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ON THE INTEGRAL HOMOLOGY OF THE SYMMETRIC GROUP

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§1. Introduction

In [C-L; C-L1; T] results are obtained on the integral cohomology ring, $H^*(\Sigma_n; Z)$, for the symmetric group on n letter when $n \leq 4$. As n increases the relations in these cohomology rings become prohibitive. In this note we determine the homology Bockstein spectral sequence for $H_*(\Sigma_n; Z/p)$ for each n and each prime p. This determines the homology groups $H_*(\Sigma_n; Z)$. For $H_i(\Sigma_n; Z)$ is a finite torsion group for all $i > 0$ and for each prime p the dimension of the boundaries ($r \geq 1$)

$$\dim_{Z/p}(d_r : E_{m+1}{}^r(\Sigma_n; Z/p) \to E_m{}^r(\Sigma_n; Z/p))$$

is equal to the number of Z/p^r summands in

$$H_m(\Sigma_n; Z) = \oplus_p \oplus_i (Z/p^i)^{\alpha_i}.$$

The purpose of this note can be explained as follows. Although we originally did our computations by imitating [M], our results are presented here as easy deductions from results of J. P. May. We were prompted to write this note by the fact that those who study cohomology of symmetric groups algebraically may be unfamiliar with the infinite loopspace results and techniques involved in the results of [C-M-T; Ma] which we use.

§2

Let p be a prime.

Recall that there is a map

$$B\Sigma_\infty \to Q_0S^0 \qquad (2.1)$$

which induces isomorphisms in homology with any coefficients. Here $QS^0 = \varinjlim \Omega^n S^n$ and Q_0S^0 denotes the component of maps of degree zero.

Let us recall the mod p homology structure of the spaces in (2.1). If $p = 2$ a sequence $I = (s_1, \cdots, s_k)$ of non-negative integers has [Ma, p. 16]

$$\begin{aligned} &\textit{length} \quad \ell(I) = k \\ &\textit{degree} \quad d(I) = \textstyle\sum_{j=1}^k s_j \\ &\textit{excess} \quad e(I) = s_1 - \textstyle\sum_{j=2}^k s_j \end{aligned}$$

and is *admissible* if $2s_j \geq s_{j-1} (2 \leq j \leq k)$. When $p > 2$ a sequence $I = (\epsilon_1, s_1, \epsilon_2,$

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$s_2, \cdots, e_k, s_k)$ ($\epsilon_j = 0$ or 1; $s_j \geq \epsilon_j$) has

$$\textit{length} \quad \ell(I) = k$$

$$\textit{degree} \quad d(I) = \textstyle\sum_{j=1}^{k} 2s_j(p-1) - \epsilon_j$$

$$\textit{excess} \quad e(I) = 2s_k - \epsilon_1 - (\textstyle\sum_{j=2}^{k} 2ps_j - \epsilon_j - 2s_{j-1})$$

and is *admissible* if $ps_j - \epsilon_j \geq s_{j-1} (2 \leq j \leq k)$. Define $b(I) = 0$ when $p = 2$ and $b(I) = \epsilon_1$ for an odd prime, p.

Let $[1] \in \tilde{H}^0(S^0; Z/p)$ denote the class of the non-basepoint then [Ma, p. 47] $H_*(Q_0S^0; Z/p)$ is the free, graded-commutative algebra on $\{Q^I[1]*[-p^{\ell(I)}]$; I admissible, $e(I) + b(I) > 0$, $d(I) > 0\}$. Here $Q^{(s_1 \cdots k)} = Q^{s_1}Q^{s_2} \cdots Q^{s_k}$ if $p = 2$ and $Q^{(\epsilon_1, s_1, \epsilon_2, s_2 \cdots)} = \beta^{\epsilon_1}Q^{s_1}\beta^{\epsilon_2} \cdots Q^{s_k}$ if $p > 2$ where Q^i is the Dyer-Lashof operation [Ma, p. 6; K-P] and β is the Bockstein.

