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KLEIN SURFACES AS ORBIT SPACES OF NEC GROUPS

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1. Introduction

Let Γ be an NEC group, then $X = D/\Gamma$ is a Klein surface (D is the complex upper half plane) and X can be represented as D/Γ', where Γ' is a surface group. Moreover, if Γ_1 is an NEC group such that $\Gamma \lhd \Gamma_1$ ($\lhd$ denotes normal subgroup), the group $G = \Gamma_1/\Gamma$ is a group of automorphisms of X, and so there exists an NEC group Γ_1' such that $\Gamma' \lhd \Gamma_1'$ and $G' = \Gamma_1'/\Gamma'$ is a group of automorphisms of X isomorphic to G.

In this paper we investigate the relationship between Γ and Γ', and when the order of G is odd, we also study the relationship between Γ_1 and Γ_1'.

The corresponding problem for Reimann surfaces has been studied by Moore in [4].

2. NEC groups and Klein surfaces

By a *non-Euclidean crystallographic (NEC) group* [8], we shall mean a discrete subgroup Γ of the group of isometries of the non-Euclidean plane with compact quotient space, including those reversing orientation, reflections and glide reflections.

NEC groups are classified according to their *signatures*. The signature of an NEC group Γ is either of the form:

(*) $$(g\,;+;\,[m_1 \cdots m_\tau];\quad \{(n_{i1} \cdots n_{is_i})i = 1 \cdots k\})$$

or

(**) $$(g\,;-;\,[m_1 \cdots m_\tau];\quad \{(n_{i1} \cdots n_{is_i})i = 1 \cdots k\})$$

A group Γ with signature (*) has the presentation given by generators:

i) x_i, $\quad i = 1 \cdots \tau$
ii) c_{ij}, $\quad i = 1 \cdots k \quad j = 0 \cdots s_1$
iii) e_i, $\quad i = 1 \cdots k$
iv) a_j, b_j, $\quad j = 1 \cdots g$

and relations:

1) $x_i^{m_i} = 1 \quad i = 1 \cdots \tau$
2) $c_{is_i} = e_i^{-1} c_{i0} e_i \quad i = 1 \cdots k$
3) $1 = c_{i(j-1)}{}^2 = c_{ij}{}^2 = (c_{i(j-1)} \cdot c_{ij})^{n_{ij}} \quad i = 1 \cdots k \quad j = 0 \cdots s_i$
4) $x_1 \cdots x_\tau e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1$

A group Γ with signature (**) has a similar presentation to that of a group with signature (*), changing the generators iv) for d_j $j = 1 \cdots g$ and the relations 4) by $x_1 \cdots x_\tau e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1$.

From now on we will denote by x_i, e_i, c_{ij}, a_i, b_i, d_j the above generators associated with an NEC group Γ.

We will say that an NEC group Γ is *the group of a surface* with k boundary components if it has the following signature

$$(g\,;\pm;\,[-];\,\{(-),\,\overset{k}{\cdots\cdots},\,(-)\})$$

$+$ for the orientable case and $-$ for the non orientable, and where $(-)$ indicates the empty symbol; i.e., $\tau = 0$, $s_i = 0$ for $i = 1, \cdots, k$.

By a *Klein surface* we shall mean a surface X with or without boundary together with an open covering by a family of sets $U = \{U_i\}$ with the properties:

1) For each $U_i \in U$ there exists a homeomorphism ϕ_i of U_i onto an open set in either $\mathbb{C}$ or $\mathbb{C}^+$.

2) If U_i, $U_j \in U$ and $U_i \cap U_j \neq \emptyset$ then $\phi_i \circ \phi_j^{-1}$ is an analytic or antianalytic mapping defined on $\phi_j(U_i \cap U_j)$. (See [1] and [6]).

A homeomorphism $f{:}X \rightarrow X$ is called an *automorphism* if $\phi_i{\cdot}f{\cdot}\phi_j^{-1}$ is either an analytic or antianalytic mapping in its domain of definition.

