Boletín de la Sociedad Matemática Mexicana Vol. 28, No. 1, 1983

ON NORMAL DILATION AND SPECTRUM OF SOME CLASSES OF SECOND ORDER PROCESSES*

BY H. SALEHI AND M. SLOCINSKI

Introduction

In recent years there has been a great interest in the study of stationary dilation as well as the spectrum of some classes of nonstationary processes including harmonizable and in particular square summable sequences. Stationary dilation of harmonizable processes are studied by J. L. Abreu [2], H. Niemi [17], A. G. Miamme and H. Salehi [14], S. G. Chatterji [5]. The spectrum of a second order process was introduced and developed by Yu. A. Rozanov in [20]. With some modification of the notion of the spectrum J. L. Abreu [3] recently obtained valuable results on stationary approximation of nonstationary processes. The main purpose of this paper is to extend the notion of stationary dilation and stationary approximation to that of subnormal (normal) dilation and subnormal (normal) approximation of a wider class of nonstationary processes. Results on dilation are in Sections 1 and 3; and the work on spectrum and approximation are in Sections 2 and 3. In Section 1 the concept of a generalized harmonizable process is introduced and spectral characterization for such a process is obtained, c.f. Theorems 1.12 and 1.13. This section also include a subnormal (normal) dilation results, c.f. Theorem 1.22 for generalized harmonizable sequences. The idea of subnormal processes occurs in the work of R. K. Getoor [10] where he gives a spectral characterization of such processes. More constructive dilation results extending Niemi's work on dilation of square summable sequences are given in Section 3, c.f. Theorems 3.1 and 3.2. In Theorem 3.1 the norm of X_n need not converge to zero, indeed the norm of X_n may tend to ∞. In Theorem 3.2 the square summability with respect to Lebesgue measure occuring in Niemi's work is replaced by square summability with respect to an arbitrary nonnegative σ-finite measure. Section 2 is about the Rozanov type spectrum of generalized harmonizable processes with spectral measure of bounded variation. The result is stated in Theorem 2.2. Theorem 2.3 deals with subnormal (normal) approximation of generalized harmonizable sequences, and is based on the existence of the type of spectrum established in Theorem 2.2. This includes the generalized harmonizable case where the spectral measure is concentrated on a countable number of concentric circles centered at the origin. Using Abreu's version of ρ-spectrum with $\rho \equiv 1$ we give a proof of a subnormal approximation of weighted square summable sequences.

---

* Presented at the "Workshop on the prediction theory of non-stationary processes and related topics", held at the Centro de Investigación en Matemticas (CIMAT), Guanajuato, México, June 20–26, 1982.

In general dilation theorems provide an upper bound for the extrapolation error, c.f. Remark 1.27. But when a process has an spectrum a least upper bound can be obtained for a mean-type extrapolation error which leads to a mean-type subnormal approximation, c.f. Theorem 2.2 and 2.3. For the special case of weighted square summable sequences the dilating subnormal process can be chosen so that its spectral distribution is exactly the same as the spectrum of the original sequence. This enables us to obtain a stronger type subnormal approximation in this case, c.f. Theorem 3.4.

§1. Generalized harmonizable processes

In this section generalized harmonizable processes are introduced and a spectral characterization for such processes is given. Also subnormal (normal) dilation for generalized harmonizable processes is obtained. We are given a separable complex Hilbert space H, with the usual notation $(\bullet, \bullet)$ and $\|\bullet\|$, for the inner product and the norm. $L(H)$ will denote the class of bounded linear operators on H into H. S is an abelian semigroup with 0 as its neutral element and + as its operation. X is a function on S into H. We first recall some definitions.

Definition (1.1). A function $X: S \to H$ is said to be a subnormal process (briefly subnormal) if there exists a subnormal semigroup of operators $N(t)$, $t \in S$, on H (We will write this by $N: S \to L(H)$) such that $X(t) = N(t)X(0)$, $t \in S$.