For a sequence I let q^I denote the element of $H_*(\Sigma_\infty; Z/p)$ which hits $Q^I[1]*[-p^{\ell(I)}]$ under (2.1).

Thus we have

(2.2) Under the product induced by disjoint union of sets, $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, $H_*(\Sigma_\infty; Z/p)$ is the free commutative algebra on $\{q^I$; $e(I) + b(I) > 0$, $d(I) > 0$, I admissible$\}$.

The Bockstein spectral sequence for Q_0S^0 is determined in [Ma, §4.13]. By means of (2.1) we may use this to write down the Bockstein spectral sequence of the infinite symmetric group

$$\{E_*^r(\Sigma_\infty); d_r, r \geq 1\}$$

in which $E_s^2 = H_s(\Sigma_\infty; Z/p)$.

THEOREM (2.3). *If $p = 2$ let*

$$\mathcal{S} = \{q^I \mid I = (2s, J), e(I) > 1, \ell(I) > 0, d(I) \textit{ even}\}$$

and when $p > 2$ let

$$\mathcal{S} = \{q^I \mid b(I) = 0, d(I) \equiv 0 \pmod 2, \ell(I) > 0\}.$$

Then for all primes p and integers $r \geq 1$ we have

$$E_*^{r+1}(\Sigma_\infty) \cong P(y^{p^r} \mid y \in \mathcal{S}) \ominus E(d_{r+1}(y^{p^r}) \mid y \in \mathcal{S})$$

where

$$d_{r+1}(y^{p^r}) = \begin{cases} y^{p^r-1}\beta y & \text{if } p > 2 \\ y^{2^r-2}(y\beta y + Q^{2q}\beta y) & \text{if } p = 2, \deg(y) = 2q. \end{cases}$$

Remarks (2.4).

(a) The d_1-differential is given by β which is easily computed, using at $p = 2$ the relation $\beta Q^{2s} = Q^{2s-1}$.

(b) Originally we had computed the above Bockstein spectral sequence by means of Pontrjagin power operations, imitating the treatment of [M].

(2.5). In [C-M-T] splittings in the stable homotopy category are constructed which generalise the second author's splitting of QX [Sn]. In particular it is shown that there is an S-map $\beta\Sigma_\infty \to B\Sigma_n$ which splits the natural map $B\Sigma_n \to B\Sigma_\infty$. Hence we obtain the Bockstein spectral sequence for each finite symmetric group as a summand in the Bockstein spectral sequence of Theorem 2.3. To write down explicitly what this amounts to we will need a little notation. Recall that q^I belongs to $H_*(\Sigma_{p^{\ell(I)}}; Z/p)$ and that

$$q^{I_1}q^{I_2}\cdots q^{I_m} \in H_*(\Sigma_i; Z/p)$$

where $I = \ell(I_1) + \ell(I_2) + \cdots + \ell(I_m)$, call this number the *weight* of the above monomial and write it $wt(q^{I_1}\cdots q^{I_m})$. Hence, by way of example, $wt(y^{p^r}) = p^r wt(y)$ which equals the weight of each monomial in $d_{r+1}(y^{p^r})$. Finally define the *weight of a polynomial* to be the maximum weight of its non-zero monomial terms.

Hence we have shown the following.

THEOREM (2.6). *For $1 \leq n \leq \infty$ and p a prime, the Bockstein spectral sequence for Σ_n, $[E_*{}^r(\Sigma_n), d_r, r \geq 1]$ is isomorphic to the sub-spectral sequence of* §2.3 *consisting of elements of weight $\leq n$.*

In order to complete the description of the Bockstein spectral sequences for $H_*(\Sigma_n; Z/p)$ it suffices by §§2.3 and 2.6 to describe $(E_*{}^1(\Sigma_\infty), d_1)$. This is trivial when $p \neq 2$ since

$$q^{(1,s_1,\epsilon_2,s_2,\cdots,\epsilon_k,s_k)} = d_1(q^{(0,s_1,\epsilon_2,s_2\cdots,\epsilon_k,s_k)}), \tag{2.7}$$

since $d_1 = \beta$, the Bockstein.

