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ASYMPTOTIC STATIONARITY OF NONSTATIONARY L^2-PROCESSES WITH APPLICATIONS TO LINEAR PREDICTION*

BY HANNU NIEMI

Introduction

An L^2-process $x_k \in H$, $k \in Z$ (H = a Hilbert space) is called asymptotically stationary, if the limits

$$(*) \qquad r(h) = \lim_{T\to\infty} \frac{1}{2T+1} \sum_{j=-T}^{T} (x_{k+h}, x_k)_H$$

exist for all $h \in Z$. In this case $r: Z \to C$ is positive definite and, *a fortiori*,

$$r(h) = \int_0^{2\pi} e^{ih\lambda}\, d\nu(\lambda), \qquad h \in Z.$$

In the present paper we are mainly concerned with the analysis of the asymptotic stationarity of weakly harmonizable L^2-processes x_k, $k \in Z$, i.e. when x_k, $k \in Z$, admits a spectral representation

$$x_k = \int_0^{2\pi} e^{ik\lambda}\, d\mu(\lambda), \qquad k \in Z.$$

It appears that not all weakly harmonizable L^2-processes are asymptotically stationary. This gives a motivation to consider stationarization procedures that are weaker than the one defined by (*). In the present paper two such methods are proposed, based on the use of invariant means on time domain and, respectively, on spectral domain.

The proposed stationarization methods are analyzed in more detail in the special case of uniformly bounded linearly stationary (UBLS) L^2-processes, introduced by D. Tjøstheim and J. B. Thomas (cf. [15] and the references given therein). A uniqueness result concerning a minimal "asymptotic stationarization" of a given UBLS L^2-process, x_k, $k \in Z$, is presented. This result is analogous to the uniqueness of the minimal p-majorant of a p-majorizable vector measure, obtained by A. Pietsch [10].

Applications to linear prediction, especially to the stationary prediction of nonstationary L^2-processes proposed by J. L. Abreu [1], are presented.

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1. Asymptotically stationary sequences

Let H be a complex Hilbert space (one can choose e.g. $H = L^2(\Omega, \boldsymbol{A}, P)$ or $H = \{\xi \in L^2(\Omega \cdot \boldsymbol{A} \cdot P) \mid E\xi = 0\}$). Recall that a sequence $x_k \in H$, $k \in \mathrm{Z}$, is a (weakly) *stationary L^2-process*, if

$$(x_j, x_k)_H = r(j-k), \qquad j, k \in \mathrm{Z}. \tag{1.1}$$

For any stationary L^2-process $x_k \in H$, $k \in \mathrm{Z}$, the corresponding covariance kernel $r: \mathrm{Z} \to \mathrm{C}$ satisfying (1.1) is positive definite, i.e.

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k r(h_j - h_k) \geq 0$$

for all $a_j \in \mathrm{C}$, $h_j \in \mathrm{Z}$, $j = 1, \cdots, n$; $n \in \mathrm{N}$. Thus,

$$r(h) = \int_0^{2\pi} e^{ih\lambda}\, d\nu(\lambda), \qquad h \in \mathrm{Z}, \tag{1.2}$$

for a uniquely determined bounded nonnegative measure ν on $[0, 2\pi[$.

The following definition is essentially the same as in [12; p. 337] (cf. [11; p. 175]).

Definition. An L^2-process $x_k \in H$, $k \in \mathrm{Z}$, is called *asymptotically stationary*, if the limit

$$r(h) = \lim_{T\to\infty} \frac{1}{2T+1} \sum_{k=-T}^{T} (x_{k+h}, x_k)_H \tag{1.3}$$

exists for all $h \in \mathrm{Z}$.

An extensive survey on asymptotically stationary L^2-processes on R^1 has been presented by M. M. Rao [12] (cf. [11]). For a different approach and applications cf. E. Parzen [9].

Suppose $x_k \in H$, $k \in \mathrm{Z}$, is an asymptotically stationary L^2-process. Then the function $r: \mathrm{Z} \to \mathrm{C}$, defined by (1.3) is positive definite. Thus, it admits a representation in the form (1.2). The corresponding uniquely determined bounded nonnegative measure ν, appearing on the *R.H.S.* of (1.2), is called the *associated spectral distribution* of the L^2-process x_k, $k \in \mathrm{Z}$.

In this paper we are mainly concerned with the asymptotic stationarity of non-stationary L^2-processes admitting a spectral representation.

Recall that an L^2-process $x_k \in H$, $k \in \mathrm{Z}$, is

(i) *weakly harmonizable*, if there exists a (uniquely determined) bounded H-valued vector measure μ on the Borel σ-algebra $\boldsymbol{B}$ of $[0, 2\pi[$ such that

$$x_k = \int_0^{2\pi} e^{ik\lambda}\, d\mu(\lambda), \qquad k \in \mathrm{Z}; \tag{1.4}$$

(ii) *strongly harmonizable*, if

$$(x_j, x_k)_H = \int_0^{2\pi} \int_0^{2\pi} e^{i(j\lambda - k\theta)}\, dF(\lambda, \theta), \qquad j, k \in \mathrm{Z}, \tag{1.5}$$

where F is a bounded (possibly complex valued) measure on $[0, 2\pi[\times [0, 2\pi[$, which is of positive definite type, i.e.

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k F(E_j, E_k) \geq 0$$

for all $a_j \in \mathrm{C}$ and disjoint $E_j \in \boldsymbol{B}$, $j = 1, \cdots, n$; $n \in \mathrm{N}$;

(iii) *uniformly bounded linearly stationary* (UBLS), if there exists a constant $K \geq 1$ such that

$$\left\| \sum_{j=1}^{n} a_j x_{k_j+h} \right\|_H^2 \leq K \left\| \sum_{j=1}^{n} a_j x_{k_j} \right\|_H^2 \tag{1.6}$$

for all $h, k_j \in \mathrm{Z}$, $a_j \in \mathrm{C}$, $j = 1, \ldots, n$; $n \in \mathrm{N}$.

