Boletín de la Sociedad Matemática Mexicana Vol. 26, No. 1, 1981

EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A CLASS OF SUPERLINEAR PROBLEMS*

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1. Introduction

In this paper we develop a method with which we prove that a certain class of superlinear problems has infinitely many solutions. Though our main result can be stated for an abstract operator equation we prefer to present it in the specific context of a variational system of ordinary differential equations.

Throughout this paper $F: \mathbf{R}^n \to \mathbf{R}$ denotes a function of class C^1 such that:

(A) There exist a positive integer N_0, $C \in R$ and a sequence of functions $F_N: \mathbf{R}^n \to \mathbf{R}$, $N \geq N_0$, of class C^1 such that **i)** $F_N(x) = F(x)$ for $\| x \| \leq \exp(N^2 + 2N - C)$, **ii)** $(\nabla F_N(x) - \nabla F_N(y), x - y) \leq (N^2 + 2N)\| x - y \|^2$ for all $x, y \in \mathbf{R}^n$, **iii)** for all $x \in \mathbf{R}^n$, $F_N(x) \geq ((N^2 + 1)\| x \|^2/2) - \exp(N^2 + C) - C)$, and **iv)** $\| \nabla F_N(x) \| \leq \exp(N)(\| x \| + 1)$.

If $F: \mathbf{R}^n \to \mathbf{R}$ is a function of class C^1 with $F(x) = \| x \|^2(\log(\| x \|^2) - 1)/4$ for $\| x \|$ large, then F satisfies condition (A). In fact, in this case we may take $F_N(x) = \| x \|^2(2N^2 + 4N - 1)/4$ for $\| x \| \geq \exp(N^2 + 2N)$. More examples of functions satisfying condition (A) can be obtained by using that condition (A) is stable under Lipshitzian perturbations.

We observe that condition (A) implies that F is superlinear, i.e., $F(x)/\| x \|^2$ tends to infinity as $\| x \| \to \infty$. Also condition (A) implies that $\nabla F(x)/\| x \|$ does not tend to infinity too fast. In fact, in the one dimensional case, $F'(x)/| x |$ tends to infinity as fast as $\log(| x |)$. Unfortunately our techniques do not cover the case where $F'(x)/| x |$ goes to infinity as $| x |^\alpha$, with $\alpha > 0$. The results announced in [12] cover this case but do not include the case $F(x) = x^2 \log(| x |)$.

For the sake of clarity we present our main result in the specific context of the problem of finding 2π-periodic weak solutions of

$$u'' + \nabla F(u) = p(t), \tag{1.1}$$

where p is 2π-periodic and $p \in (L_2(0, 2\pi))^n$.

The results of this paper hold under weaker forms of condition (A). However, we have stated condition (A) as above in order to make the computations as simple as possible keeping F superlinear. Our main result is:

Theorem 1. *If F satisfies property* (A) *then* (1.1) *has infinitely many 2π-periodic weak solutions.*

We prove Theorem 1 as follows. Using a well known minimax argument (see [4], [5]) we prove that for each $N \geq N_0$,

$$u'' + \nabla F_N(u) = p(t) \tag{1.2}$$

with p 2π-periodic and $p \in (L_2(0, 2\pi))^n$, has a 2π-periodic weak solution u_N.

* This research was supported by CONACYT grant PNCB 236.

Next, using the minimax characterization we obtain an a priori estimate on the functions u_N and we show that $u_N \neq u_m$ for $N < m$ and N sufficiently large. Finally, using this a priori estimate we show that infinitely many of the u_N are actually weak solutions of (1.1).

For other studies which consider related superlinear problems, see [2], [3], [6], [7], [8], [9], [10] and [11].