We mention that a semigroup $N(t)$, $t \in S$, on H said to be subnormal if there exists a space $\tilde{H} \supset H$ and a semigroup of normal operators $\tilde{N}(t)$ on $\tilde{H}$, $t \in S$, such that $\tilde{N}(t)H \subseteq H$, $t \in S$ and $N(t)x = \tilde{N}(t)x$, $x \in H$, $t \in S$. This differs from the concept of a semigroup of subnormal operators, in a sense that each subnormal semigroup of operators is a semigroup of subnormal operators, but the converse is not true in general. For more information on related topics see [1], [12], [21].

Definition (1.2). X is said to be minimal if and only if (iff) $H = V_{t\in S}X(t)$, V means "the subspace spanned by".

Definition (1.3). X is normal iff S is a group, N is a group of normal operators on H and $X(t) = N(t)X(0)$.

Remark (1.4). Let $X_t = N(t)X_0$, $t \in S$. Then X is minimal iff X_0 is a cyclic vector for N meaning $V_{t\in S}N(t)X_0 = H$.

Remark (1.5). The above definition of subnormal processes is in some sense a modification of Getoor's definition [10], S is any abelian semigroup, no topological assumption is made, but we assume that the operators are bounded. The special cases of interest to us are $S = Z$ = the group of integers and $S = Z^+$ = the semigroup of nonnegative integers.

Remark (1.6). For the case $S = Z$ or Z^+, X is subnormal $\Leftrightarrow X(n) = T^nX_0$ with T being subnormal.

The following theorem provides a spectral characterization for subnormal processes. (For the case $S = Z$ or Z^+) we may write X_n in place of X or $X(n)$).

THEOREM (1.7). *Let $S = Z$ or Z^+. X_n is subnormal $\Leftrightarrow$ there exists a compact set $K \subseteq \mathbf{C}$ = complex numbers and there exists a finite measure μ on K such that*

$$(X_n, X_m) = \int_K z^n \bar{z}^m d\mu(z).$$

Proof. ($\Rightarrow$) $X_n = T^n X_0$. Let N be a normal extension of T and $K = \sigma(N) =$ spectrum of N which is compact. N normal $\Rightarrow N^n = \int_k z^n dE(z)$, E being the spectral measure of N.
Let $\mu(\Delta) = (E(\Delta)X_0, X_0)$, $\| \mu \| \leq \| X_0 \|^2$. Now

$$(X_n, X_m) = (T^n X_0, T^m X_0) = (N^n X_0, N^m X_0) = (N^{*m} N^n X_0, X_0)$$
$$= \int_K \bar{z}^m z^n d(EX_0, X_0) = \int_K z^n \bar{z}^m d\mu(z).$$

($\Leftarrow$) Let $\tilde{H} = L^2(\mu)$. Define $\tilde{N}$ on $\tilde{H}$ by

$$(\tilde{N}f)(z) = zf(z),\ f \in L^2(\mu).$$

Clearly the operator $\tilde{N}$ is normal on $L^2(\mu)$. Let $H_1 = V_{n \in S} X_n$ and define

$$\tilde{\Phi}\colon \sum \alpha_n X_n \to \sum \alpha_n z^n, \text{ finite sums.}$$

We note that

$$(\textstyle\sum_n \alpha_n X_n, \sum_k \beta_k X_k) = \sum_{n,k} \alpha_n \bar{\beta}_k (X_n, X_k) = \sum_{n,k} \alpha_n \bar{\beta}_k \int_K z^n \bar{z}^m d\mu(z)$$
$$= (\textstyle\sum_n \alpha_n z^m, \sum_k \beta_k z^k)_{L^2(\mu)}.$$

Therefore, the map $\tilde{\Phi}$ can be extended uniquely to a unitary map Φ of H_1 onto $\tilde{H}_1$ = the closed span (in $L^2(\mu)$) of the set $\{z^n, n \in S\}$. $\tilde{H}_1$ is invariant under $\tilde{N}$, and $\tilde{T} = \tilde{N}_{|\tilde{H}_1}$ = restriction of $\tilde{N}$ to $\tilde{H}_1$, is subnormal. Let $T = \Phi^{-1}\tilde{T}\,\Phi$. Then T is subnormal and

$$X_n = \Phi^{-1} z^n = \Phi^{-1}(z^n \cdot 1) = \Phi^{-1}(z^n \cdot \Phi X_0) = \Phi^{-1} \tilde{T}^n \Phi X_0 = T^n X_0.$$

This finishes the proof for the case when X_n is minimal. If X_n is not minimal define T on $H \ominus H_1$ as a unitary operator.

