty])$ and $f(z) = f(\sigma' : \alpha', \beta'; z')$ where $z' = z^{1/2}$. With $\{f_\nu\} = \mathbf{MS}(\sigma; \alpha, \beta)$, $\{f_\nu'\} = \mathbf{MS}(\sigma'; \alpha', \beta')$, $f_{2\nu}' = f_\nu$, $f_{2\nu+1}' = 0 (\nu = 0, 1, \cdots)$ and, denoting the approximating fractions derived from the series $\Sigma f_\nu' z'^\nu$ by $\{ P_{i,j}' \}$,

$$P_{i,j}(z) = P_{2i+\nu, 2j+\tau}'(z') \quad (i, j = 0, 1, \cdots; \nu, \tau = 0, 1).$$

The results of the second part of the theorem now follow from those concerning the sequences of the form $\{ P_{i,i+2m-1}' \}$ and $\{ P_{i+2m-1}' \}$ of the first.

When $n = 0$, the above theorem yields no information concerning sequences of the form $\{P_{i+m,i}\}$ with $m > 1$, lying below the sub-principal diagonal in the Padé table. Nevertheless, something concerning the sequence $\{ P_{i+2,i} \}$ can be deduced in this case by the use of special methods. Carleman ([4] Ch. 8) showed that with $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([-\infty, \infty])\}$ and σ nondegenerate, the continued fraction associated with $\mathscr{F}$ converges completely for fixed finite z $(\mathrm{Im}(z) \neq 0)$ if, in the notation of formula (5), the series $\Sigma_1 a_\nu^{(0)^{-1/2}}$ diverges (the $a_\nu^{(0)}$ are all positive real numbers). He then showed that this series diverges if the series (4) with $n = 0$ diverges. The partial numerators of the continued fraction associated with the series $\Sigma g_\nu z^\nu$ of formulae (7–9) are $a_{\nu+1}^{(0)} (\nu = 1, 2, \cdots)$. If the series (4) with $n = 0$ diverges, the series $\Sigma_1 a_{\nu+1}^{(0)^{-1/2}}$ diverges. Hence, in particular, with f and g being as in the proof of Theorem 1, the sequence $\{ \tilde{P}_{i,i} \}$ converges uniformly to g over $\mathbf{BE}([-\infty, \infty])$ and, from formulae (7, 10), the sequence $\{P_{i+2,i}\}$ converges uniformly to f over $\mathbf{BE}([-\infty, \infty])$. In the Stieltjes case in which $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([0, \infty])\}$, no additional information is obtained in this way.

Basing his analysis on an examination of the convergence behaviour of series whose terms are derived from the coefficients $\{a_\nu^{(2m)}\}$ and $\{b_\nu^{(2m)}\}$ occurring in formula (5), Wall [19, 18] has given accounts of the convergence behaviour of the approximating fractions $\{P_{i,j}\}$ derived from the series $\mathscr{F}$ in the two cases in which $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([-\infty, \infty])\}$ and $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([0, \infty])\}$.

Using the notation of this paper, the description in the former case is as follows: for integers $n, n' \geq -1$ $\sigma^{(2m)} \in \mathbf{DH}(m = 0, 1, \cdots, n)$ and $\tilde{\sigma}^{(2m)} \in \mathbf{DH}(m = 0, 1, \cdots, n')$ and, with $\hat{f}(z) = f(\sigma: -\infty, \infty; z)$, the forward diagonal sequences $\{P_{i,i+m-1}\}$ $(m = 0, 1, \cdots, 2n + 1)$ and $\{P_{i+m,i}\} (m = 2, 3, \cdots, 2n')$ converge uniformly to $\hat{f}$ over $\mathbf{BE}([-\infty, \infty])$; the remaining diagonal sequences, although behaving in a regular manner, may not converge as just described (details are given in [19]). In the latter case, and again with integers $n, n' \geq -1$, the Stieltjes moment problems associated with the functions $\sigma^{(m)}(m = 0, 1, \cdots, n)$ and $\tilde{\sigma}^{(m)}(m = 0, 1, \cdots, n')$, where

