Boletín de la Sociedad Matemática Mexicana Vol. 27, No. 2, 1982

# SOLVABILITY OF INFINITE SYSTEMS OF INFINITE LINEAR EQUATIONS

BY ALEXANDER ABIAN AND ESFANDIAR ESLAMI

This paper is in the setting of real numbers and is concerned with the solvability of infinite systems such as $S$ given by $\sum_{i=1}^{\infty} a_{ki}x_i = c_k$ with $k = 1, 2, \cdots$ of linear equations each with infinitely many unknowns $x_i$ with $i = 1, 2, \cdots$. In general it is not true that if every subsystem of $S$ has a solution then $S$ has a solution (of course, the converse is always true). However, as shown in this paper, under certain conditions imposed on $S$ the solvability of every finite subsystem of $S$ implies the solvability of $S$. These conditions were given by F. Riesz [3, p. 61] and we use them throughout this paper. Our treatment differs from that of Riesz' in that for the case, say $k = 2$ we consider (see Lemma 1) the function $F(u, v)$ motivated by the fact that $F(u, v)$ gives the distance (in $\ell^q$ sense) of the origin from the hyperplane whose equation is a linear combination (with coefficients $u$ and $v$) of the two equations of the system. This gives further motivation to consider and invoke the existence of a local maximum of $F(u, v)$ which is one of the crucial points in this paper. Also (unlike Riesz) we prove the existence of a solution of the entire system $S$ as the intersection of a family $F$ of closed subsets of a compact topological space, where $F$ has the finite intersection property. The final result (as stated in Theorem 4) is that if $\{a_{ki} \mid i = 1, 2, \cdots\}$ is an element of $\ell^q$ for every $k = 1, 2, \cdots$ and if $M$ is a preassigned nonnegative real number then $S$ has a solution $x_i = r_i$ for $i = 1, 2, \cdots$ with the property $\sum_{i=1}^{\infty} | r_i |^p \leq M^p$ if and only if every finite subsystem of $S$ has a solution with the same property. In what follows we assume that $M > 0$ since $M = 0$ implies the trivial solution $x_i = 0$.

Throughout this paper by "infinity" we always mean "denumerable infinity". Also, $p$ and $q$ always stand for real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1 \quad with \quad p > 1. \tag{1}$$

As usual, $\ell^p$ stands for the set of all infinite sequences $(a_i)_{i=1,2,\cdots}$ of real numbers such that $\sum_{i=1}^{\infty} |a_i|^p < \infty$. For the sake of convenience we denote $(a_i)_{i=1,2,\cdots}$ simply as $(a_i)$ and, very often, we denote $\sum_{i=1}^{\infty} | a_i |^p$ simply as $\Sigma_i | a_i |^p$. Moreover, if $(a_i) \in \ell^p$, then as usual the $\ell^p$-norm of $(a_i)$ is denoted by $\|(a_i)\|_p$ where

$$\|(a_i)\|_p = (\Sigma_i | a_i |^p)^{p^{-1}}. \tag{2}$$

For every real number $r$ we define sgn $r$ (read: *sign of r*) as follows:

$$\operatorname{sgn} r = 1 \quad if \quad r > 0,$$
$$\operatorname{sgn} r = -1 \quad if \quad r < 0,$$
$$\operatorname{sgn} r = 0 \quad if \quad r = 0.$$

Let us also recall that if $(a_i) \in \ell^p$ and $(b_i) \in \ell^q$ then by the Holder's inequality $| \Sigma_1 a_i b_i | \leq \| a_i \|_p \cdot \| b_i \|_q$ which also implies that $\Sigma_i a_i b_i$ converges absolutely.

We first prove the following lemma which is of major significance for our purpose.

LEMMA 1. *Let $(a_i) \in \ell^q$ and $(b_i) \in \ell^q$ and let $c_1$ and $c_2$ be real numbers not both zero and let $M$ be a positive real number such that*

$$(3) \qquad F(u, v) = \frac{| c_1 u + c_2 v |}{\Sigma_1 | a_i u + b_i v |^q} \leq M^q$$

*for every real number $u$ and $v$ not both zero. Then there exist real numbers $u_0$ and $v_0$ such that*

$$(4) \qquad u_0^2 + v_0^2 = 1$$

*and such that*

$$(5) \qquad \frac{\partial F}{\partial u}(u_0, v_0) = \frac{\partial F}{\partial v}(u_0, v_0) = 0 \quad \textit{with} \quad c_1 u_0 + c_2 v_0 \neq 0.$$