If X is orientable and without boundary we will say that it is a *Riemann surface.*

If Γ is an NEC group, then the quotient space D/Γ has a unique dianalytic structure such that the quotient map $p{:}D \rightarrow D/\Gamma$ is a morphism of Klein surfaces [Alling, Greenleaf 1].

Let X be a compact Klein surface of algebraic genus $g \geq 2$. Then X can be represented in the form D/Γ, where Γ is a surface group [Singerman 7] and [Preston 5]. Moreover, G is a group of automorphisms of the Klein surface D/Γ if and only if $G = \Gamma'/\Gamma$ where Γ' is an NEC group such that $\Gamma \subset \Gamma' \subset N(\Gamma)$ [Singerman 7] and [May 3].

3. Klein surfaces as orbit spaces

Let Γ be an NEC group, then $X = D/\Gamma$ is a Klein surface and X can be represented in the form D/Γ', where Γ' is a surface group. The following theorem studies the relationship between Γ and Γ'.

THEOREM (3.1). *Let Γ be an NEC group and let $X = D/\Gamma$ be the Klein surface associated with Γ. Let M be the subgroup of Γ generated by the elliptic transformations of Γ. Then:*

1) *The orbit space $D_1 = D/M$ is conformally equivalent to D and $D_1/(\Gamma/M)$ is isomorphic to X.*

2) *The surface group for X is the factor group Γ/M, with $X = D_1/(\Gamma/M)$*

Proof. As Γ is an NEC group, it has an associated signature

$$(g\,;\pm;\,[m_1\,\cdots\,m_\tau];\quad \{(n_{i1}\,\cdots\,n_{is_i})i = 1\,\cdots\,k\}).$$

We will prove the theorem in the $+$ case; the $-$ case being similar.

Let Γ' be an NEC group with signature

$$(g\,;+;\,[-];\ \{(-)\overset{k}{\cdots\cdots}(-)\})$$

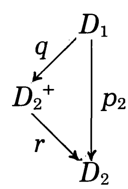
and let $\theta:\Gamma \to \Gamma'$ be the epimorphism defined by

$$\begin{array}{lll}
\theta(a_j) = a_j' & j = 1 \cdots g & \\
\theta(b_j) = b_j' & j = 1 \cdots g & \\
\theta(x_i) = 1 & i = 1 \cdots \tau & \\
\theta(e_i) = e_i' & & \\
\theta(c_{ij}) = c_i' & i = 1 \cdots k & j = 0 \cdots s_i
\end{array}$$

Let M be the kernel of θ, M is a normal subgroup of Γ generated by the elliptic transformations of Γ, and $\Gamma/M \simeq \Gamma'$.

Let Γ^+ be the canonical Fuchsian group [7] associated with Γ and let $D_1 = D/M$. By [1] the canonical map $p_1:D \to D_1$ gives to D_1 a structure of Klein surface, and Γ/M is a group of automorphisms of D_1 acting discontinuously on D_1, then by [1] $D_2 = D_1/(\Gamma/M)$ is a Klein surface, and the double cover of D_2 is the Riemann surface $D_2^+ = D_1/(\Gamma^+/M)$.

Let the following diagram be commutative



where q and p_2 are the canonical maps and r is the map of the double cover. As the map $q:D_1 \to D_2^+$ fulfill the conditions of the main theorem of [4], D_1 is the universal covering space of D_2^+ and D_1 is conformally equivalent to the upper half plane.

To prove the first part of the theorem, we only need to prove that D_2 is isomorphic to $X = D/\Gamma$.

As the double cover of X is D/Γ^+, and by [4], D/Γ^+ is isomorphic to D_2^+. Then it is easy to verify that this isomorphism fulfills the conditions of (1.9.3) of [1], and so D_2 is isomorphic to X.

Let us prove the second part. Identifying D_1 with the upper half plane, Γ/M is a discrete group of automorphisms of the upper half plane with compact quotient space. Hence, Γ/M is an NEC group. As Γ/M is isomorphic to Γ', Γ/M is a surface group and $X = D_1/(\Gamma/M)$ □.