$$f(\sigma:0, \infty; z) = \frac{f_0}{1 - zf(\hat{\sigma}: 0, \infty; z)}$$

are determinate and, with $\tilde{f}(z) = f(\sigma: 0, \infty; z)$, the forward diagonal sequences $\{P_{i,i+m}\}(m = 0, 1, \cdots, n)$ and $\{P_{i+m,i}\}(m = 1, 2, \cdots, n' + 1)$ converge uniformly to $\tilde{f}$ over $\mathbf{BE}([0, \infty])$; thereafter, the above Stieltjes moment problems are indeterminate for $m = n + 1, n + 2, \cdots$ and $m = n' + 1, n' + 2, \cdots$ respectively, and each of the remaining diagonal sequences converges uniformly over $\mathbf{BE}([0, \infty])$, although no two limit functions associated with adjacent sequences are equal. (Before commenting further, we remark that the above theory may slightly be extended (see the proof of Theorem 1): if in the first of the above cases σ is devoid of points of increase in $[-\infty, \infty]\backslash[\alpha, \beta]$, then the even order sequences $\{P_{i,i+2m-1}\}(m = 0, 1, \cdots, n)$ and $P_{i+2m-1,i}(m = 2, 3, \cdots, n')$ converge uniformly to $\hat{f}$ over $\mathbf{BE}([\alpha, \beta]^{-1})$ (i.e. if $\alpha > -\infty$ or $\beta < \infty$, a more extensive domain than $\mathbf{BE}([-\infty, \infty])$); if in the second of the above cases σ is devoid of points of increase in $[0, \infty]\backslash[\alpha, \beta]$, then all sequences $\{P_{i,i+m}\}(m = 0, 1, \cdots, n)$ and $\{P_{i+m,i}\}$ $(m = 1, 2, \cdots, n' + 1)$ converge uniformly to $\tilde{f}$ over $\mathbf{BE}([\alpha, \beta]^{-1})$). In the first of the above cases it may occur that $n = -1$ and the first band of diagonal sequences is missing; alternatively it may occur that $n = \infty$, and all sequences $\{P_{i,i+m-1}\}(m = 0, 1, \cdots)$ converge uniformly as described; mutatis mutandis, the same considerations hold with regard to n'; a similar comment may be made upon the second case.

In the first part of Theorem 1, the condition that the series (4) should diverge has been imposed (this implies that $\sigma^{(2m)} \in \mathbf{DH}(m = 0, 1, \cdots, n)$) and it has then been demonstrated that, subject to this additional restraint, $n' \geq n$ in the first of the above descriptions; similarly for the second part. Carleman's criterion (namely that the divergence of the series (4) implies that $\sigma^{(2n)} \in \mathbf{DH}$) is a sufficient condition for the determinacy of the Hamburger moment problem, although no example is known (at least to the author) for which the series (4) converges when $\sigma^{(2n)} \in \mathbf{DH}$. If it transpires that Carleman's condition is also necessary, Wall's account as given above will have to be refined by the additional remark that the band of convergent sequences lying below the principal diagonal of the Padé table is at least as extensive as that lying upon and above it, in the sense that $n' \geq n$. Again, similar considerations hold with regard to the Stieltjes case in which $[\alpha, \beta] \subseteq [0, \infty]$. (For the sake of completeness, it is mentioned in this context that if, under the conditions of the first part of Theorem 1, the series (11) converges, then the series (4) does likewise.)

4. Linear transformations of power series

Adapting the version according to Hardy ([9] §4.13) of le Roy's method [11], the power series $\mathscr{F}$ with complex number coefficients $\{f_\nu\}$ is transformed over the sector $\bar{\Delta}(\theta_1, \theta_2)$ as follows: it is assumed that for each $z \in \bar{\Delta}(\theta_1, \theta_2)$, a) the series $\Sigma f_\nu (zu)^\nu/(\chi\nu)!$ converges for small values of u, b) the function $f_\chi(zu)$ defined by analytic continuation of the sum of this series is regular in u for all $u \in (0, \infty)$ and c) the integral

(13) $$S_\chi(z) = \chi^{-1} \int_0^\infty \exp(-u^{1/\chi}) u^{(1/\chi)-1} f_\chi(zu)\, du$$

exists; $\mathscr{F}$ is then said to be summable $(\mathbf{B}', \chi)$ to S_χ over $\bar{\Delta}(\theta_1, \theta_2)$. The case in which $\chi = 1$ was studied in detail by Borel ([3] Ch. 3).

Hardy ([9] §13.16) remarks that under certain conditions the Euler-Maclaurin series is $\mathbf{B}^2$-summable (i.e. summable by a repetition of Borel's method). He does not provide an explicit definition of $\mathbf{B}^2$-summability but this is easily done: with regard to the transformation of the series $\mathscr{F}$ it is assumed that for each $z \in \bar{\Delta}(\theta_1, \theta_2)$, a) the series $\Sigma f_\nu (zvw)^\nu/(\nu!)^2$ converges for small values of v and w, b) the function $\tilde{f}(zvw)$ defined by analytic continuation of the sum of this series is regular in v and w for all $v, w \in (0, \infty)$ and c) the double integral

$$S(z) = \int_0^\infty \int_0^\infty e^{-v-w} \tilde{f}(zvw)\, dv\, dw$$

exists; $\mathscr{F}$ is then said to be summable $(\mathbf{B}^2)$ to S over $\bar{\Delta}(\theta_1, \theta_2)$.