An extensive account on the historical development and characteristic properties of the classes (i)–(ii) has been presented by M. M. Rao [12] (cf. [6], [5]). The class of UBLS L^2-processes was introduced by D. Tjøstheim and J. B. Thomas [16], (cf. [7], [15] and the references therein).

One of the basic results concerning the asymptotic stationarity of non-stationary L^2-processes is the following one; essentially due to Rozanov [13; p. 283] (cf. [1; Theorem 3.1], [7; Theorem 14]). For a proof cf. [1; pp. 6–7].

THEOREM (1.1). *Suppose $x_k \in H$, $k \in \mathrm{Z}$, is a strongly harmonizable L^2-process and suppose*

$$(x_j, x_k)_H = \int_0^{2\pi} \int_0^{2\pi} e^{i(j\lambda - k\theta)}\, dF(\lambda, \theta), \qquad j, k \in \mathrm{Z},$$

is a representation in the form (1.5). *Then:*
(i) *x_k, $k \in \mathrm{Z}$, is asymptotically stationary*;
(ii) *furthermore*

$$\begin{aligned} r(h) &= \lim_{T\to\infty} \frac{1}{2T+1} \sum_{k=-T}^{T} (x_{k+h}, x_k)_H \\ &= \lim_{T\to\infty} \frac{1}{T+1} \sum_{k=0}^{T} (x_{k+h}, x_k)_H \\ &= \int_0^{2\pi} \int_0^{2\pi} \chi_\Delta(\lambda, \theta) e^{i(j\lambda - k\theta)}\, dF(\lambda, \theta), \qquad k \in \mathrm{Z}, \end{aligned}$$

where

$$\Delta = \{(\lambda, \theta) \in [0, 2\pi[\times [0, 2\pi[\mid \lambda = \theta\}.$$

Example 1.4 below shows that there exist weakly harmonizable, in fact, even UBLS L^2-processes $x_k \in H$, $k \in \mathrm{Z}$, which are not asymptotically stationary. The same example indicates that Theorem 8.1 in [12] cannot be true without additional assumptions.

Example 1.4 combined with Theorem 1.1 and the well-known fact that all UBLS L^2-processes are weakly harmonizable (cf. [7; Theorem 4.3]) gives the following result.

THEOREM (1.2) (i) *There exists weakly harmonizable, in fact,* UBLS *L^2-processes $x_k \in H$, $k \in \mathrm{Z}$, which are not asymptotically stationary*;

(ii) *there exists* UBLS *processes $x_k \in H$, $k \in \mathrm{Z}$, which are not strongly harmonizable.*

Example 1.4 is based on the following characterization due to D. Tjøstheim and J. B. Thomas [15].

THEOREM (1.3). *Let $x_k \in H$, $k \in Z$, be an L^2-process. The following three statements are equivalent*:
(i) x_k, $k \in Z$, *is* UBLS;
(ii) *there exist a stationary L^2-process $y_k \in H$, $k \in Z$, and a bounded linear mapping $A: H \to H$ with a bounded inverse such that*

$$x_k = Ay_k, \qquad k \in Z$$

(iii) *there exist a constant $K \geq 1$ and a stationary L^2-process $y_k \in H$, $k \in Z$, such that*

$$(1.7) \qquad 1/K \,\| \textstyle\sum_{j=1}^{n} a_j y_{k_j} \|_H^2 \leq \| \sum_{j=1}^{n} a_j x_{k_j} \|_H^2 \leq K \,\| \sum_{j=1}^{n} a_j y_{k_j} \|_H^2$$

for all $a_j \in C$, $k_j \in Z$, $j = 1, \cdots, n$; $n \in N$.

Remark. It follows from Theorem 1.3 (i)–(ii) that a weakly harmonizable L^2-process

$$x_k = \int_0^{2\pi} e^{ik\lambda}\, d\mu(\lambda), \qquad k \in Z,$$

is UBLS, if and only if there exist a bounded orthogonally scattered H-valued vector measure μ_0 on $[0, 2\pi[$, i.e.

$$(\mu_0(E), \mu_0(E'))_H = 0 \qquad \text{for all disjoint } E, E' \in \boldsymbol{B},$$

and a bounded linear mapping $A: H \to H$ with a bounded inverse such that

$$(1.8) \qquad \mu(E) = A\mu_0(E) \qquad \text{for all } E \in \boldsymbol{B}.$$

Example (1.4). Let H be a separable complex Hilbert space and let e_k, $k \in Z$, be a complete orthogonal basis in H. Define a weakly stationary L^2-process $x_k \in H$, $k \in Z$, by

$$x_k = e_k, \qquad k \in Z.$$

Define a sequence of real numbers a_k, $k \in Z$, by

$$a_k = \begin{cases} 1, & k = 0 \\ 1, & 2^{2n} \leq |k| < 2^{2n+1}, \qquad k \in Z, n \geq 0 \\ 2, & 2^{2n+1} \leq |k| < 2^{2n+2}, \qquad k \in Z, n \geq 0 \end{cases}$$

and another L^2-process $y_k \in H$, $k \in Z$, by

$$y_k = a_k x_k, \qquad k \in Z.$$

It is obvious that there exists a bounded linear mapping $A: H \to H$ (with a bounded inverse) such that $Ae_k = a_k e_k$, $k \in Z$, and, *a fortiori*, $y_k = Ax_k$, $k \in Z$, showing that y_k, $k \in Z$, is a UBLS L^2-process (cf. Theorem 1.3 (i)–(ii)).

However, in this case the supposed-to-be limit

$$r(0) = \lim_{T\to\infty} \frac{1}{2T+1} \textstyle\sum_{j=-T}^{T} \| y_j \|_H^2$$

fails to exist.

Example 1.4 gives a motivation to consider stationarization procedures, which are weaker than the one defined by (1.3). One possibility is to apply invariant means on Z. Another approach, based on the use of invariant means on the spectral domain is presented in Section 2.