*Proof.* From (3) it follows that by the hypothesis of the Lemma $(a_i)$ and $(b_i)$ are such that except for $u = v = 0$, the denominator of the fraction appearing in (3) is never zero (e.g., in particular $b_i$'s are not proportional to $a_i$'s). Thus, from (3) it follows that $F(u, v)$, except at $(0, 0)$, is an everywhere continuous and bounded nonnegative real valued function of the real variables $u$ and $v$ (i.e., the domain of definition of $F(u, v)$ is the entire $(u, v)$-plane minus the origin). Also, since $c_1$ and $c_2$ are not both zero, from (3) it follows that on the circumference $C$ of the unit circle given by $u^2 + v^2 = 1$, the function $F(u, v)$ is not identically zero. Hence, there exists $(u_0, v_0) \in C$ such that $F(u, v)$ attains its maximum $F(u_0, v_0) > 0$. Thus, for every $u$ and $v$ not both zero, we have:

$$(6) \qquad F(u_0, v_0) \geq F\left(\frac{u}{(u^2 + v^2)^{1/2}}, \frac{v}{(u^2 + v^2)^{1/2}}\right) \quad \textit{with} \quad F(u_0, v_0) > 0$$

However, by (3) we see that the second member of the first inequality in (6) is equal to $F(u, v)$. Consequently, for every $u$ and $v$ not both zero we have:

$$(7) \qquad F(u_0, v_0) \geq F(u, v) \quad \textit{with} \quad F(u_0, v_0) > 0 \quad \textit{and} \quad u_0^2 + v_0^2 = 1.$$

Obviously, from (7) and (3), we obtain:

$$(8) \qquad c_1 u_0 + c_2 v_0 \neq 0.$$

Next we show that the partial derivatives $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ of the function $F(u, v)$, as given by (3), exist at the interior point $(u_0, v_0)$ of the domain of definition of $F(u, v)$. To this end it suffices to show that the partial derivatives of the demoniator $\Sigma_i | a_i u + b_i v |^q$ of the fraction appearing in (3) exist in a bounded neighborhood $D$ of $(u_0, v_0)$ where $D$ is given, say by $(u - u_0)^2 + (v - v_0)^2 = (0.5)^2$. To accomplish this, in view of [2, p. 396] it is enough to show that $\Sigma_i | a_i u + b_i v |^q$ converges on $D$ and that the series

$$q\Sigma_i a_i | a_i u + b_i v |^{q-1}\mathrm{sgn}(a_i u + b_i v) \tag{9}$$

and the series

$$q\Sigma_i b_i | a_i u + b_i v |^{q-1}\mathrm{sgn}(a_i u + b_i v), \tag{10}$$

which are respectively obtained by termwise differentiation of $\Sigma_i | a_i u + b_i v |^q$ with respect to $u$ and $v$ are uniformly convergent on $D$.

Since $(a_i) \in \ell^q$ and $(b_i) \in \ell^q$ we see that $(a_i u + b_i v) \in \ell^q$ and therefore $\Sigma_i | a_i u + b_i v |^q$ converges (in fact uniformly) on $D$. On the other hand, obviously, $((a_i u + b_i v)^{q-1}) \in \ell^p$ and therefore by the Holder's inequality for every $(u, v) \in D$ we have:

$$\Sigma_i | a_i \| a_i u + b_i v |^{q-1} \leq \|(a_i)\|_q \cdot \|((a_i u + b_i v)^{q-1})\|_p \leq H, \tag{11}$$

where $H$ is a positive real number which exists since $(u, v) \in D$.

Clearly, (11) implies that (9) converges uniformly on $D$. Similarly, we can show that (10) also converges uniformly on $D$. Hence the partial derivatives of the denominator of the fraction appearing in (3) exist at $(u_0, v_0)$. Consequently, the partial derivatives of $F(u, v)$, as given by (3), exist at $(u_0, v_0)$. Indeed, from (9), (10) and (3), for every $(u, v) \in D$ we have:

$$\frac{\partial F}{\partial u} = \frac{\begin{array}{l} c_1 | c_1 u + c_2 v |^{q-1}\mathrm{sgn}(c_1 u + c_2 v)\Sigma_i | a_i u + b_i v |^q \\ - | c_1 u + c_2 v |^q \Sigma_i a_i | a_i u + b_i v |^{q-1}\mathrm{sgn}(a_i u + b_i v) \end{array}}{q^{-1}(\Sigma_i | a_i u + b_i v |^q)^2} \tag{12}$$

and

$$\frac{\partial F}{\partial v} = \frac{\begin{array}{l} c_2 | c_1 u + c_2 v |^{q-1}\mathrm{sgn}(c_1 u + c_2 v)\Sigma_i | a_i u + b_i v |^q \\ - | c_1 u + c_2 v |^q \Sigma_i b_i | a_i u + b_i v |^{q-1}\mathrm{sgn}(a_i u + b_i v) \end{array}}{q^{-1}(\Sigma_i | a_i u + b_i v |^q)^2} \tag{13}$$