A more concise definition of $(\mathbf{B}^2)$ summability, involving an auxiliary series in one variable and a single integral, may be given: it is assumed that for each $z \in \bar{\Delta}(\theta_1, \theta_2)$, a) the series $\Sigma f_\nu (zu)^\nu/(\nu!)^2$ converges for small values of u, b) the function $\hat{f}(zu)$ defined by analytic continuation of the sum of this series is regular in u for all $u \in (0, \infty)$ and c) the integral

(14) $$\hat{S}(z) = 2 \int_0^\infty K_0(2u^{1/2}) \hat{f}(zu)\, du$$

exists (K_0 being a modified Bessel function of the second kind); the series $\mathscr{F}$ is then said to be summable $(\mathbf{B}^2)$ to $\hat{S}$ over $\bar{\Delta}(\theta_1, \theta_2)$. (As is easily verified by use of the formula

$$\int_0^\infty v^{-1} \exp\left(-v - \frac{u}{v}\right) dv = 2K_0(u^{1/2}) \qquad (0 < u < \infty)$$

the above two definitions of $\mathbf{B}^2$-summability are equivalent.)

By varying the path of integration in formulae (13, 14) it is possible to extend the definitions of (B', χ) and $\mathbf{B}^2$-summability in such a way that certain series are summable over sectors larger than those for which the above methods are effective. The extended definitions are as follows. Firstly it is assumed that the series $\Sigma f_\nu x^\nu/(\chi\nu)!$ converges in the neighbourhood of the origin, and that the function $\tilde{f}_\chi$ obtained by analytic continuation of the sum of this series is uniformly bounded over $\bar{\Delta}(\phi_1, \phi_2)$; set

(15) $$R_\chi(z) = \chi^{-1} \int_0^{\infty \exp\{i\phi(\chi, z)\}} \exp(-u^{1/\chi}) u^{(1/\chi)-1} \tilde{f}_\chi(zu)\, du$$

where

(16) $$\phi(\chi, z) = \frac{\chi\pi\{\frac{1}{2}(\phi_1 + \phi_2) - \arg(z)\}}{\chi\pi + \phi_2 - \phi_1};$$

the series $\mathscr{F}$ is then said to be summable $(\bar{\mathbf{B}}, \chi)$ to R_χ over $\Delta(\phi_1 - \frac{1}{2}\chi\pi, \phi_2 + \frac{1}{2}\chi\pi)$. Secondly, it is assumed that the series $\Sigma f_\nu x^\nu/(\nu!)^2$ converges in the neighbourhood of the origin and that the function $\hat{f}$ obtained by analytic continuation of the sum of this series is uniformly bounded over $\bar{\Delta}(\phi_1, \phi_2)$; set

$$\hat{R}(z) = 2 \int_0^{\infty \exp\{i\phi(2,z)\}} K_0(2u^{1/2})\hat{f}(zu)\, du;$$

the series $\mathscr{F}$ is then said to be summable $(\bar{\mathbf{B}}^2)$ to $\hat{R}$ over $\Delta(\phi_1 - \pi, \phi_2 + \pi)$. (That the above functions R_χ and $\hat{R}$ are well defined over their associated sectors is easily demonstrated: with regard to the first function, $-\frac{1}{2}\chi\pi < \phi(\chi, z) < \frac{1}{2}\chi\pi$ and $zu \in \Delta(\phi_1, \phi_2)$ when $z \in \Delta(\phi_1 - \frac{1}{2}\chi\pi, \phi_2 + \frac{1}{2}\chi\pi)$; the second function is dealt with similarly.) For any fixed z in the appropriate summability sector, the integral (15) also exists if $\phi(\chi, z)$ is replaced by other sufficiently close values. However, by adopting the value given by formula (16), $(\bar{\mathbf{B}}, \chi)$-summability over the largest possible sector deriving from the sector $\bar{\Delta}(\phi_1, \phi_2)$ of boundedness of $\tilde{f}_\chi$ is ensured. Similar considerations hold with regard to $\bar{\mathbf{B}}^2$-summability.

THEOREM 2. *Subject to further conditions, let* $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([\alpha, \beta])\}$; *for fixed* $\chi, \xi, \kappa \in (0, \infty)$ *and finite integer* $r \geq 0$ *let* $f_{2^r\nu} = O((2^r(\chi\nu + \kappa))!\xi^\nu)$; *let* $f(z) = f(\sigma : \alpha, \beta; z)$; *let* $\{P_{i,j}\}$ *be the approximating fractions derived from the series* $\mathscr{F}$.