Recall that for any commutative semigroup G there exists at least one invariant mean M on G, i.e. M is a linear mapping defined on $C(G) = \{h: G \rightarrow \mathrm{C} \mid h \text{ bounded}\}$ having the properties:

(i) $\inf_{g\in G} h(g) \leq M(h) \leq \sup_{g\in G} h(g)$

for all real valued $h \in C(G)$ and

(ii) $M(h) = \overline{M(\bar{h})}$,

(iii) $M(t_{g'} h) = M(h)$

for all $h \in C(G)$ and $g' \in G$; here $t_{g'} h(g) = h(g'g)$, $g, g' \in G$ (cf. [2; pp. 108–109], [4; p. 5]).

Remark. Let $x_k \in H$, $k \in \mathrm{Z}$, be any bounded L^2-process, i.e. $\| x_k \|_H < C$, $k \in \mathrm{Z}$, for some constant $C > 0$. Then

$$r(h) = M(x_{k+h}, x_k)_H, \qquad h \in \mathrm{Z}, \tag{1.9}$$

is a positive definite function for any invariant mean M on Z. The relationship between x_k, $k \in \mathrm{Z}$, and the spectral representation (cf. (1.2)) of r, defined by (1.9), or equivalently, between x_k, $k \in \mathrm{Z}$, and any stationary L^2-process $y_k \in H$, $k \in \mathrm{Z}$, with

$$r(h) = (y_{k+h}, y_k)_H, \qquad h \in \mathrm{Z},$$

has not yet been fully analyzed. A partial solution can be obtained in the case of UBLS L^2-processes, as indicated in Theorem 1.5 below.

The statement (i) of Theorem 1.5 can be proved by following the proof of Theorem 11 in [7]. The proofs of (ii)–(iii) are presented at the end of this paper.

THEOREM (1.5). *Let $x_k \in H$, $k \in \mathrm{Z}$, be a* UBLS *L^2-process and*

$$K_0 = \inf K,$$

where the infimum is formed over all $K \geq 1$ satisfying (1.6). *Then*: (i) *for any invariant mean $M: C(\mathrm{Z}) \rightarrow \mathrm{C}$ there exists a stationary L^2-process $y_k \in H$, $k \in \mathrm{Z}$, satisfying* (1.7) *with $K = K_0$ and*

$$(y_{j+h}, y_j)_H = M(x_{k+h}, x_k), \qquad \textit{for all } j, h \in \mathrm{Z}.$$

(ii) *Define*

$$r_{\inf}(0) = \inf \| y_0 \|_H^2,$$

where the infimum is formed over all stationary L^2-processes $y_k \in H$, $k \in \mathrm{Z}$, satisfying (1.7) *with $K = K_0$. Then, there exists a stationary L^2-process $y_k \in H$,*

$k \in \mathrm{Z}$, uniquely determined up to unitary equivalence, satisfying (1.7) *with* $K = K_0$ *and*

$$\| y_0 \|_H^2 = r_{\inf}(0).$$

2. Diagonal measure of a positive definite bimeasure

Suppose x_k, $k \in \mathrm{Z}$, is a strongly harmonizable sequence and suppose

$$(x_j, x_k)_H = \int_0^{2\pi} \int_0^{2\pi} e^{i(j\lambda - k\theta)} \, dF(\lambda, \theta), \qquad j, k \in \mathrm{Z}, \tag{2.1}$$

is the representation in the form (1.5). Theorem 1.1 (ii) shows that the part of F concentrated on the main diagonal Δ of $[0, 2\pi[\times [0, 2\pi[$ gives the associated spectral distribution of x_k, $k \in \mathrm{Z}$.

In this section we are concerned with the problem of how to find an analogue of Theorem 1.1 (ii) for weakly harmonizable L^2-processes x_k, $k \in \mathrm{Z}$. In this case the representation (1.5) must be interpreted as an integral with respect to the bimeasure defined by the spectral measure of x_k, $k \in \mathrm{Z}$ (cf. [6; Theorem 3.2.6], [5], [12]).

We are thus led to the notion of a diagonal measure of a positive definite bimeasure. We follow here the same approach as in [8].

Definition. Let $\boldsymbol{S}$ be a σ-algebra on a space S. A mapping $B : \boldsymbol{S} \times \boldsymbol{S} \to \mathrm{C}$ is a *bounded bimeasure*, if it is separately countably additive and if its semivariation is bounded, i.e.

$$\sup \left| \sum_{j=1}^m \sum_{k=1}^n a_j b_k B(E_j, E_k') \right| < \infty,$$

where the supremum is formed over all finite $\boldsymbol{S}$-measurable partitions $S = E_1 + \cdots + E_m$, $S = E_1' + \cdots + E_n'$ and $a_j, b_k \in \mathrm{C}$; $|a_j| \leq 1$, $|b_k| \leq 1$; $j = 1, \cdots, m$, $k = 1, \cdots, n$; $m, n \in \mathrm{N}$.

A bounded bimeasure $B : \boldsymbol{S} \times \boldsymbol{S} \to \mathrm{C}$ is *positive definite*, if

$$\sum_{j=1}^m \sum_{k=1}^m a_j \bar{a}_k B(E_j, E_k) \geq 0$$

for all $a_j \in \mathrm{C}$, $E_j \in \boldsymbol{S}$, $j = 1, \cdots, m$; $m \in \mathrm{N}$.

Remark. A mapping $B : \boldsymbol{S} \times \boldsymbol{S} \to \mathrm{C}$ is a bounded positive definite bimeasure, if and only if there exists a Hilbert space H_0 and a bounded vector measure $\mu : \boldsymbol{S} \to H_0$ such that

$$B(E, E') = (\mu(E), \mu(E'))_{H_0}, \qquad E, E' \in \boldsymbol{S}.$$

Definition. Suppose $\boldsymbol{S}$ is generated by a finite partition $d = \{E_1, \cdots, E_n\}$ of S (i.e., $S = E_1 + \cdots + E_n$). The *diagonal measure* of a positive definite bimeasure $B : \boldsymbol{S} \times \boldsymbol{S} \to \mathrm{C}$ is the nonnegative measure $\Delta_B : \boldsymbol{S} \to \mathrm{R}^+$ defined by setting

$$\Delta_B(E_k) = B(E_k, E_k), \qquad k = 1, \cdots, n.$$

Let $\boldsymbol{S}$ be an arbitrary field of subsets in S. In what follows by $\mathrm{D}(\boldsymbol{S})$ we denote the set of all finite $\boldsymbol{S}$-measurable partitions of S. The set $\mathrm{D}(\boldsymbol{S})$ can be considered as a commutative semigroup (with identity), if the semigroup operation associates to $d, d' \in \mathrm{D}(\boldsymbol{S})$ the refinement partition, denoted by dd', induced by d and d'.