In the hypothesis of the beginning of the section, if Γ_1 is an NEC group such that $\Gamma \lhd \Gamma_1$, $G = \Gamma_1/\Gamma$ is a group of automorphisms of the Klein surface D/Γ, then by (§2) there exists an NEC group Γ_1' such that $\Gamma' \lhd \Gamma_1'$ and G'

$= \Gamma_1'/\Gamma'$ is an automorphism group of D/Γ', isomorphic to G. Now we study the relationship between Γ_1 and Γ_1' in the case that the order of G is odd.

THEOREM (3.2). *Let Γ be an NEC group and $X = D/\Gamma$ the Klein surface associated to Γ. If $\Gamma \lhd \Gamma_1$, $[\Gamma_1:\Gamma] = N$, N odd and M is the subgroup of Γ generated by the elliptic transformations of Γ, then Γ_1/M becomes an NEC group when D_1 is identified with D (where $D_1 = D/M$). Moreover $\Gamma/M \lhd \Gamma_1/M$ and $(\Gamma_1/M)/(\Gamma/M)$ is the group of automorphisms of $X = D_1/(\Gamma/M)$ isomorphic to Γ_1/Γ.*

Proof. Let us suppose that the NEC group Γ has in its signature the sign +, the other case is similar.

By [2] Γ_1 and Γ have the same sign in their signatures, therefore the signature of Γ_1 is of the form

$$(g' ;+; [\mu_1 \cdots \mu_t]; \quad \{(p_{i1} \cdots p_{ir_i})i = 1 \cdots s\})$$

Suppose that n_i is the least n, such that $x_i{}^n$ is in Γ, and k_i is the least k, such that $e_i{}^k$ is in Γ.

Let Γ_1' be an NEC group with signature

$$(g' ;+; [n_1 \cdots n_t]; \quad \{(-)\overset{s}{\cdots\cdots}(-)\})$$

and let $\theta:\Gamma_1 \to \Gamma_1'$ be the epimorphism defined by

$$\begin{aligned}
&\theta(a_i) = a_i' \\
&\theta(b_i) = b_i' \qquad i = 1 \cdots g' \\
&\theta(x_i) = x_i' \qquad i = 1 \cdots t \\
&\theta(e_i) = e_i' \qquad i = 1 \cdots s \\
&\theta(c_{ij}) = c_i' \qquad i = 1 \cdots s \qquad j = 0 \cdots r_i
\end{aligned}$$

As $\Gamma \lhd \Gamma_1$, $\theta(\Gamma) \lhd \Gamma_1'$ and $M = \ker \theta$ is generated by the elliptic transformations of Γ, then $\ker \theta \lhd \Gamma$, so

$$\frac{\Gamma}{M} \simeq \theta(\Gamma) \quad \text{and} \quad \frac{\Gamma_1}{M} \simeq \Gamma_1'$$

By the second isomorphism theorem,

$$\frac{\Gamma_1}{\Gamma} \simeq \frac{\Gamma_1}{\theta(\Gamma)}$$

so $[\Gamma_1':\theta(\Gamma)] = N$. If the signature of Γ is

$$(g ;+; [m_1 \cdots m_\tau]; \quad \{(n_{11} \cdots n_{1s_1}) \cdots (n_{k1} \cdots n_{ks_k})\})$$

as x_i' has exponent n_i modulo $\theta(\Gamma)$ and e_i' has exponent k_i modulo $\theta(\Gamma)$ then by [2] the signature of $\theta(\Gamma)$ is

$$(g\,;+;\,[-];\ \ \{(-),\ \overset{k}{\cdots\cdots},\ (-)\})$$

If we consider θ restricted to Γ, then $\theta\colon\Gamma \to \Gamma' = \theta(\Gamma)$ is a homeomorphism in the conditions of the previous theorem, hence $X = D_1/(\Gamma/M)$. As $\Gamma/M \lhd \Gamma_1/M$ and Γ/M is an NEC group, and the normalizer of an NEC group is an NEC group, then Γ_1/M is an NEC group, since Γ_1/M is a group of automorphisms of D_1. Moreover $(\Gamma_1/M)/(\Gamma/M) \simeq \Gamma_1/\Gamma$ □.

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