(i) *Let* $[\alpha, \beta\} \subseteq [-\infty, \infty]$ *and* $\chi \in (0, 1]$.
a) $\mathscr{F}$ *is summable* $(\mathbf{B}', \chi)$ *to* f *over both*

$$\bar{\Delta}\left(\tfrac{1}{2}\chi\pi, \frac{\pi}{2}(2 - \chi)\right)$$

and

$$\bar{\Delta}\left(\frac{\pi}{2}(\chi - 2), -\tfrac{1}{2}\chi\pi\right).$$

b) $\mathscr{F}$ *is summable* $(\bar{\mathbf{B}}, \chi)$ *to* f *over both* $\Delta(0, \pi)$ *and* $\Delta(-\pi, 0)$.
c) *For increasing* i, *the forward diagonal sequences* $\{P_{i,i+2m+1}\}$ $(m = 0, 1, \cdots)$ *and* $\{P_{i+2m-1,i}\}$ $(m = 1, 2, \cdots)$ *converge uniformly to* f *over* $BE([\alpha, \beta]^{-1})$, *and the forward diagonal sequences* $\{P_{i,i+2m}\}$ $(m = 0, 1, \cdots)$ *and* $\{P_{i+2m,i}\}$ $(m = 1, 2, \cdots)$ *converge uniformly to* f *over* $\mathbf{BE}([-\infty, \infty])$.

(ii) *Let* $[\alpha, \beta] \subseteq [0, \infty]$ *and* $\chi \in (0, 2]$.
a) $\mathscr{F}$ *is summable* $(\mathbf{B}', \chi)$ *to* f *over* $\bar{\Delta}\left(\tfrac{1}{2}\chi\pi, \frac{\pi}{2}(4 - \chi)\right)$.
b) $\mathscr{F}$ *is summable* $(\bar{\mathbf{B}}, \chi)$ *to* f *over* $\Delta(0, 2\pi)$.
c) *For increasing* i, *the forward diagonal sequences* $\{P_{i,i+m}\}$ $(m = 0, 1, \cdots)$ *and* $\{P_{i+m,i}\}$ $(m = 1, 2, \cdots)$ *converge uniformly to* f *over* $\mathbf{BE}([\alpha, \beta]^{-1})$.

(iii) *Let* $[\alpha, \beta] \subseteq [0, \infty]$ *and* $\chi = 2$.
a) $\mathscr{F}$ *is summable* $(\mathbf{B}^2)$ *to* f *over* $(-\infty, 0]$.
b) $\mathscr{F}$ *is summable* $(\bar{\mathbf{B}}^2)$ *to* f *over* $\Delta(0, 2\pi)$.

Proof. Set $f_\nu^{(r)} = f_{2^r\nu}(\nu = 0, 1, \cdots)$. When $r > 1$ we have $\{f_\nu^{r-1}\} = \mathbf{MS}(\sigma_{r-1}; 0, \infty)$ where $d\sigma_{r-1}(t) = \sigma'(t^{1/2^{r-1}}) + \sigma'(-(t^{1/2^{r-1}}))(0 < t \leq \infty)$, $d\sigma_{r-1}(0) = d\sigma'(0)$, σ' being the extensional $\sigma' = e(\sigma; -\infty, \infty)$ of the function σ defined under the conditions of part (i). Since $\sigma_{r-1} \in \mathbf{BN}([0, \infty])$, $f_{2\nu+1}{}^{(r-1)} \leq (f_{2\nu}{}^{(r-1)} f_{2\nu}{}^{(r-1)})^{1/2}$ $(\nu = 0, 1, \cdots)$. Also $f_{2\nu}{}^{(r-1)} = f_\nu^{(r)} = O((2^r(\chi\nu + \kappa))!\xi^\nu)$. Hence, as is easily verified, $f_\nu^{(r-1)} = O((2^{r-1}(\chi\nu + \kappa'))!\xi'^\nu)(0 < \kappa', \xi' < \infty)$. Proceeding in this way, it is shown that $f_{2\nu} = O((2(\chi\nu + \hat{\kappa}))!\hat{\xi}^\nu)(0 < \hat{\kappa}, \hat{\xi} < \infty)$. Since $\sigma \in \mathbf{BN}([\alpha, \beta] \subseteq [-\infty, \infty])$, $f_{2\nu+1} \leq (f_{2\nu} f_{2\nu+2})^{1/2}$ $(\nu = 0, 1, \cdots)$, and hence $f_\nu = O((\chi\nu + \hat{\kappa})!\hat{\xi}^\nu)(0 < \hat{\kappa}, \hat{\xi} < \infty)$. By choosing η slightly larger than $\hat{\xi}$, it follows that $f_\nu = O((\chi\nu)!\eta^\nu)$.