Remark. Let $B:\boldsymbol{S} \times \boldsymbol{S} \rightarrow \mathrm{C}$ be a bounded positive definite bimeasure and let $\boldsymbol{S}(d)$ be the field generated by $d \in \mathrm{D}(\boldsymbol{S})$. In what follows, by $B(d):\boldsymbol{S}(d) \times \boldsymbol{S}(d) \rightarrow \mathrm{C}$ we denote the restriction of $B:\boldsymbol{S} \times \boldsymbol{S} \rightarrow \mathrm{C}$ to $\boldsymbol{S}(d) \times \boldsymbol{S}(d)$; and by $\Delta_{B(d)}$ we denote the diagonal measure of $B(d)$.

Suppose $B:\boldsymbol{S} \times \boldsymbol{S} \rightarrow \mathrm{C}$ is a bounded positive definite bimeasure. It was shown in [8; Theorem 2] that for any invariant mean $M:C(\mathrm{D}(\boldsymbol{S})) \rightarrow \mathrm{C}$ the mapping $\Delta_B{}^M:\boldsymbol{S} \rightarrow \mathrm{C}$ defined by

$$\Delta_B{}^M(E) = M(\Delta_{B(d)}(E)), \qquad E \in \boldsymbol{S}, \tag{2.2}$$

is a bounded nonnegative measure on S.

Definition. Let $B:\boldsymbol{S} \times \boldsymbol{S} \rightarrow \mathrm{C}$ be a bounded positive definite bimeasure. The bounded nonnegative measure $\Delta_B{}^M:\boldsymbol{S} \rightarrow \mathrm{R}^+$ defined by (2.2) is called a *diagonal measure* of B for any invariant mean $M:C(\mathrm{D}(\boldsymbol{S})) \rightarrow \mathrm{C}$.

Example 2.1. (i) Let $x_k \in H$, $k \in \mathrm{Z}$, be a stationary L^2-process. Thus, its spectral measure μ is orthgonally scattered, i.e.

$$(\mu(E), \mu(E'))_H = 0 \qquad \text{for all disjoint } E, E' \in \boldsymbol{B}.$$

In this case the diagonal measure $\Delta_B{}^M$ of the corresponding bimeasure B satisfies

$$\Delta_B{}^M(E) = \| \mu(E) \|_H^2, \qquad E \in \boldsymbol{B},$$

for all invariant means $M:C(\mathrm{D}(\boldsymbol{B})) \rightarrow \mathrm{C}$.

(ii) Let $x_k \in H$, $k \in \mathrm{Z}$, be a strongly harmonizable L^2-process. In this case, with notation as in Theorem 1.1,

$$\Delta_B{}^M(E) = \int_0^{2\pi} \int_0^{2\pi} \chi_{\Delta \cap E}(\lambda, \theta)\, dF(\lambda, \theta), \qquad E \in \boldsymbol{B},$$

for all invariant means $M:C(\mathrm{D}(\boldsymbol{B})) \rightarrow \mathrm{C}$.

The properties of diagonal measures of an arbitrary bounded positive definite bimeasure or the bimeasure defined by the spectral measure of an arbitrary weakly harmonizable L^2-process $x_k \in H$, $k \in \mathrm{Z}$, have not been fully analyzed. The case of UBLS processes is much clearer as indicated in Theorems 2.2 and 2.3 below (cf. the remark following Theorem 1.4).

Theorem 2.2 was first presented in [8; Theorem 3]. Theorem 2.3 gives a uniqueness property similar to the uniqueness of the minimal p-majorant of p-majorizable vector measures obtained by A. Pietsch [10; Satz 2]. The proof of Theorem 2.3 is presented at the end of this paper.

THEOREM (2.2). *Let $\mu:\boldsymbol{S} \rightarrow H$ be a bounded vector measure with values in a*

Hilbert space H. The following four conditions are equivalent:

(i) *there exists a bounded orthogonally scattered vector measure* $\mu_0:\boldsymbol{S}\to H$ *and a bounded linear mapping* $A:H\to H$ *with a bounded inverse such that*

$$\mu(E) = A\mu_0(E), \qquad E \in \boldsymbol{S}; \tag{2.3}$$

(ii) *there exists a constant* $K \geq 1$ *such that for any finite* $\boldsymbol{S}$*-measurable partition* $\{E_1, \dots, E_n\}$ *of S one has*

$$\frac{1}{K}\sum_{j=1}^{n} |a_j|^2 \|\mu(E_j)\|_H^2 \leq \|\sum_{j=1}^{n} a_j\mu(E_j)\|_H^2 \leq K\sum_{j=1}^{n} |a_j|^2 \|\mu(E_j)\|_H^2 \tag{2.4}$$

for all $a_j \in \mathrm{C}$, $j = 1, \dots, n$;
(iii) *there exists a constant* $K' \geq 1$ *and a bounded nonnegative measure* ν *on S such that*

$$\frac{1}{K'}\int_S |\phi|^2\, d\nu \leq \|\int_S \rho\, d\mu\|_H^2 \leq K'\int_S |\phi|^2\, d\nu \tag{2.5}$$

for all $\boldsymbol{S}$*-measurable simple functions* $\phi:\boldsymbol{S}\to \mathrm{C}$;
(iv) *there exist a constant* $K'' \geq 1$ *and a bounded orthogonally scattered vector measure* $\mu_0:\boldsymbol{S}\to H$ *such that for any finite* $\boldsymbol{S}$*-measurable partition* $\{E_1, \dots, E_n\}$ *of S*

$$\frac{1}{K''}\|\sum_{j=1}^{n} a_j\mu_0(E_j)\|_X^2 \leq \|\sum_{j=1}^{n} a_j\mu(E_j)\|_H^2 \leq K''\|\sum_{j=1}^{n} a_j\mu_0(E_j)\|_H^2 \tag{2.6}$$

for all $a_j \in \mathrm{C}$, $j = 1, \dots, n$.