Finally, from (7) it follows that the maximum value $F(u_0, v_0) > 0$ of $F(u, v)$ is attained at the interior point $(u_0, v_0)$ of the domain of definition of $F(u, v)$. Hence, at $(u_0, v_0)$ the partial derivatives $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ are equal to zero. But the latter together with (7) and (8) imply (4) and (5).

Thus, the Lemma is proved.

*Remark* 1. Let us replace $u$ and $v$ respectively by $u_0$ and $v_0$ in (12) and (13) and let us divide both sides by the nonzero real number $| c_1 u_0 + c_2 v_0 |^{q-1}\Sigma_i | a_i u_0 + b_i v_0 |^q$. But then, by virtue of (5), (12) and (13), we obtain:

$$c_1 = \Sigma_i a_i \frac{(c_1 u_0 + c_2 v_0) | a_i u_0 + b_i v_0 |^{q-1}\mathrm{sgn}(a_i u_0 + b_i v_0)}{\Sigma_i | a_i u_0 + b_i v_0 |^q} \tag{14}$$

and

$$c_2 = \Sigma_i b_i \frac{(c_i u_0 + c_2 v_0)|\, a_i u_0 + b_i v_0 \,|^{q-1}\mathrm{sgn}(a_i u_0 + b_i v_0)}{\Sigma_i |\, a_i u_0 + b_i v_0 \,|^q}. \tag{15}$$

To simplify the expressions of $c_1$ and $c_2$ given in (14) and (15), we introduce real numbers $s_1$ and $s_2$ defined by:

$$s_1 = u_0/(\Sigma_i |\, a_i u_0 + b_i v_0 \,|^q)^{1/q} \quad \textit{and} \quad s_2 = v_0/(\Sigma_i |\, a_i u_0 + b_i v_0 \,|^q)^{1/q} \tag{16}$$

But then, based on (3), (14), (15) and (16), we prove:

COROLLARY 1. *Let $(a_i) \in \ell^q$ and $(b_i) \in \ell^q$ and let $c_1$ and $c_2$ be real numbers and let $M$ be a positive real number such that*

$$|\, c_1 u + c_2 v \,|^q \le M^q \Sigma_i |\, a_i u + b_i v \,|^q \tag{17}$$

*for every real numbers $u$ and $v$. Then there exist real numbers $s_1$ and $s_2$ such that:*

$$c_1 = \Sigma_i a_i (c_1 s_1 + c_2 s_2)|\, a_i s_1 + b_i s_2 \,|^{q-1}\mathrm{sgn}(a_i s_1 + b_i s_2) \tag{18}$$

*and*

$$c_2 = \Sigma_i b_i (c_1 s_1 + c_2 s_2)|\, a_i s_1 + b_i s_2 \,|^{q-1}\mathrm{sgn}(a_i s_1 + b_i s_2). \tag{19}$$

*Proof.* Let us observe that the hypothesis of Corollary 1 differs from the hypothesis of Lemma 1 in that $c_1$ and $c_2$ may both be equal to zero and that the denominator of the fraction appearing in (3) may also be equal to zero for some $u_1$ and $v_1$. Thus, to prove the Corollary, in view of Lemma 1, it suffices to prove the validity of (18) and (19) for the above two cases. However, if $c_1$ and $c_2$ are both zero then (18) and (19) are valid for any choice of $s_1$ and $s_2$. The same is true if $(a_i)$ and $(b_i)$ are both equal to the zero sequence. Next, let there exist $u_1$ and, say, $v_1 \ne 0$ such that $|\, a_i u_1 + b_i v_1 \,| = 0$ for every $i = 1, 2, \cdots$. Clearly, this implies that there exists $w$ such that $b_i = w a_i$ for every $i = 1, 2, \cdots$. Obviously, (17) also implies that $c_1 u_1 + c_2 v_2 = 0$ and $c_2 = w c_1$. But then, assuming (without loss of generality) that $(a_i)$ is not the zero sequence, it can be readily verified that (18) and (19) are valid for $s_1 = (\Sigma_i |\, a_i \,|^q)^{-1/q}$ and $s_2 = 0$.

