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A NOTE ON THE CORRESPONDENCE BETWEEN FUCHSIAN GROUPS AND ALGEBRAIC CURVES

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Let Γ be a torsion free Fuchsian group acting on the unit disc $D = \{z \in \mathbf{C}; |z| < 1\}$ such that D/Γ is a compact surface of genus n. It is known that D/Γ is conformally equivalent to an algebraic curve given by a polynomial $p(z, w)$. In general there is no known method of determining the polynomial $p(z, w)$ from a given Fuchsian group Γ. For certain cases, however, it is possible to obtain information about this correspondence (see for example Burnside [2]). In this note we determine the Fuchsian groups corresponding to polynomial equations of the form $w^2 = z(z^{2n} - 1)$, $n > 1$. We remark that Burnside [2] considers the case when $n = 2$. Our method is to study a certain class of Fuchsian groups Γ such that D/Γ is hyperelliptic (see also Whittaker [6]). Then, by analyzing a subgroup of the group of conformal automorphisms of D/Γ, we are able to determine the associated polynomial.

The unit disc D carries a hyperbolic metric

$$\left(ds^2 = \frac{|dz|}{1 - |z|^2}\right)$$

in which geodesics are arcs of circles orthogonal to the unit circle (including straight lines through the origin). The Fuchsian groups we will consider are obtained, for each $n > 1$, from the regular, $4n$ sided, hyperbolic polygon P with interior angles that add up to 2π (see [1]).

We label the sides of P by A_m, $m = 1, \cdots, 4n$, in counterclockwise order. Since the A_m have equal hyperbolic lengths, there exist non-elliptic Möbius transformations T_m, $m = 1, \cdots, 2n$, which fix D and identify opposite sides of P (i.e. $T_m(A_m) = A_{m+2n}$, $m = 1, \cdots, 2n$). It now follows from Poincaré's theorem (see Maskit [4]) that the group Γ generated by the T_m is Fuchsian and that P is a fundamental polygon for Γ.

At this point we need to state some elementary facts about hyperelliptic surfaces. A compact Riemann surface S of genus n is *hyperelliptic* if there exists a conformal map $J: S \mapsto S$ (called the *hyperelliptic involution*) such that $J^2 = \text{id.}$ and J has $2n + 2$ fixed points (called the *Weierstrass points*). Let $\langle J \rangle$ be the group generated by J and let $S/\langle J \rangle$ be the surface obtained by identifying points in S via the action of J. Since the natural projection $\tilde{J}: S \to S/\langle J \rangle$ is branched to order two at the Weierstrass points, it follows from the Riemann-Hurwitz relation that $S/\langle J \rangle$ is conformally the Riemann sphere $\hat{\mathbf{C}}$. For a thorough exposition the reader is refered to a standard text such as [5].

Given $z \in D$, we denote by $\{z\}$ the orbit of z under the group Γ. We note

[1] The results of this paper are included in the author's doctoral dissertation at the State University of New York, under the direction of Professor Irwin Kra.

that P is invariant under the transformation $j(z) = -z$. Define map $J: D/\Gamma \to D/\Gamma$ by $J(\{z\}) = \{j(z)\}$. W give a brief sketch of the proof that J is well defined. Clearly we need only show that $j\circ\Gamma\circ j = \Gamma$. Since each generator T_m, $m = 1, \cdots, 2n$, fixes D, it is of the form

$$T_m(z) = \frac{\alpha_m z + \beta_m}{\bar{\beta}_m z + \bar{\alpha}_m}, \; \alpha_m, \beta_m \in \mathbf{C}.$$

Let $a_m{}^1$ and $a_m{}^2$ be the vertices of the side A_m. Then $T_m(a_m{}^1) = -a_m{}^2$ and $T_m(a_m{}^2) = -a_m{}^1$. A short computation now yields that $\alpha_m = \bar{\alpha}_m$ and $\alpha_m \in \mathbf{R}$. It follows easily that j conjugates each generator to its inverse. An induction argument then gives that $j\circ\Gamma\circ j = \Gamma$ (see [3]).

Since the Euler-Poincaré characteristic of P is $2 - 2n$ it follows that D/Γ is a compact Riemann surface of genus n. It is trivial to verify that all the vertices of P are equivalent under Γ. Hence $J(\{v\}) = \{v\}$ where v is any vertex of P. Let c_m, $m = 1, \cdots, 2n$ be the hyperbolic midpoint of the side A_m. Since T_m is a hyperbolic isometry it follows that $T_m(c_m) = -c_m$. Hence $J(\{c_m]\}) = \{c_m\}$. Clearly $J(\{0\}) = \{0\}$. Since P is a fundamental polygon and identifications occur in pairs, it is obvious that $\{0\}$, $\{v\}$ and $\{c_m\}$ are distinct fixed points of J. It is well known that a non-trivial conformal automorphism of a compact surface has at most $2n + 2$ fixed points. Hence J is a hyperelliptic involution. For full details see [3].