Mittag-Leffler's function

$$E_\chi(x) = \Sigma x^\nu/(\chi\nu)!$$

is, for $0 < \chi \leq 2$, uniformly bounded over $\bar{\Delta}\left(\frac{1}{2}\chi\pi, \frac{\pi}{2}(4 - \chi)\right)$ ([12] Th. 8.a). Under the conditions of part (i), the series $\Sigma f_\nu(zu)^\nu/(\chi\nu)!$ converges for any fixed finite z and $|u| < 1/(\eta|z|)$ to

$$f_\chi(zu) = \textstyle\int_{-\infty}^{\infty} E_\chi(zut)\, d\sigma(t). \tag{17}$$

When $z \in \bar{\Delta}\left(\frac{1}{2}\chi\pi, \frac{\pi}{2}(2 - \chi)\right) \cup \bar{\Delta}\left(\frac{\pi}{2}(\chi - 2), -\frac{1}{2}\chi\pi\right)$, $f_\chi(zu)$ is regular for $0 \leq u < \infty$, and the integral (17) is absolutely convergent. Furthermore, in the notation of formula (13), $S_\chi(z) = f(\sigma: \alpha, \beta; z)$ for all $z \in \bar{\Delta}\left(\frac{1}{2}\chi\pi, \frac{\pi}{2}(2 - \chi)\right) \cup \bar{\Delta}\left(\frac{\pi}{2}(\chi - 2), -\frac{1}{2}\chi\pi\right)$; clause (ia) has been proved. Clauses (ib), (iia, b) are demonstrated in the same way.

Under the conditions of part (i), all series (4) of Theorem 1 diverge for $n = 0, 1, \cdots$. The result of clause (ic) now follows directly from part (i) of Theorem 1; the result of clause (iic) is a consequence of part (ii) of that theorem.

Under the conditions of part (iii), the series $\Sigma f_\nu(zu)^\nu/(\nu!)^2$ converges for any fixed finite z and $u < 1/(\eta|z|)$ to

$$\hat{f}(zu) = \textstyle\int_0^\infty J_0\,\{2i(zut)^{1/2}\}\, d\sigma(t)$$

(J_0 being a Bessel function of the first kind). $|J_0\{2i(zut)^{1/2}\}|$ is uniformly bounded for $z \in (-\infty, 0]$ and $u, t \in [0, \infty)$, and hence $\hat{f}(zu)$ is uniformly bounded for all $z \in (-\infty, 0]$ and $u \in [0, \infty)$: hence, in the notation of formula (14),

$$\hat{S}(z) = f(\sigma: \alpha, \beta; z)$$

for all $z \in (-\infty, 0]$. The result of clause (iiia) has been proved, and that of clause (iiib) is derived in the same way. (We remark that the result of part (iic) holds a fortiori under the conditons of part (iii).)

It will have been noted that the series of the class considered in Theorem 2 are (B', χ) or B^2-summable not over two half-planes or a single cut plane, but over sectors belonging to them. That this corresponds to the facts may be shown by means of simple examples. One of the series to which clause (ia) refers is that obtained by taking the function σ to have salti of magnitude $\frac{1}{2}$ at the two points $\pm t'(t' \in (0, \infty))$ and no other points of increase when, with $[\alpha, \beta] = [-\infty, \infty]$, $f_{2\nu} = t'^{2\nu}$ $(\nu = 0, 1, \cdots)$. This series satisfies the stated conditions with, in particular, $\chi = 1$. Since $E_1(x) = e^x$, $f_1(zu) = \cosh(zut')$, and

$$S_1(z) = \int_0^\infty e^{-u}\cosh(zut')\,du.$$

This integral exists for $|\mathrm{Re}(z)| < 1/t'$. For any z for which $\mathrm{Re}(z) \neq 0$, a sufficiently large t' can be found (i.e. a series to which clause (ia) refers can be constructed) for which the above integral fails to exist: $(B', 1)$ summability of all series of the type just considered holds only over the finite part of the imaginary axis, as the result of clause (ia) indicates. Similar critical examples may be constructed for the more general processes of (B', χ) and B^2-summation considered in the theorem. Naturally, certain series to which Theorem 2 refers are summable over larger sectors than the stated results suggest. For example, when $\sigma(t) = \sigma(-t) = \frac{1}{2}(1 - e^{-t})(0 \leq t < \infty)$ and $[\alpha, \beta] = [-\infty, \infty]$, $f_{2\nu} = (2\nu)!$, $f_{2\nu+1} = 0$ $(\nu = 0, 1, \cdots)$. These coefficients satisfy the conditions of clause (ia) with, in particular, $\chi = 1$. In this case $f_1(zu) = (1 - z^2u^2)^{-1}$, and S_1 exists (i.e. the series in question is $(\mathbf{B}', 1)$ summable) over the maximal open sectors $\Delta(0, \pi)$ and $\Delta(-\pi, 0)$.