Corollary 1 can be applied to any finite number of $\ell^q$ sequences instead of just two $(a_i)$ and $(b_i)$. Thus, we prove:

THEOREM 1. *Let $n \ge 1$ be an integer. Let $(a_{ki}) \in \ell^q$ and $c_k$ be real numbers for every $k = 1, \cdots, n$ and let $M$ be a positive real number such that*

$$|\, \textstyle\sum_{k=1}^n c_k u_k \,|^q \le M^q \Sigma_i |\, \sum_{k=1}^n a_{ki} u_k \,|^q \tag{20}$$

*for every $n$ real numbers $u_1, \cdots, u_n$. Then there exist real numbers $s_k$ such that*

$$c_k = \Sigma_i a_{ki}((|\, \textstyle\sum_{k=1}^n a_{ki} s_k \,|^{q-1}\mathrm{sgn} \sum_{k=1}^n a_{ki} s_k) \sum_{k=1}^n c_k a_k) \tag{21}$$

*for every $k = 1, \cdots, n$.*

*Proof.* To prove the Theorem for $n = 1$, it is enough to establish (18) for the case where $(b_i)$ is the zero sequence and $c_2 = s_2 = 0$. But then it can be readily verified that (18) is valid for $s_1 = (\Sigma_i |a_i|^q)^{-1/q}$. For $n > 2$, the proof of the Theorem is the verbatim version of its proof for $n = 2$ as given by the proof of Corollary 1.

A most significant application of Theorem 1 is given as follows.

THEOREM 2. *Let $n \geq 1$ be an integer. Let $(a_{ki}) \in \ell^q$ and $c_k$ be real numbers for every $k = 1, \cdots, n$ and let $M$ be a positive real number. Then the system of $n$ linear equations in infinitely many unknowns $x_i$*

$$\sum_{i=1}^{\infty} a_{ki}x_i = c_k \qquad \textit{for } k=1, \cdots, n \tag{22}$$

*has a solution*

$$x_i = r_i \quad \textit{with} \quad i = 1, 2, \cdots \ \textit{such that} \sum_{i=1}^{\infty} |r_i|^p \leq M^p \tag{23}$$

*if and only if*

$$\left| \sum_{k=1}^{n} c_k u_k \right|^q \leq M^q \sum_{i=1}^{\infty} \left| \sum_{k=1}^{n} a_{ki}u_k \right|^q \tag{24}$$

*for every $n$ real numbers $u_1, \cdots, u_n$.*

*Proof.* First we prove that (24) implies (23). Since (20) implies (21), we see that (24) implies (21). But then comparing (21) with (22), it is obvious that the coefficients of $a_{ki}$ appearing in (21) give a solution of the system (22). Thus,

$$x_i = \left( \left| \sum_{k=1}^{n} a_{ki}s_k \right|^{q-1} \operatorname{sgn} \sum_{k=1}^{n} a_{ki}s_k \right) \sum_{k=1}^{n} c_k s_k \qquad \textit{with } i=1,2, \cdots \tag{25}$$

give a solution of system (22). To prove (23), let us choose $r_i = x_i$ as given by (25). Observing by (1) that $p(q-1) = q$, from (25) we obtain

$$\Sigma_i |x_i|^p = \Sigma_i |r_i|^p = \left| \sum_{k=1}^{n} c_k s_k \right|^p \Sigma_i \left| \sum_{k=1}^{n} a_{ki}s_k \right|^q. \tag{26}$$

On the other hand, (16) for the general case $(k = 1, \cdots, n)$ yields

$$s_k = u_k \left( \Sigma_i \left| \sum_{k=1}^{n} a_{ki}u_k \right|^q \right)^{-1/q} \qquad \textit{with } k = 1, \cdots, n \tag{27}$$

Substituting (27) in (26), we obtain

$$\Sigma_i |x_i|^p = \Sigma_i |r_i|^p = \left| \sum_{k=1}^{n} c_k u_k \right|^p \left( \Sigma_i \left| \sum_{k=1}^{n} a_{ki}u_k \right|^q \right)^{-p/q}$$

which by (24) implies (23), as desired.