Remark. Let $\mu:\boldsymbol{S}\to H$ and $\nu:\boldsymbol{S}\to \mathrm{R}^+$ be a bounded H-valued vector measure and, respectively, a bounded nonnegative measure on S satisfying (2.5). Then there exist a bounded orthogonally scattered vector measure $\mu_0:\boldsymbol{S}\to H$ with $\|\mu_0(E)\|_H^2 = \nu(E)$, $E \in \boldsymbol{S}$, and a bounded hermitean linear mapping $A:H\to H$ with a bounded inverse satisfying (2.3) and

$$\frac{1}{K'^{1/2}}\,\mathrm{Id}_H \leq A \leq K'^{1/2}\,\mathrm{Id}_H$$

(cf. [8; Theorem 3]).

THEOREM (2.3). *Suppose* $\mu:\boldsymbol{S}\to H$ *is a bounded vector measure with values in a Hilbert space H satisfying* (2.4), $B:\boldsymbol{S}\times\boldsymbol{S}\to \mathrm{C}$ *is the bimeasure defined by* μ *and*

$$K_0 = \inf K,$$

where the infimum is formed over all $K \geq 1$ *satisfying* (2.4). *Then*:
(i) *the inequalities* (2.5) *are satisfied with*

$$K' = K_0 \textit{ and } \nu = \Delta_B{}^M$$

for any invariant mean $M: C(\mathrm{D}(\boldsymbol{S})) \to \mathrm{C}$.

(ii) *Define*

$$\nu_{\inf} = \inf \nu(S),$$

where the infimum is formed over all bounded nonnegative measures ν *on* S *satisfying* (2.5) *with* $K = K_0$. *Then there exists a uniquely determined bounded nonnegative measure* ν_0 *on* S *satisfying the inequalities* (2.5) *with* $K = K_0$ *and* $\nu_0(S) = \nu_{\inf}$.

(iii) *There exists a bounded orthogonally scattered vector measure* $\mu_0: \boldsymbol{S} \to H$, *uniquely determined up to unitary equivalence, satisfying* (2.6) *with* $K'' = K_0$ *and*

$$\| \mu_0(E) \|_H{}^2 = \nu_0(E), \qquad E \in \boldsymbol{S}.$$

3. Prediction theoretical applications

In certain special cases the prediction theoretical properties of an asymptotically stationary L^2-process $x_k \in H$, $k \in \mathrm{Z}$, are reflected in the properties of the associated spectral distribution of x_k, $k \in \mathrm{Z}$ (cf. [7; Theorem 13]). In this section we present an interpretation of the "stationary prediction"-approach introduced by J. L. Abreu [1] in terms of the associated spectral distribution of a UBLS L^2-process $x_k \in H$, $k \in \mathrm{Z}$.

The associated spectral distribution of an arbitrary, say, strongly harmonizable L^2-process $x_k \in H$, $k \in \mathrm{Z}$, seems not to be very informative from the prediction theoretical point of view as is indicated in Example 3.2, due to J. Veilahti [16]. However, the use of the ρ-spectrum, introduced by J. L. Abreu [1], might prove to be a fruitful approach.

Let $x_k \in H$, $k \in \mathrm{Z}$, be an, say, asymptotically stationary L^2-process and let ν be the associated spectral distribution. Furthermore, suppose $y_k \in H$, $k \in \mathrm{Z}$, is a stationary L^2-process with the property

$$(y_{k+h}, y_k)_H = \int_0^{2\pi} e^{ih\lambda}\, d\nu(\lambda), \qquad h \in \mathrm{Z}.$$

It is well-known that for e.g. the optimal linear one-step predictions

$$\hat{y}_k(1) = \mathrm{Proj}_{sp\{y;k-1\}} y_k, \qquad k \in \mathrm{Z}$$

(i.e. $\hat{y}_k(1)$ is the orthogonal projection of y_k onto the closed linear subspace spanned by y_j, $j \leq k-1$) there exist approixmating sequences

$$y_{k,n} = \textstyle\sum_{h=1}^{N_n} a_{n,h} y_{k-h}, \qquad n = 1, 2, \ldots, \tag{3.1}$$

such that

$$\lim_{n\to\infty} y_{k,n} = \hat{y}_k(1) \qquad \text{for all} \quad k \in \mathrm{Z}.$$

Furthermore, for every $n \in \mathrm{Z}$

$$\inf_{a_1,\ldots,a_N} \lim_{T\to\infty} \frac{1}{2T+1} \sum_{j=-T}^{T} \| x_n - \sum_{j=1}^{N} a_j x_{n-j} \|_H^2$$
$$= \| y_n - \hat{y}_n(1) \|_H^2$$
$$= \exp \int_0^{2\pi} \log f(\lambda)\, d\lambda,$$

where $f(\lambda)\, d\lambda$ is the absolutely continuous part of ν and the infimum is formed over all $a_1, \ldots, a_N \in \mathrm{C}$; $N \in \mathrm{N}$ (cf. [14; pp. 111–116]).

The stationary prediction problem of x_k, $k \in \mathrm{Z}$, consists of finding criteria (i) for the convergence of the corresponding sequences

$$x_{k,n} = \sum_{h=1}^{N_n} a_{n,h} x_{k-h}, \qquad n = 1, 2, \ldots,$$

for all $k \in \mathrm{Z}$ (cf. (3.1)); and

(ii) for the goodness of the approximation of x_k with $\lim_{n\to\infty} x_{n,k}$, $k \in \mathrm{Z}$.