2. Main results

Throughout this paper $\int$ means integral over the interval $[0, 2\pi]$ unless otherwise indicated. Let H denote the Sobolev space of 2π-periodic functions $u: \mathbf{R} \to \mathbf{R}^n$ belonging to $(L_2(0, 2\pi))^n$ and having a generalized first order derivative in $(L_2(0, 2\pi))^n$. The inner product in H is given by

$$\langle u, v\rangle = \int ((u(t), v(t)) + (u'(t), v'(t)))\, dt \tag{2.1}$$

where $(\,,)$ denotes the usual inner product in $\mathbf{R}^n$. Let $\|\ \|_1$ denote the norm in H. Since for every $u \in H$ with $\int u(t)\, dt = 0$ we have $\int (u(t), u(t))\, dt \leq \int (u'(t), u'(t))\, dt$, we see that on the subspace of functions with mean value zero the norm $\|\ \|_1$ is equivalent to the norm given by

$$\| u \|_2 = (\textstyle\int (u'(t), u'(t))\, dt)^{1/2}. \tag{2.2}$$

By the Sobolev embedding theorem (see [1, pp. 97]) it follows that there exists a constant C_1 such that

$$\max\{\| u(t)\|; t \in \mathbf{R}\} \leq C_1 \| u \|_1 \tag{2.3}$$

for all $u \in H$. Moreover, the Sobolev embedding theorem implies that every element of H is a continuous function.

We define the functionals $J, J_N: H \to \mathbf{R}$ in the following form:

$$J(u) = \int (((u'(t), u'(t))/2) - F(u(t)) + (p(t), u(t)))\, dt; \tag{2.4}$$

$$J_N(u) = \int (((u'(t), u'(t))/2) - F_N(u(t)) + (p(t), u(t)))\, dt. \tag{2.5}$$

A simple computation shows that

$$\begin{aligned}\langle \nabla J(u), v\rangle &\equiv \lim_{t\to 0}((J(u + tv) - J(u))/t)\\ &= \int ((u'(t), v'(t)) - (\nabla F(u(t)) - p(t), v(t)))\, dt\end{aligned} \tag{2.6}$$

Hence an element $u \in H$ is a critical point of J iff u is a 2π-periodic weak solution of (1.1). Similarly, $u \in H$ is critical point of J_N iff u is a 2π-periodic weak solution of (1.2).

For each positive integer N we define

$$X_N = \{\textstyle\sum_{K=0}^{N} (a_K \sin(Kt) + b_K \cos(Kt));\ a_K, b_K \in \mathbf{R}^n\}$$

and Y_N the orthogonal complement of X_N in H. It is easily seen that Y_N is generated by the elements of the form $a \sin(Kt) + b \cos(Kt)$, with $a, b \in \mathbf{R}^n$ and $K \geq N + 1$. Clearly for each $x \in X_N$ we have

$$\int (x'(t), x'(t))\, dt \leq N^2 \int (x(t), x(t))\, dt, \tag{2.7}$$

and for $y \in Y_N$ we have

$$\int (y'(t), y'(t))\, dt \geq (N+1)^2 \int (y(t), y(t))\, dt. \tag{2.8}$$

By $(A - \mathbf{ii})$ and (2.8) we obtain

$$\langle \nabla J_N(x+y) - \nabla J_N(x+y_1), y - y_1 \rangle \geq \| y - y_1 \|_2^2/(N+1)^2 \tag{2.9}$$

for all $x \in X_N$, y, $y_1 \in Y_N$. Therefore, by Lemma 2.1 of [4], for each $x \in X_N$ there exists a unique $\Phi_N(x) \in Y_N$ such that

$$J_N(x + \Phi_N(x)) = \min\{J_N(x+y); y \in Y_N\}. \tag{2.10}$$

Moreover; the function $\Phi_N : X_N \to Y_N$ is continuous, the functional $\tilde{J}_N : X_N \to \mathbf{R}$, $x \to J_N(x + \Phi_N(x))$ is of class C^1 and

$$\langle \nabla \tilde{J}_N(x), x_1 \rangle = \langle \nabla J_N(x + \Phi_N(x)), x_1 \rangle \tag{2.11}$$

for all x, $x_1 \in X_N$.