Next, we show that (23) implies (24). From (22) and (23) for any $n$ real numbers $u_1, \cdots, u_n$ we obtain

$$\sum_{i=1}^{\infty} \left( \sum_{k=1}^{n} a_{ki}u_k \right) r_i = \sum_{k=1}^{n} c_k u_k \tag{28}$$

However, since $(a_{ki}) \in \ell^q$ we see that $(\sum_{k=1}^{n} a_{ki}u_k) \in \ell^q$. Also, by (23) we have $(r_i) \in \ell^p$. Thus, (28) by Holder's inequality and again by (23) yields

$$\left| \sum_{k=1}^{n} c_k u_k \right| = \left| \sum_{i=1}^{\infty} \left( \sum_{k=1}^{n} a_{ki}u_k \right) r_i \right|$$
$$\leq \left\| \left( \sum_{k=1}^{n} a_{ki}u_k \right) \right\|_q \| (r_i) \|_p \leq M \left\| \left( \sum_{k=1}^{n} a_{ki}u_k \right) \right\|_q$$

which, by (2), implies (24).

As shown below, Theorem 1 is directly applied to the case of a system of infinitely (denumerable) many linear equations.

THEOREM 3. *Let* $(a_{ki}) \in \ell^q$ *and* $c_k$ *be real numbers for every* $k = 1, 2, \cdots$ *and let* $M$ *be a positive real number. Then the system of infinitely (denumerable) many linear equations in infinitely many unknowns* $x_i$

$$\sum_{i=1}^{\infty} a_{ki}x_i = c_k \qquad \text{for} \quad k = 1, 2, \cdots \tag{29}$$

*has a solution*

$$x_i = r_i \quad \text{with } i = 1, 2, \cdots \quad \text{such that } \sum_{i=1}^{\infty} | r_i |^p \leq M^p \tag{30}$$

*if and only if*

$$| \textstyle\sum_{k=1}^{n} c_k u_k |^q \leq M^q \sum_{i=1}^{\infty} | \sum_{k=1}^{n} a_{ki}u_k |^q \tag{31}$$

*for every* $n$ *(finitely many) real numbers* $u_1, \cdots, u_n$.

*Proof.* First we show that if the infinite system (29) has a solution such as given by (30) then (31) is satisfied. But if (29) has a solution such as given by (30) then (22) and (23) are satisfied and therefore, by Theorem 1 we see that (24) as well as (31) is satisfied.

It remains to show that if (31) is satisfied for every $n$ (finitely many) real numbers $u_1, \cdots, u_n$ then the infinite system (29) has a solution satisfying (30). To this end, in view of Theorem 1 it suffices to prove that if for every positive integer $n$ the finite subsystem (22) of the first $n$ equations of the infinite system (29) has a solution satisfying (23), then the infinite system (29) has a solution satisfying (30). We prove this as follows.

Let us consider the infinite (denumerable) product $I^\infty$ of the closed interval $I = [-M, M]$ as a subset of the infinite (denumerable) product topological space $R^\infty$ where $R$ is the set of all real numbers in its usual (metric) topology. It is well known [4, p. 162] that $I^\infty$ is a compact subset of $R^\infty$. Now let $x_i = r_i$ with $i = 1, 2, \cdots$ be a solution of a single equation $\sum_{i=1}^{\infty} a_{ki}x_i = c_k$ such that $x_i = r_i$ with $i = 1, 2, \cdots$ satisfy (23). Clearly, $(r_1, r_2, \cdots) \in I^\infty$ since $\sum_{i=1}^{\infty} | r_i |^p \leq M^p$ implies that $| r_i | \leq M$ and therefore $r_i \in [-M, M]$ for every $i = 1, 2, \cdots$. Next, let

$$S_k = \{(r_1, r_2, r_3, \cdots), \quad (s_1, s_2, s_3, \cdots), \quad (t_1, t_2, t_3, \cdots), \cdots\} \tag{32}$$

be the set of all solutions of the single equation $\sum_{i=1}^{\infty} a_{ki}x_i = c_k$ where each solution satisfies (23). We show that $S_k$ as given by (32) is a closed subset of the compact subset $I^\infty$ of $R^\infty$. To this end we show that if a sequence of elements of $S_k$ converges coordinatewise to, say, $(h_1, h_2, h_3, \cdots)$ then the latter is an element of $S_k$, i.e.,

$$\sum_{i=1}^{\infty} a_{ki}h_i = c_k \quad \text{and} \quad \sum_{i=1}^{\infty} | h_i |^p \leq M^p \tag{33}$$

Let the sequence of elements of $S_k$ which converge coordinatewise to $(h_1, h_2,$