We extend the group Γ by the map $j(z)$ to obtain the group $\langle j, \Gamma\rangle$. Since D/Γ is hyperelliptic, it follows that $D/\langle j, \Gamma\rangle$ is conformally a sphere with $2n + 2$ distinguished points which are the fixed points of elliptic elements of order two in $\langle j, \Gamma\rangle$ (see [3]).

We now look more closely at a group of conformal automorphisms of D/Γ determined by symmetries of P.

Let $\gamma: D \to D$ be the map defined by

$$\gamma(z) = \left(\exp \frac{i2\pi}{4n}\right)z.$$

Clearly the sides of P satisfy the following relations:

$$A_2 = \gamma(A_1),\ A_3 = \gamma^2(A_1),\ \cdots,\ A_m = \gamma^{m-1}(A_1)$$

$$m \leq 4n. \text{ Thus } T_m = \gamma^{m-1} \circ T_1 \circ \gamma^{-(m-1)},\ m = 2, \cdots, 2n.$$

It now follows that $\gamma^{-1} \circ \Gamma \circ \gamma = \Gamma$. Therefore γ induces an automorphism γ^* of D/Γ (where $\gamma^*(\{z\}) = \{\gamma(z)\}$) such that the following diagram commutes

$$\begin{array}{ccc} D & \xrightarrow{\ \gamma\ } & D \\ \Big\downarrow p & & \Big\downarrow p \\ D/\Gamma & \xrightarrow{\ \gamma^*\ } & D/\Gamma \end{array}$$

(here p is the natural projection).

Each map $(\gamma^*)^k$, $1 \leq k < 4n$, $k \neq 2n$, fixes the points $\{0\}$ and $\{v\}$, and no other points. When $k = 2n$, $(\gamma^*)^k$ is the hyperelliptic involution J and its fixed points are $\{0\}$, $\{v\}$ and $\{c_m\}$, $m = 1, \cdots, 2n$.

Finally, we look at the projection of γ^* as a map on the Rieman sphere $\hat{\mathbf{C}}$. It is clear that $\gamma^{-1} \circ \langle j, \Gamma\rangle \circ \gamma = \langle j, \Gamma\rangle$. Hence γ^* induces an automorphism $\hat{\gamma}$ of $\hat{\mathbf{C}}$ such that the following diagram commutes

$$\begin{array}{ccc} D/\Gamma & \xrightarrow{\ \gamma^*\ } & D/\Gamma \\ \downarrow{\tilde{J}} & & \downarrow{\tilde{J}} \\ \hat{\mathbf{C}} & \xrightarrow{\ \hat{\gamma}\ } & \hat{\mathbf{C}} \end{array}$$

(where $\hat{\gamma}(\{\{z\}\}) = \{\gamma^*(\{z\})\}$).

Since $(\gamma^*)^{2n} = J$, it follows that $\hat{\gamma}^{2n} = \text{id}$. Hence $\hat{\gamma}$ is elliptic of order $2n$ and its fixed points are $\tilde{J}(\{0\})$ and $\tilde{J}(\{v\})$. Moreover,

$$\tilde{J}(\{c_{m+1}\}) = \tilde{J}(\gamma\{c_m\}) = \hat{\gamma}(\tilde{J}(\{c_m\}), \quad m = 1, \cdots, 2n-1 \tag{1}$$

and

$$\tilde{J}(\{c_1\}) = \tilde{J}(\gamma^*(\{c_{2n}\}) = \hat{\gamma}(\tilde{J}(\{c_{2n}\})). \tag{2}$$

By normalizing, we may assume that $\tilde{J}(\{0\}) = 0$, $\tilde{J}(\{v\}) = \infty$, $\tilde{J}(\{c_1\}) = 1$ and $\hat{\gamma}$ is a rotation about the origin by the angle $\frac{\pi}{n}$. From (1) and (2) we obtain that $\tilde{J}(\{c_{m+1}\}) = \exp\frac{\mathrm{i}m\pi}{n}$, $m = 0, \cdots, 2n-1$.

The surface corresponding to the polynomial $w^2 - z\prod_{m=0}^{2n-1}\left(z - \exp\frac{\mathrm{i}m\pi}{n}\right)$ is branched over 0, ∞ and the $2n$ points $\exp\frac{\mathrm{i}m\pi}{n}$, $m = 0, \cdots, 2n-1$. These are, up to a Möbius transformation precisely the points over which the twofold cover $\tilde{J}: D/\Gamma \to \hat{\mathbf{C}}$ is branched. It is known classically that the conformal structure of a hyperelliptic surface is determined by the branch points of the associated twofold cover of the sphere. Thus we have that the polynomial corresponding to

$$D/\Gamma \text{ is } w^2 - z\prod_{m=0}^{2n-1}\left(z - \exp\frac{\mathrm{i}m\pi}{\mathrm{n}}\right).$$

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