Now, using $(A - \mathbf{iii})$ and (2.10), we have

$$\begin{aligned} 2\tilde{J}_N(x) &= 2J_N(x + \Phi_N(x)) \leq 2J_N(x) \\ &\leq \int ((x'(t), x'(t)) - (N^2+1)(x(t), x(t)))\, dt \\ &+ 2\|p\|_{L^2}\|x\|_{L^2} + 4\pi C + 4\pi \exp(N^2+1). \end{aligned} \tag{2.12}$$

Combining (2.7) and (2.12) we see that $\tilde{J}_N(x) \to -\infty$ as $\|x\|_1 \to \infty$. Consequently, there exists $x_N \in X_N$ which is a point of maximum of $\tilde{J}_N$. Hence

$$\begin{aligned} J_N(x_N + \Phi_N(x_N)) &= \tilde{J}_N(x_N) = \max\{\tilde{J}_N(x); x \in X_N\} \\ &= \max_{x \in X_N}(\min\{J_N(x+y); y \in Y_N\}). \end{aligned} \tag{2.13}$$

Since X_N and Y_N are complementary subspaces in H, by (2.10) and (2.11) we see that $u_N \equiv x_N + \Phi_N(x_N)$ is a critical point of J_N. Hence u_N is a 2π-periodic weak solution of (1.2).

We summarize the above discussion in:

LEMMA 2.1. *For each integer $N \geq N_0$ the equation* (1.2) *has a 2π-periodic weak solution satisfying the variational characterization* (2.13).

Now, using (2.13), we obtain an a priori estimate for the functions u_N.

LEMMA 2.2. *If u_N are as above, then there exists a real number C_2, independent of N, such that*

$$\|u_N\|_{L^2} < C_2 N \exp(N^2 + (C/2)). \tag{2.14}$$

PROOF: We denote $y_N = \Phi_N(x_N)$. From $(A$–$\mathbf{ii})$ it follows that there exists a constant C_3, independent of N, such that

$$F_N(x) \leq C_3(1 + \|x\|) + (N^2 + 2N)\|x\|^2/2 \tag{2.15}$$

Indeed, C_3 can be taken to be $\|\nabla F(0)\| + |F(0)|$. From (2.10) we infer that

$J_N(x_N + y_N) \leq J_N(x_N)$. Hence

$$\begin{aligned} \|y_N\|_2^2 &\leq \int(2F_N(x_N(t) + y_N(t)) - 2F_N(x_N(t)))\,dt + 2\,\|p\|_{L^2}\|y_N\|_{L^2} \\ &\leq \int(2C_3(1 + \|x_N(t)\| + \|y_N(t)\|) \\ &\quad + (N^2 + 2N)\|x_N(t) + y_N(t)\|^2)\,dt \\ &\quad + 4\pi C + 4\pi\exp(N^2 + C) + 2\|p\|_{L^2}\|y_N\|_{L^2}. \end{aligned} \tag{2.16}$$

In the foregoing set of inequalities we have used (2.15) and (A–**iii**). Replacing (2.8) in (2.16) we have

$$\begin{aligned} \|y_N\|_{L^2}^2 - (2\|p\|_{L^2} + 4\pi C_3)\|y_N\|_{L^2} &\leq 4\pi C_3 + 4\pi C_3\|x_N\|_{L^2} \\ &\quad + (N^2 + 2N)\|x_N\|_{L^2}^2 + 4\pi C + 4\pi\exp(N^2 + C). \end{aligned} \tag{2.17}$$

From this we see that there exists a constant C_4, independent of N, such that

$$\|y_N\|_{L^2} \leq C_4(1 + (N + 1))\|x_N\|_{L^2} + 2\pi\exp(N^2 + (C/2)). \tag{2.18}$$

By (2.15) we have

$$\begin{aligned} J_N(y) &\geq (\|y\|_2^2/2) - \int F_N(y(t))\,dt - \|p\|_{L^2}\|y\|_{L^2} \\ &\geq (\|y\|_2^2/2) - \|p\|_{L^2}\|y\|_{L^2} - 2\pi C_3 \\ &\quad - 2\pi C_3\|y\|_{L^2} - (N^2 + 2N)\,\|y\|_{L^2}^2/2 \\ &\geq \quad -(\|p\|_{L^2} + 2\pi C_3)^2 - 2\pi C_3. \end{aligned} \tag{2.19}$$

Hence (A-**iii**), (2.13) and (2.19) imply

$$\begin{aligned} &-2\pi C_3 - (\|p\|_{L^2} + 2\pi C_3)^2 \leq \inf\{J_N(y); y \in Y_N\} \\ &\leq J_N(x_N + \Phi_N(x_N)) \leq J_N(x_N) \\ &\leq (\|x_N\|_2^2/2) - ((N^2 + 1)\,\|x_N\|_{L^2}^2/2) + \|p\|_{L^2}\|x_N\|_{L^2} \\ &+ 2\pi\exp(2N^2 + C) + 2\pi C. \end{aligned} \tag{2.20}$$

From (2.7) and (2.20) it follows that there exists a constant C_5, independent of N, such that

$$\|x_N\|_{L^2} \leq C_5(1 + \exp(N^2(C/2))). \tag{2.21}$$

Combining (2.18) and (2.21) we have (2.14) and Lemma 2.2 has been proved.