$h_3, \cdots)$ be given by:

$$(34) \qquad (r_1, r_2, r_3, \cdots), \quad (e_1, e_2, e_3, \cdots), \quad (w_1, w_2, w_3, \cdots), \quad \cdots$$

Thus, we have:

$$(35) \qquad \lim(r_1, e_1, w_1, \cdots) = h_1, \quad \lim(r_2, e_2, w_2, \cdots) = h_2, \quad \lim(r_3, e_3, w_3, \cdots) = h_3, \cdots$$

Moreover,

$$(36) \qquad c_k = \Sigma_i a_{ki} r_i = \Sigma_i a_{ki} e_i = \Sigma_i a_{ki} w_i = \cdots \quad \text{and } \Sigma_i | r_i |^p \leq M^p, \Sigma_i | e_i |^p \leq M^p, \Sigma_i | w_i |^p \leq M^p, \cdots$$

where, as always,

$$(37) \qquad \Sigma_i | a_{ki} |^q < \infty$$

The inequality in (33) follows readily from (35) and (36) by considering the partial sums $\sum_{i=1}^{m} | h_i |^p$. From this and (37), in view of the Holder's inequality, we see that $\sum_{i=1}^{\infty} a_{ki} h_i$ converges absolutely and, because of (36), it also converges uniformly. But then it can be verified that $\Sigma_i a_{ki} h_i = \Sigma_i a_{ki} r_i = c_k$ which establishes the equality in (33). Thus, the set $S_k$, as given by (32), of all the solutions of the single equation $\sum_{i=1}^{\infty} a_{ki} x_i = c_k$ where each solution satisfies (23) is a closed subset of the compact subset $I^\infty$ of $R^\infty$. Now, if for every positive integer $n$ the finite subsystem (22) of the first $n$ equations of the infinite system (29) has a solution satisfying (23) then the set of all such solutions (of that subsystem) is a nonempty closed subset of $I^\infty$ since the intersection of finitely many closed subsets of $I^\infty$ is again a closed subset of $I^\infty$. From this we conclude that our assumption that every finite subsystem of (29) has a solution satisfying (23), implies that the family $F = \{S_k \mid k = 1, 2, \cdots\}$, where $S_k$ is given by (32), is a family of the closed subsets $S_k$ of a compact subset $I^\infty$ of $R^\infty$ such that $F$ has the finite intersection property [4, p. 117–118]. Thus, $\cap F$ is nonempty and therefore the system (29) has a solution satisfying (30). This completes what was remaining to be proved.

Finally, from Theorem 2 it follows that Theorem 3 can be rephrased as follows:

THEOREM 4. *Let $(a_{ik}) \in \ell^q$ and $c_k$ be real numbers for every $k = 1, 2, \cdots$ and let $M$ be a positive real number. Then the system $\sum_{i=1}^{\infty} a_{ki} x_i = c_k$ for $k = 1, 2, \cdots$ of infinitely (denumerable) many equations in infinitely many unknowns $x_i$ has a solution $x_i = r_i$ for $i = 1, 2, \cdots$ with the property $\sum_{i=1}^{\infty} | r_i |^p \leq M^p$ if and only if every finite subsystem of the system has a solution which has the same property.*

*Remark* 2. Theorem 4 gives another instance of a case where the solvability of an infinite system of linear equations is ensured when every finite subsystem of it has a solution. However, contrary to the case [1] where each equation of an infinite system has finitely many unknowns, we note that for the general case (as Theorem 3 shows) some rather stringent conditions are imposed on the system. In particular, we observe that condition (31) states that every hyperplane whose equation is given as a linear combination of finitely many equations of the system is such that its $\ell^q$ distance from the origin is $< M$. This condition not only ensures that every finite subsystem (Theorem 2) has a solution but also implies that the entire system has a solution.

IOWA STATE UNIVERSITY, AMES, IOWA, U.S.A.
KERMAN UNIVERSITY, KERMAN, IRAN

REFERENCES

[1] A. ABIAN, *Solvability of infinite systems of linear equations*, Arch. Math. (Brno) **12** (1976) 43–44.
[2] R. COURANT, *Differential and integral calculus*, Vol. 1, Interscience Publishers, Inc., New York, 1947.
[3] F. RIESZ, *Les systèmes d'equations lineaires*, Gauthier-Villars, Paris, 1913.
[4] S. WILLARD, *General topology*, Addison-Wesley Pub., Co., Reading, 1968.