The restricted regions over which the processes of $(\mathbf{B}', \chi)$ and $\mathbf{B}^2$ summation are effective have a significant interest with regard to the series concerned and their related generating functions. The theory is illustrated with reference to an example. Denote the function defined by an expression of the form (2) with $\alpha = -\infty$, $\beta = \infty$, $\sigma \in \mathbf{BN}([-\infty, \infty])$ over the half-plane $\mathrm{Im}(z) > 0$ by f_+, and that similarly defined over the half-plane $\mathrm{Im}(z) < 0$ by f_-. For simplicity, let $[\alpha', \beta'] \subset (0, \infty)$ and $[\alpha'', \beta''] = [1/\beta', 1/\alpha']$, and let ω be a function of a complex variable, which is analytic in a domain $\mathbf{D}$ containing the segment $[\alpha', \beta']$ in its interior. Let $d\sigma(t) = \omega(t)\,dt$ for all $t \in [\alpha', \beta']$. Regard expression (2) as an (incomplete) contour integral, and distend that part of the contour coincident with the segment $[\alpha', \beta']$ in such a way that the deformed part lies in the upper half-plane but still in $\mathbf{D}$ (the value of $f_+(z)$ for $\mathrm{Im}(z) > 0$ is unchanged by this operation). Allow z^{-1} (which when $\mathrm{Im}(z) > 0$ lies in the lower half-plane) to cross the segment (α', β') with z^{-1} still in $\mathbf{D}$. $f_+(z)$ has now been expressed for a new value of z for which $\mathrm{Im}(z) < 0$ by means of a contour integral. Deform the distended part of the contour in such a way that it becomes the segment $[\alpha', \beta']$ again, together with a small circle about the new value of z^{-1}. Since, for the new value of z, $\mathrm{Im}(z) < 0$, the integrals over the segments $[-\infty, \alpha']$, $[\alpha', \beta']$, $[\beta', \infty]$ together yield the value of $f_-(z)$. Evaluating the integral over the small circle, the relationship

$$f_+(z) = f_-(z) - 2\pi i z^{-1}\omega(z^{-1}), \tag{18}$$

which offers a basis for the analytic continuation of f_+ across the segment $[\alpha', \beta']$, is obtained. The same formula may also be derived from the representation

$$
\begin{aligned}
f_+(z) &= f(\sigma:-\infty, \alpha'; z) + \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{\omega(u)}{1-zu} \, ln\left\{\frac{\alpha'-u}{\beta'-u}\right\} du \\
&\quad -z^{-1}\omega(z^{-1}) ln\left\{\frac{1-\alpha' z}{1-\beta' z}\right\} + f(\sigma:\beta', \infty; z)
\end{aligned} \tag{19}
$$

where $\mathscr{C}$ is a simple loop surrounding all points of $[\alpha', \beta']$ but lying entirely within **D**, and z^{-1} lies within $\mathscr{C}$. If σ is equivalent to other analytic functions of a complex variable over other segments of the real axis, further branches of f_+ are determined by relationships of the form (18), and further representations of the form (19) hold.

Let $d\sigma(t) = \exp\,(-\,|t|^{1/\chi})\; dt\; (0 < \chi < 1;\; -\infty < t < \infty)$ so that $d\sigma(t) = \exp(-t^{1/\chi})\, dt\; (0 \leq t < \infty)$ and $d\sigma(t) = \exp(-(-t)^{1/\chi})\, dt\; (-\infty < t \leq 0)$. For the branch of f_- obtained by analytic continuation across the positive real axis