Proof of Theorem 1. By (A − **iv**) and (2.14) we have

$$\begin{aligned} \|\nabla F_N(u_N)\|_{L^2} &\leq \exp(N)(4\pi\|u_N\|_{L^2} + 4\pi) \\ &\leq \exp(N)(4\pi C_2(N + 1)(\exp(N^2 + (C/2) + 1) + 4\pi) \\ &\leq C_6(1 + (N + 1)\exp\,(N^2 + N + 1)). \\ &\leq C_6 N\exp\,(N^2 + N + (C/2)), \end{aligned} \tag{2.22}$$

where C_6 is a constant independent of N.

Since u_N is a weak solution of (1.2), we have $\int(u_N'(t), u_N'(t))dt = \int(\nabla F_N(u_N(t)), u_N(t))\, dt - \int(p(t), u_N(t))\, dt \leq \|\nabla F_N u_N)\|_{L^2} \|u_N\|_{L^2} + \|p\|_{L^2} \|u_N\|_{L^2}$. Hence, using (2.14) and (2.22), we obtain

$$\begin{aligned} \|u_N\|_1^2 = \|u_N\|_2^2 + \|u_N\|_{L^2}^2 \\ \leq C_2 N(\exp(N^2 + (C/2)))(C_6 N\exp(N^2 + N + (C/2))) \\ + (1 + \|p\|_{L^2}) C_2(N+1)(\exp(N^2 + (C/2))). \end{aligned}$$

The latter inequality implies that there exists a constant C_7, independent of N, such that

$$\|u_N\|_1 \leq C_7 N\exp(N^2 + N + (C/2)). \tag{2.23}$$

Then by (2.3) and (2.23), we see that there exists a positive integer N_1 such that

$$\max\{\|u_N(t)\|;\, t \in \mathbf{R}\} \leq \exp(N^2 + 2N) - 1 \tag{2.24}$$

for all $N \geq N_1$. Consequently, by (A–**i**), we see that for $N \geq N_1$, u_N is actually a 2π-periodic weak solution of (1.1).

Now we claim that if N, $m \geq N_1$ and $N \neq m$ then $u_N \neq u_m$. Indeed, let $N < m$ and suppose $u_N = u_m$. If $x_N \neq x_m$, then by (A–**ii**), (2.3), (2.9) and (2.24), for some $\epsilon > 0$ small enough we have

$$\begin{aligned} \tilde{\mathcal{J}}_m(x_m) &= \mathcal{J}_m(x_m + \Phi_m(x_m)) = \mathcal{J}_N(x_N + \Phi_N(x_N)) \\ &< \mathcal{J}_N(x_N + (1+\epsilon)(x_m - x_N) + \Phi_m(x_N + (1+\epsilon)(x_m - x_N))) \\ &= \tilde{\mathcal{J}}_m(x_N + (1+\epsilon)(x_m - x_N)). \end{aligned}$$

But this contradicts that $\tilde{\mathcal{J}}_m(x_m)$ is the maximum value of $\tilde{\mathcal{J}}_m$. If on the other hand $x_m = x_N$, then $\Phi_N(x_N) = \Phi_m(x_m)$, and replacing $(1+\epsilon)(x_m - x_N)$ by $\epsilon v(t) \equiv \epsilon(\sin((N+1)t), 0, \cdots, 0) \in X_m \cap Y_N$ in the latter inequality gives $\tilde{\mathcal{J}}_m(x_m) < \tilde{\mathcal{J}}_m(x_N + \epsilon v(t))$, again a contradiction. Thus we conclude that $u_N \neq u_m$, and the proof of Theorem 1 is complete.

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