The stationary prediction problem can be solved e.g. in the case of UBLS processes (cf. [7; Remark 10]) by using e.g. (1.9) or (2.2) instead of (1.3) as the method to define the associated spectral distribution.

THEOREM (3.1). *Suppose $x_k \in H$, $k \in \mathrm{Z}$, is a* UBLS *process and suppose $y_k \in H$, $k \in \mathrm{Z}$, is a stationary L^2-process such that there exists a bounded linear mapping $A: H \to H$ with a bounded inverse satisfying*

$$x_k = A y_k \qquad \text{for all} \qquad k \in \mathrm{Z}.$$

Then:

(i) *for any sequences*

$$y_{k,n} = \sum_{j=1}^{N_n} a_{n,j} y_{k-j}, \qquad n = 1, 2, \ldots,$$

with $\lim_n y_{k,n} = \hat{y}_k(1)$ *for all* $k \in \mathrm{Z}$ *one has*

$$\lim_{n\to\infty} \sum_{j=1}^{N_n} a_{n,j} x_{k-j} = A\hat{y}_k(1), \qquad k \in \mathrm{Z};$$

(ii) *for all* $k \in \mathrm{Z}$

$$1/\| A^{-1} \|^2 \, \| y_k - \hat{y}_k(1) \|_H^2 \leq \| x_k - \hat{x}_k(1) \|_H^2$$
$$\leq \| x_k - A\hat{y}_k(1) \|_H^2$$
$$\leq \| A \|^2 \, \| y_k - \hat{y}_k(1) \|_H^2.$$

Proof. The statement (i) follows immediately from the properties of the bounded linear mapping $A: H \to H$.

To prove (ii), consider for a fixed $k \in \mathrm{Z}$ a sequence

$$y_{k,n} = \sum_{j=1}^{N_n} b_{n,j} y_{k-j}, \qquad n = 1, 2, \ldots,$$

satisfying $\lim_{n\to\infty} y_{k,n} = \hat{y}_k(1)$. It clearly follows from the optimality properties of $\hat{x}_k(1)$ and $\hat{y}_k(1)$ that e.g.

$$\begin{aligned}\| x_k - \hat{x}_k(1) \|_H^2 &\leq \lim_{n\to\infty} \| x_k - \textstyle\sum_{j=1}^{N_n} b_{n,j} x_{k-j} \|_H^2 \\ &= \| x_k - A\hat{y}_k(1) \|_H^2 \\ &\leq \| A \|^2 \, \| y_k - \hat{y}_k(1) \|_H^2.\end{aligned}$$

In a similar way one obtains

$$\begin{aligned}\| y_k - \hat{y}_k(1) \|_H^2 &\leq \| y_k - A^{-1}\hat{x}_k(1) \|_H^2 \\ &\leq \| A^{-1} \|^2 \, \| x_k - \hat{x}_k(1) \|_H^2.\end{aligned}$$

We close this section by presenting an example showing that, in general, the associated spectral distribution of an asymptotically stationary, in fact, even strongly harmonizable L^2-process $x_k \in H$, $k \in Z$, is hardly informative from the prediction theoretical point of view. The example is due to J. Veilahti [16].

Example (3.2). Consider stationary L^2-processes $x_k^{(1)} \in H$, $k \in Z$, and $x_k^{(2)} \in H$, $k \in Z$, having the properties $sp\{x^{(1)}\} \perp sp\{x^{(2)}\}$ and

$$(x_{k+h}^{(s)}, x_k^{(s)}) = \int_0^{2\pi} e^{ih\lambda} h_s(\lambda)\, d\lambda, \qquad h \in Z;\ s = 1, 2, \ldots,$$

with

$$h_1 = \mathrm{Ind}_{[0,\pi[}, \qquad h_2 = \mathrm{Ind}_{[\pi,2\pi[}.$$

Furthermore, define

$$\begin{cases} y_{2k} = x_k^{(1)} \\ y_{2k+1} = x_k^{(2)}, \end{cases} \qquad k \in Z.$$

Then, $y_k \in H$, $k \in Z$, is a periodically correlated L^2-process and, *a fortiori*, it is strongly harmonizable (cf. E. G. Gladyshev [3]).

It is obvious that both of the stationary L^2-processes $x_k^{(1)}$, $k \in Z$, and $x_k^{(2)}$, $k \in Z$, are deterministic and, *a fortiori*, also $y_k \in H$, $k \in Z$, is deterministic (or linearly singular, cf. [14; pp. 115–116]).

However, a simple calculation shows that the associated spectral distribution of y_k, $k \in Z$, is

$$\nu(d\lambda) = \tfrac{1}{2}\cdot d\lambda,$$

i.e. in this case the covariance kernel

$$\begin{aligned} r(h) &= \lim_{T\to\infty} \frac{1}{2T+1} \textstyle\sum_{j=-T}^{T} (y_{k+h}, y_k)_H \\ &= \tfrac{1}{2} \textstyle\int_0^{2\pi} e^{ih\lambda}\, d\lambda, \qquad h \in Z,\end{aligned}$$

is defined by a purely non-deterministic, or linearly regular, stationary L^2-process (cf. [14; pp. 115–116]).

4. Proofs of Theorems 2.3 and 1.5.

The proof of Theorem 2.3 is analogous to the proof of the uniqueness of the minimal p-majorant of a p-majorizable vector measure presented by Pietsch [10]. The proof is based on the following Lemma by Pietsch [10; p. 244]: For $p \geq 1$ and $x, y \geq 0$ define

$$F(x, y) = \begin{cases} 2^p xy(x^{1/p} + y^{1/p})^{-p}, & x > 0, y > 0, \\ 0, & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

Then

$$R(x, y) = (x + y)/2 - F(x, y) \geq 0,$$

and especially

$$R(x, y) = 0, \quad \text{if and only if} \quad x = y. \tag{4.1}$$

Proof of Theorem 2.3. (i) The statement (i) follows immediately from Theorem 3 (ii) in [8].