$$f_-(z) = f_+(z) + 2\pi i z^{-1}\exp(-z^{-1/\chi}) \tag{20}$$

and for that obtained by analytic continuation across the negative real axis

$$f_-(z) = f_+(z) + 2\pi i z^{-1}\exp(-(-z)^{-1/\chi}). \tag{21}$$

In the case being considered $\sigma \in \mathbf{BN}([-\infty, \infty])$, and with $\{f_\nu\} = \mathbf{MS}(\sigma; -\infty, \infty)$, $f_+(z) \sim \Sigma f_\nu z^\nu$ as z tends to zero in $\Delta(0, \pi)$. However, $\lim z^{-r}\exp(-z^{-1/\chi}) = 0$ $(r = 0, 1, \cdots)$ as z tends to zero in $\Delta(0, \frac{1}{2}\chi\pi)$ so that, from relationship (20), $f_-(z) \sim \Sigma f_\nu z^\nu$ also as z tends to zero in this open sector. Similarly, with f_- now defined by formula (21), $f_-(z) \sim \Sigma f_\nu z^\nu$ as z tends to zero in $\Delta\left(\frac{\pi}{2}(2-\chi), \pi\right)$. Over the two sectors $\Delta(0, \frac{1}{2}\chi\pi)$ and $\Delta\left(\frac{\pi}{2}(2-\chi), \pi\right)$ the series $\mathscr{F}$ represents f_+ as directly defined by formula (2), and also functions of the two branches of f_-. Also $f_{2\nu} = 2\chi(2\chi\nu + \chi)!$ $(\nu = 0, 1, \cdots)$: the series $\mathscr{F}$ is $(\mathbf{B}', \chi)$ summable to f_+ over $\bar{\Delta}\left(\frac{1}{2}\chi\pi, \frac{\pi}{2}(2-\chi)\right)$. This sector is precisely the complement in $\Delta(0, \pi)$ of the two sectors over which $\mathscr{F}$ represents two functions, and is the sector in which $\mathscr{F}$ represents one function, namely f_+, alone.

The above observations are complemented by the remark that although certain of the functions to which part (i) of Theorem 2 refers are defined by analytic continuation over sectors larger than $\Delta(0, \pi)$ and $\Delta(-\pi, 0)$, nevertheless these sectors are optimal $(\bar{B}, \chi)$ summability sectors with respect to the class of relevant series. When $\sigma \in \mathbf{BN}([-\infty, \infty])$ and σ is non-analytic at every point of $(-\infty, \infty)$ then (see [16] §59, [17] §19 and the footnote to p. 268 of part I of [7]) the real axis is a natural barrier for the functions $f_\pm(\sigma: -\infty, \infty; z)$, and clearly an integral expression of the form (15) whose use would permit analytic continuation across the real axis does not exist. Similar remarks can be made concerning the further parts of the theorem.

As a matter of further interest it is pointed out that the relationships between the rates of growth of the coefficients $\{f_\nu\}$ and $\{g_\nu\}$ established in the proof of Theorem 1 lead to the following result: if $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([-\infty, \infty])\}$, $f(z) = f(\sigma: -\infty, \infty; z)$, for $\chi \in (0, 1]$ the $\{f_\nu\}$ satisfy the order relationship stated at the commencement of Theorem 2, and $\{h_\nu\}$ are the coefficients of the power series reciprocal to $\mathscr{F}$ (so that $h_0 = f_0^{-1}$, $h_r = -f_0^{-1}\Sigma_0^{r-1}h_\nu f_{r-\nu}$ $(r = 1, 2, \cdots)$), then the series $\Sigma h_\nu z^\nu$ is (B', χ) summable to $1/f$ over the sectors $\bar{\Delta}\left(\frac{1}{2}\chi\pi, \frac{\pi}{2}(2-\chi)\right)$ and $\bar{\Delta}\left(\frac{\pi}{2}(\chi-2), -\frac{1}{2}\chi\pi\right)$. Similar results may be formulated concerning $(\bar{B}, \chi)$ summability of the above reciprocal series and, when $[-\infty, \infty]$ in the above is replaced by $[0, \infty]$, with regard to $(\mathbf{B}', \chi)$ and $(\bar{\mathbf{B}}', \chi)$ and summability when $\chi \in (0, 2]$, and to $\mathbf{B}^2$- and $\bar{\mathbf{B}}^2$-summability.

In a classic memoir [6], Hamburger investigated the convergence of the continued fraction associated with the power series $\mathscr{F}$ in the case in which $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([-\infty, \infty])\}$ and $f_\nu = O(\nu!\, \xi^\nu)$; he showed that this continued fraction converges to the $(B', 1)$ sum of $\mathscr{F}$ over the finite part of the imaginary axis and, since the convergents are uniformly bounded over $\mathbf{BE}([-\infty, \infty])$, that convergence holds over this domain. The convergents ($\{C_i(z, 0)\}$ in the notation of formula (5)) in question are the approximating fractions $\{P_{i,i-1}(z)\}$ (of a single diagonal sequence) derived from $\mathscr{F}$. Part (i) of Theorem 2 is an extension of Hamburger's work in two senses: firstly, a relationship between $(\mathbf{B}', \chi)$ summability and convergence of an associated continued fraction has been established for general values of χ in the range $(0, 1]$; secondly convergence has been shown to hold for all forward diagonal sequences of approximating fractions.