PROPOSITION (2.7). *If $p > 2$ then $E_*{}^1(\Sigma_\infty)$ is isomorphic to*

$$(\otimes_{I\in\mathcal{T}} P(q^I) \otimes E(d_1 q^I)) \otimes (\otimes_{J\in\mathcal{S}} E(q^J) \otimes P(d_1 q^J))$$

where $\mathcal{T} = [I \mid \ell(I) > 0, d(I) \equiv 0\ (2), b(I) = 0$ and $\ell(I) > 0]$. $\mathcal{S}$ is as in §2.3.

When $p = 2$ the situation is more complicated. However we may choose generators in the following manner (imitating [M]) to see what is happening.

Let $p = 2$ then we encounter the problem that q^1 with $e(I) = 0$ is a square, which complicates the algebra. For $I = (s_1, s_2, \cdots, s_k) \in \mathcal{S}$ (as in §2.3) let

$$I - \Delta_1 = (s_1 - 1, s_2, \cdots, s_k)$$

$$a(I) = (d(I), s_1 - 1, s_2, \cdots, s_k)$$

$$b_1(I) = (2d(I), d(I), s_1 - 1, s_2, \cdots).$$

$$b_2(I) = (4d(I), 2d(I), d(I), s_1 - 1, s_2, \cdots)$$

and in general $b_r(I) = (2^r d(I), b_{r-1}(I))$. Set $x_{a(I)} = q^{a(I)} + q^I q^{I-\Delta_1}$, $x_{b_1(I)} = q^{b_1(I)} + (q^I)^3 q^{I-\Delta_1}$ and in general $x_{b_r(I)} = q^{b_r(I)} + (q^I)^{2^{r+1}} - 1_q{}^{I-\Delta_1}$. Set $(\mathcal{A}(I), d_1)$ equal to

$Z/2[q^I, q^{I-\Delta_1}, x_{a(I)}, x_{b_1(I)}, x_{b_2(I)}, \cdots]$ *with*

$$d_1(q^I) = q^{I-\Delta_1},$$
$$d_1(x_{a(I)}) = 0,$$
$$d_1(x_{b_1(I)}) = x_{a(I)}{}^2 \text{ and}$$
$$d_1(x_{b_r(I)}) = x_{b_{r-1}(I)}{}^2 \text{ for } r \geq 2.$$

For $I = (s_1, s_2, \cdots, s_k)$ with $e(I) > 1$, s_1 even and $d(I)$ odd write

$$c_1(I) = (d(I) + 1, s_1, s_2, \cdots, s_k)$$
$$c_2(I) = (2d(I) + 2, d(I) + 1, s_1, s_2, \cdots, s_k)$$

and in general

$$c_r(I) = (2^{r-1} d(I) + 2^{r-1}, c_{r-1}(I)).$$

Set $(\mathscr{B}(I), d_1)$ equal to $Z/2[q^I, q^{I-\Delta_1}, q^{c_1(I)}, q^{c_2(I)}, \cdots]$ with

$$d_1(q^I) = q^{I-\Delta_1},$$
$$d_1(q^{c_j(I)}) = (q^{c_{j-1}(I)})^2.$$

It is clear that the Algebras $\mathscr{A}(I)$ and $\mathscr{B}(I)$ generate $H_*(\Sigma_\infty; Z/2)$. It is easy to check, via the Nishida and Adem relations that

PROPOSITION (2.8). *When $p = 2$, $(E_*{}^1(\Sigma_\infty), d_1)$ is isomorphic to*

$$\otimes_{I \in \mathscr{S}}(\mathscr{A}(I), d_1) \otimes (\otimes_J (\mathscr{B}(J), d_1))$$

where J runs over sequences $J = (s_1, \cdots, s_k)$ with $e(J) > 1$, s_1 even and $d(J)$ is odd.

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