(ii) It follows from the first part of this theorem, that there exists at least one bounded nonnegative measure ν on S satisfying (2.5) with $K' = K_0$. Choose a sequence ν_n of bounded nonnegative measures on S satisfying (2.5) with $K = K_0$ and

$$\nu_n(S) = \nu_{\inf} + 1/n, \qquad n = 1, 2, \ldots.$$

Since the set ν_n, $n = 1, 2, \ldots$, is bounded, i.e. $\nu_n(S) \leq \nu_{\inf} + 1$ for all n, it follows that the set ν_n, $n = 1, 2, \ldots$, is weakly sequentially complete as a subset in the space of all bounded measures on S. Thus, there exists a subsequence of ν_n, $n = 1, 2, \ldots$, converging weakly to a bounded nonnegative measure ν_0 on S. Clearly, $\nu_0(S) = \nu_{\inf}$.

To show the uniqueness, consider two bounded nonnegative measures ν_1 and ν_2 on S satisfying (2.5) with $K' = K_0$ and

$$\nu_1(S) = \nu_2(S) = \nu_{\inf}.$$

Define $\nu = \nu_1 + \nu_2$ and choose the functions $f_s = d\nu_s/d\nu$, $s = 1, 2$, to be nonnegative and bounded. Then for the uniformly bounded $\boldsymbol{S}$-measurable functions

$$f_\epsilon = F(f_1 + \epsilon, f_2 + \epsilon), \qquad 0 < \epsilon < 1,$$

one has

$$f_\epsilon^{1/2}\{(f_1 + \epsilon)^{-1/2} + (f_2 + \epsilon)^{-1/2}\} = 2$$

(cf. [10; p. 244]). Thus for any bounded $\boldsymbol{S}$-measurable function $\phi : S \to \mathrm{C}$

$$\begin{aligned} 2 \| \textstyle\int_S \phi \, d\mu \|_H &\leq \| \textstyle\int_S \phi \, f_\epsilon^{1/2} (f_1 + \epsilon)^{-1/2} \, d\mu \|_H \\ &\quad + \| \textstyle\int_S \phi \, f_\epsilon^{1/2} (f_2 + \epsilon)^{-1/2} \, d\mu \|_H \\ &\leq K_0^{1/2} \{ \textstyle\int_S | \phi |^2 f_\epsilon (f_1 + \epsilon)^{-1} \, d\nu_1 \}^{1/2} \\ &\quad + K_0^{1/2} \{ \textstyle\int_S | \phi |^2 f_\epsilon (f_2 + \epsilon)^{-1} \, d\nu_2 \}^{1/2} \\ &\leq 2 K_0^{1/2} \{ \textstyle\int_S | \phi |^2 f_\epsilon \, d\nu \}^{1/2}. \end{aligned}$$

Then, by letting $\epsilon \to 0$ we get

$$\| \textstyle\int_S \phi \, d\mu \|_H^2 \leq K_0 \int_S | \phi |^2 f_0 \, d\nu, \tag{4.2}$$

with $f_0 = F(f_1, f_2)$.

On the other hand, for all $x, y > 0$ and $p \geq 1$

$$\frac{2^p xy}{(x^{1/p} + y^{1/p})^p} \leq \frac{x + y}{2}$$

(cf. [10; p. 244]). Thus for all $\epsilon > 0$

$$\begin{aligned} \textstyle\int_S | \phi |^2 f_\epsilon \, d\nu &\leq \tfrac{1}{2} \textstyle\int_S | \phi |^2 \, (f_1 + f_2 + 2\epsilon) \, d\nu \\ &\leq K_0 \| \textstyle\int_S \phi \, d\mu \|_H^2 + C\epsilon \qquad (C = \text{const.}); \end{aligned}$$

proving that

$$\textstyle\int_S | \phi |^2 f_0 \, d\nu \leq K_0 \| \int_S \phi \, d\mu \|_H^2. \tag{4.3}$$

The inequalities (4.2) and (4.3) show that the bounded measure

$$\nu_0(E) = \textstyle\int_E f_0 \, d\nu, \qquad E \in \boldsymbol{S},$$

satisfies (2.5) with $K' = K_0$. Thus,

$$\nu_0(S) \geq \nu_{\text{inf}}.$$

On the other hand,

$$\begin{aligned} \nu_0(S) &= [\nu_1(S) + \nu_2(S)]/2 - \textstyle\int_S R(f_1, f_2) \, d\nu \\ &= \nu_{\text{inf}} - \textstyle\int_S R(f_1, f_2) \, d\nu. \end{aligned}$$

Thus,

$$\textstyle\int_S R(f_1, f_2) \, d\nu = 0$$

and, *a fortiori*,

$$R(f_1, f_2) = 0 \qquad \nu \text{ - } a.e.;$$

proving that $f_1 = f_2$ ν - *a.e.* (cf. (4.1)), that is $\nu_1 = \nu_2$.

(iii). The final part of the theorem follows immediately from the obvious fact that any two bounded orthogonally scattered vector measures $\mu_1: S \to H$ and $\mu_2: S \to H$ with

$$\| \mu_1(E) \|_H^2 = \| \mu_2(E) \|_H^2 \qquad \text{for all} \quad E \in S,$$

are unitarily equivalent.

Proof of Theorem 1.5. (i) As noted above, the first part of the theorem can be proved by following the proof of Theorem 11 in [7].