Carleman's criterion for the convergence of the continued fraction associated with the series $\mathscr{F}$ in the case in which $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([-\infty, \infty])\}$ (namely that the series (4) with $n = 0$ diverges) appears to be more general than Hamburger's criterion developed in the paper referred to in the preceding paragraph (namely that $f_\nu = O(\nu!\, \xi^\nu)$) in the sense that if the second criterion is satisfied, the first is also. It is nevertheless an interesting open question as to whether Hamburger moment sequences exist which satisfy the first condition and violate the second. (It is remarked that such sequences for which $f_\nu = O((\nu + \tau(\nu))!\, \xi^\nu)$ $(\xi \in (0, \infty))$ with $\tau(\nu) = O(\nu/\ln(\nu))$ satisfy the first test, but only appear to violate the second: an $\xi' \in (0, \infty)$ exists for which $f_\nu = O(\nu!\, \xi'^\nu)$.)

Wall ([20] Th. 3) has shown that if $\{f_\nu\} = \mathbf{MS}(\sigma \in \mathbf{BN}([-\infty, \infty])\}$, $f_\nu = O(\nu!\, \xi^\nu)$ and for the series $\Sigma h_\nu z^\nu$ reciprocal to $\mathscr{F}$, $h_\nu = O(\nu!\, \xi'^\nu)$, then with $f(z) = f(\sigma: -\infty, \infty; z)$ all forward diagonal sequences of approximating fractions $\{P_{i,j}\}$ derived from $\mathscr{F}$ converge uniformly to f over $\mathbf{BE}([-\infty, \infty])$. As is clear from part (i) of Theorem 2, the assumption concerning the reciprocal series can be discarded. (As has been remarked above, once the stated conditions upon the $\{f_\nu\}$ have been imposed, the coefficients $\{h_\nu\}$ necessarily satisfy the required conditions.) Wall ([20] Th. 1) has also shown that if $\{f_\nu\}$ and f are as just described but now $f_\nu = O(\xi^\nu)$, then all forward diagonal sequences of approximating fractions derived from $\mathscr{F}$ converge uniformly to f over $\mathbf{BE}([-\infty, \infty])$. As is evident from part (i) of Theorem 2, a further term of the form $\nu!$ may be inserted with impunity in the stated order relationship.

F. Bernstein [1] has shown that if $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([0, \infty])\}$ and $f_\nu = O((\chi\nu)!\ \xi^\nu)$ for some $\chi \in (0, 2)$ and $\xi \in (0, \infty)$ then, with $f(z) = f(\sigma\colon 0, \infty; z)$, $\mathscr{F}$ is summable (B', χ) to f over $\Delta\left(\frac{1}{2}\chi\pi, \frac{\pi}{2}(2-\chi)\right)$ and $\Delta\left(\frac{\pi}{2}(\chi-2), -\frac{1}{2}\chi\pi\right)$ and that the continued fraction associated with $\mathscr{F}$ converges uniformly to f over $\mathbf{BE}([0, \infty])$. In a footnote to p. 46 of [2], Bernstein remarks that it follows as a corollary to Hamburger's work referred to above, that $\mathscr{F}$ is $(B', 2)$ summable to f over the finite part of the nonpositive real axis, and that uniform convergence of the associated continued fraction over $\mathbf{BE}([0, \infty])$ also holds for $\chi = 2$. Part (ii) of Theorem 2 offers a slight extension to Bernstein's work in that (B', χ) summability is shown to hold over the closed sectors corresponding to the open sectors given above for $0 < \chi \leq 2$; furthermore, convergence also holds for all forward diagonal sequences of approximating fractions, not just that upon which the convergents of the associated continued fraction lie.

Wall ([20] Th. 3) has shown that if $\{f_\nu\} = \mathbf{MS}\{\sigma \in \mathbf{BN}([0, \infty])\}$, $f_\nu = O(\nu!\ \xi^\nu)$, and the coefficients of the series reciprocal to $\mathscr{F}$ (see above) satisfy a similar relationship, then with $f(z) = f(\sigma\colon 0, \infty; z)$ all forward diagonal sequences of approximating fractions derived from $\mathscr{F}$ converge to f uniformly over $\mathbf{BE}([0, \infty])$. As part (ii) of Theorem 2 indicates, $\nu!$ may be replaced by $(2\nu)!$ in the above order relationship and the assumption concerning the reciprocal series may be discarded.

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