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FAITHFUL REPRESENTATIONS OF BANACH-LIE ALGEBRAS

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Introduction and statement of the theorem

Here, one proves that an arbitrary Banach-Lie algebra can be faithfully represented as a subalgebra of the Banach-Lie algebra of bounded operators in some Banach space.

The theorem of Ado in the theory of finite-dimensional Lie algebras states that an arbitrary Lie algebra is isomorphic to some subalgebra of the matrix Lie algebra $gl(n, R)$ for an appropriate natural n. The theorem proved in the present paper can be treated as a counterpart of the Ado theorem in the theory of Banach-Lie algebras.

THEOREM. *Given an arbitrary Banach-Lie algebra* $(\mathbf{g}, [\cdot, \cdot])$ *one can construct a Banach space E and a continuous linear injection*:

$$j: \mathbf{g} \to L(E)$$

such that

$$j([X, Y]) = j(X)j(Y) - j(Y)j(X) =: [j(X), j(Y)]$$

for all $X, Y \in \mathbf{g}$.

In other words j is the Lie algebra homomorphism into the Banach-Lie algebra of bounded operators in the Banach space E. It should be underlined that this is not a generalization of the theorem of Ado. In the finite-dimensional case this theorem (and its proof) leads merely to the construction of a faithful representation of the algebra $\mathbf{g}$ in an infinite-dimensional Banach space.

It would be interesting to obtain a formulation containing both theorems.

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Proof of the theorem

It is a classical result (see e.g. [2] [1]) that to every Banach-Lie algebra there is associated a local Banach-Lie group U such that the Lie algebra of U is isomorphic to $\mathbf{g}$.

We will need a more precise formulation of this statement. In some neighborhood U of zero a $\mathbf{g}$-valued multiplication can be defined:

$$U \times U \ni (x, y) \to x \circ y \in \mathbf{g}$$

in such a way that $x \circ (-x) = 0$, $x \circ 0 = 0 \circ x = x$, and $(x \circ y) \circ z = x \circ (y \circ z)$, whenever the product makes sense.

The multiplication is an analytic mapping of $U \times U$ in $\mathbf{g}$. The one-parameter subgroups of this local Lie group are precisely the one-dimensional subspaces of $\mathbf{g}$:

$$a_X: R \ni t \to tX \in \mathbf{g}.$$

Let $C^\infty(U)$ be the space of C^∞-functions on U in the sense of Frechét. Every element $X \in \mathbf{g}$ determines a differential operator D_X on U by means of the formula:

$$D_X f(x) := \lim_{t \to 0} \frac{f(x \circ (tX)) - f(x)}{t}.$$

One proves that the mapping $X \to D_X$ is injective and

$$[D_X, D_Y] := D_X D_Y - D_Y D_X = D_{[X,Y]}, \tag{1}$$

that is, the mapping is the homomorphism of Lie algebras.

Now, let $A(\mathbf{g})$ be the stalk of germs of analytic functions in some neighborhood of zero in $\mathbf{g}$. (For definitions, compare [3]). Every element $s \in A(\mathbf{g})$ can be represented by a function defined in some neighborhood of zero and such that

$$f(X) = \sum_{n=0}^{\infty} \frac{1}{n!} D_X{}^n f(0). \tag{2}$$

We write $s := \tilde{f}$.

The action of the operators D_X can be projected on the stalk $A(\mathbf{g})$ as the operator $\tilde{D}_X$, which assigns to the germ s of the function f the germ of the function $D_X f$. The property (1) implies

$$\tilde{D}_{[X,Y]} = [\tilde{D}_X, \tilde{D}_Y]. \tag{3}$$

Let $D^0 f := f(0)$ and for $n = 1, 2, \cdots$

$$D^n f(X_1, \cdots, X_n) := D_{X_1} D_{X_2} \cdots D_{X_n} f(0).$$

Now, let us observe that the function

$$\times^n \mathbf{g} \ni (X_1, \cdots, X_n) \to D^n f(X_1, \cdots, X_n) \in R$$

belongs to the space $L_n(\mathbf{g})$ of all n-multilinear continuous forms on $\mathbf{g}$. The sequence $\{D^n f\}_{n=1}{}^\infty$ determines the germ of the function uniquely because

$$D_X{}^n f(0) = D^n f(X, X, \cdots, X).$$

Let us distinguish among all sequences $\ell = \{\ell_n\}_{n=1}{}^\infty$ those which satisfy

$$|||\, \ell \,||| := \sup_n \| \ell_n \| < \infty.$$

The resulting space E is a Banach space with respect to the norm $|||\cdot|||$ defined above. Let in turn $B(\mathbf{g})$ be the subspace of $A(\mathbf{g})$ consisting of germs of those functions f for which $\{D^n f\}_{n=1}{}^\infty \in E$.

The mapping $B(\mathbf{g}) \ni f \overset{p}{\longrightarrow} D^n f \in E$ is one-to-one with the inverse defined by means of (2).

We are going to construct the following commutative diagram of continuous mappings:

$$\begin{array}{ccc} B(\mathbf{g}) \ni \tilde{f} & \overset{\tilde{D}_X}{\longrightarrow} & \tilde{D}_X \tilde{f} \in B(\mathbf{g}) \\ p \downarrow & & \downarrow p \\ E \ni \{D_X{}^n f\}_{n=1}{}^{\infty} & \overset{j}{\longrightarrow} & \{D^n(D_X f)\}_{n=1}{}^{\infty} \in E. \end{array}$$

To this end we have to define the continuous mapping:

$$j(X)\colon E \to E$$

such that

$$\{D^n(D_X f)\}_{n=1}{}^{\infty} = j(X)\{D^n f\}_{n=1}{}^{\infty}.$$

Namely we define $j(X)\{\ell_n\} := \{c_n\}$, where

$$c_n(X_1, \cdots, X_n) := \ell_{n+1}(X_1, \cdots, X_n, X). \tag{4}$$

The norm of $j(X)$ can be estimated in the following way:

$$\| j(X) \| = \sup_{|||\{\ell_n\}|||<1} ||| j(X)\{\ell_n\} ||| \leq \sup_n \| \ell_{n+1} \| \cdot \| X \| \leq \| X \|,$$

hence the operator $j(X)$ is in fact continuous and continuously depending on X.

By the definition of the form D^n and the formula (1) we see that the operator $j(X)$ makes the diagram above a commutative one, what demonstrates by the way that the operator D_X leaves the space $B(\mathbf{g})$ invariant.

Now, we are able to prove that the mapping

$$\mathbf{g} \ni X \to j(X) \in L(E)$$

is a representation of the Banach-Lie algebra $\mathbf{g}$:

$$\begin{aligned} j([X, Y])P\tilde{f} &= P(\tilde{D}_{[X,Y]}\tilde{f}) = P((\tilde{D}_X\tilde{D}_Y - \tilde{D}_Y\tilde{D}_X)\tilde{f}) \\ &= (j(X)j(Y) - j(Y)j(X)P\tilde{f}. \end{aligned}$$

Since $P(B(\mathbf{g})) = E$ this implies

$$j([X, Y]) = [j(X), j(Y)].$$

In order to complete the proof of the theorem it remains to notice that the representation j is faithful i.e. that for $X \neq 0$ the operator $j(X)$ is nonzero.

To this end, given $X \in \mathbf{g}$ we chose $\phi \in \mathbf{g}'$ such that $\phi(X) \neq 0$ and then define

$$\ell := \{0, \phi, 0 \cdots\} \in B(\mathbf{g}).$$

Now, $j(X)\ell = \{\phi(X), 0, \cdots\} \neq 0$, what was to be proved.

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