(ii) Let $y_k \in H$, $k \in \mathrm{Z}$, be a stationary sequence satisfying

$$1/K_0 \| \textstyle\sum_{j=1}^n a_j y_{k_j} \|_H^2 \leqslant \| \sum_{j=1}^n a_j x_{k_j} \|_H^2$$
$$\leqslant K_0 \| \textstyle\sum_{j=1}^n a_j y_{k_j} \|_H^2$$

for all $a_j \in C$, $k_j \in Z$, $j = 1, \cdots, n$; $n \in N$, and let μ_x and μ_y be the spectral measures of x_k, k, $\in Z$, and y_k, $k \in Z$, respectively. It then follows that

$$1/K_0 \| \textstyle\sum_{j=1}^n a_j \mu_y(E_j) \|_H^2 \leq \| \sum_{j=1}^n a_j \mu_x(E_j) \|_H^2$$
$$\leq K_0 \| \textstyle\sum_{j=1}^n a_j \mu_y(E_j) \|_H^2$$

for all finite Borel measurable partitions $\{E_1, \cdots, E_n\}$ of $[0, 2\pi[$ and $a_j \in \mathrm{C}$, $j = 1, \ldots, n$ (cf. the remark following Theorem 1.3).

Since μ_y is the spectral measure of a stationary L^2-process y_k, $k \in \mathrm{Z}$, it is orthogonally scattered and, *a fortiori*,

$$\nu(E) = \| \mu_y(E) \|_H^2, \qquad E \in \boldsymbol{B},$$

is a bounded nonnegative measure on $[0, 2\pi[$. Moreover, since

$$\| y_0 \|_H^2 = \nu[0, 2\pi[,$$

it is obvious that the proof of Theorem 2.3 can be applied with $K_0' = K_0$ to show the uniqueness of the nonnegative measure ν_0 on $[0, 2\pi[$ satisfying

$$1/K_0 \int_0^{2\pi} | \phi |^2 \, d\nu_0 \leq \| \int_0^{2\pi} \phi \, d\mu_x \|_H^2$$
$$\leq K_0 \int_0^{2\pi} | \phi |^2 \, d\nu_0$$

for all bounded Borel measurable functions $\phi: [0, 2\pi[\to \mathrm{C}$ and, in addition,

$$r_{\inf}(0) = \nu_0[0, 2\pi[.$$

(Notice that, in the present case, it follows from the first part of this theorem that there exists at least one ν on $[0, 2\pi[$ satisfying (2.5) with $K' = K_0$ and $\mu = \mu_x$.)

Let μ_0 be a bounded H-valued orthogonally scattered vector measure on $[0, 2\pi[$ such that

$$\| \mu_0(E) \|_H^2 = \nu_0(E), \qquad E \in \boldsymbol{B}. \tag{4.4}$$

The L^2-process

$$y_k = \int_0^{2\pi} e^{ik\lambda} \, d\mu_0(\lambda), \qquad k \in \mathrm{Z},$$

is then stationary and

$$(y_{k+h}, y_k)_H = \int_0^{2\pi} e^{ih\lambda}\, d\nu_0(\lambda), \qquad h \in Z$$

and, *a fortiori*, $\| y_0 \|_H^2 = r_{\inf}(0)$.

As in the proof of Theorem 2.3 all the bounded orthogonally scattered vector measures μ_0 on $[0, 2\pi[$ satisfying (4.4) are unitarily equivalent, showing that all the stationary L^2-processes $y_k \in H$, $k \in Z$, satisfying (1.7) with $\| y_0 \|_H^2 = r_{\inf}(0)$ are unitarily equivalent.

UNIVERSITY OF HELSINKI
Department of Statistics
Aleksanterinkatu 7
00100 Helsinki 10, Finland

REFERENCES

[1] J. L. ABREU, *Stationary prediction of non-stationary sequences in Hilbert space.* In Prediction theory and Harmonic Analysis. (V. Mandrekar and H. Salehi, Eds.), pp. 1–12, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1983.

[2] M. M. DAY, Normed linear spaces. (Third edition) Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[3] E. G. GLADYSHEV, *On periodically correlated random sequences.* Soviet Math. Dokl. **2** (1961), 385–388.

[4] F. P. GREENLEAF, Invariant means on topological groups. Van Nostrand Reinhold Co., New York-Toronto-London-Melbourne, 1969.

[5] A. G. MIAMEE AND H. SALEHI, *Harmonizability, V-boundedness and stationary dilation of stochastic processes.* Indiana Univ. Math J. **27** (1978), 37–50.

[6] H. NIEMI, *Stochastic processes as Fourier transforms of stochastic measures.* Ann. Acad. Sci. Fenn. Ser. A I 591, 1975.

[7] ———, *On the linear prediction problem of certain non-stationary stochastic processes.* Math. Scand. **39** (1976), 146–160.

[8] ———, *Diagonal measure of a positive definite bimeasure.* In Measure Theory, Proc. Conf. Oberwolfach 1981. (D. Kölzow and D. Maharam-Stone, Eds.), pp. 327–346. Lecture Notes in Mathematics No. 945, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[9] E. PARZEN, *Spectral analysis of asymptotically stationary time series.* Bull. Inst. Internat. Statist. **39**, 2(1962), 87–103.

[10] A. PIETSCH, p-*majorisierbare vektorwertige Masse.* Wiss. Z. Friedrich-Schiller-Univ. Jena-Thüringen Math.-Natur. Reihe **18** (1969), 243–247.

[11] M. M. RAO, *Covariance analysis of nonstationary time series.* In Developments in Statistics (P. R. Krishnaiah, Ed.), Vol. 1, pp. 171–225. Academic Press, New York-San Francisco-London, 1978.

[12] ———, *Harmonizable processes: Structure theory.* Enseign. Math. (2) **28** (1982), 295–351.

[13] Y. A. ROZANOV, *Spectral analysis of abstract functions.* Theor. Probability Appl. **4** (1959), 271–287.

[14] ———, *Stationary random processes,* Holden-Day series in time series analysis, Holden-Day, San Francisco-Cambridge-London-Amsterdam, 1967.

[15] D. TJØSTHEIM, *On the analysis of a class of multivariate nonstationary stochastic processes,* In Prediction Theory and Harmonic Analysis. (V. Mandrekar and H. Salehi, Eds.), pp. 403–416, 1983.

[16] ——— AND J. B. THOMAS, *Some properties and examples of random processes that are almost wide sense stationary,* IEEE Trans. Information Theory **21** (1975), 257–262.

[17] J. VELAHTI, Licenciate Thesis, University of Helsinki, 1974 (unpublished).