COROLLARY (1.8) *If $S = Z$ then $0 \notin K$.*

THEOREM (1.9). *Let $S = Z$. Then X_n is normal and minimal $\Leftrightarrow$ there exists a compact set $K \subseteq \mathbb{C}$, $0 \notin K$ and there exists a finite measure μ on K such that*
(i) $(X_n, X_m) = \int_k z^n \bar{z}^m d\mu(z)$,
(ii) *The set $\{z^n, n \in Z\}$ is linearly dense in $L^2(\mu)$.*

Proof. ($\Rightarrow$) $X_n = N^n X_0$. Put $K_1 = \sigma(N)$, $N^{-1} \in L(H) \Rightarrow 0 \notin K_1$. Part (i) is a consequence of Theorem 1.7, consequently it suffices to show (ii). Let $\tilde{H}_1 = V_{n \in Z} z^n$ in $L^2(\mu)$, μ is concentrated on K_1. Let Φ be the unitary map on H onto $\tilde{H}_1$ as in Theorem 1.7. Let $\tilde{N} = \Phi N \Phi^{-1}$. Then $\tilde{N}$ is multiplication by z on $\tilde{H}_1$

$\subseteq L^2(\mu)$, because $\Phi N \Phi^{-1}(\sum_k \alpha_k z^k) = \Phi N \sum_k \alpha_k X_k = \Phi \sum_k \alpha_k X_{k+1} = \sum_k \alpha_k z^{k+1} = z \sum_k \alpha_k z^k$. N and $\tilde{N}$ are unitarily equivalent, hence $\tilde{N}$ is also normal on $\tilde{H}_1$. The space $\tilde{H}_1$ is invariant for M_z = multiplication operator by z in $L^2(\mu)$ and $\tilde{N}$ = Restriction of M_z to $\tilde{H}_1$. Moreover M_z is normal on $L^2(\mu)$. Consequently $\tilde{H}_1$ reduces M_z. This last statement follows from $\| P_{\tilde{H}_1} M_z^* X \| = \| \tilde{N}^* X \| = \| \tilde{N} X \| = \| M_z X \| = \| M_z^* X \|$ for all $X \in \tilde{H}_1$. Consequently there exists a $K \subseteq K_1$ such that $\tilde{H}_1 = \chi_{K_1} L^2(\mu)$. But $\mu(K_1) = \int_{K_1} d\mu = \| X_0 \|^2 = \| \Phi X_0 \|^2 = \int_K 1 \mu d = \mu(K)$. This implies $\mu(K_1 \backslash K) = 0$ or $\mu(K) = \mu(K_1)$. Consequently $\mu(\bar{K}) = \mu(K_1)$, $\bar{K}$ = closure of K. Therefore $\tilde{H}_1 = L^2(\mu)$ which finishes the proof of $(\Rightarrow)$

$(\Leftarrow)$ $0 \notin K \Rightarrow$ the multiplication by z^n, $n \in Z$, in $L^2(\mu)$ is welldefined. Following the proof of Theorem 1.7 the operator $\tilde{T}$ defined there is defined on all of $L^2(\mu)$ by assumption (ii). Consequently T, given there, is a normal operator in H. $X_n = T^n X_0$, $n \in Z$, completing the proof.

The following definition is a generalization of the usual notion of harmonizability, c.f. [20].

Definition (1.10). Let $S = Z$ or Z^+. X: $S \to H$ is said to be a generalized harmonizable process $\Leftrightarrow$ there exists a compact set $K \subseteq \mathbf{C}$ and there exists an H-valued measure η such that $X_n = \int_K z^n d\eta(z)$. (For integration with respect to H-valued measure see [8] pp. 318–329).

Remark (1.11). Let $S = Z$, then the usual definition of harmonizability for X is $X_n = \int_\Gamma z^n d\eta(z)$, where $\Gamma = \{ |z| = 1 \} \subseteq \mathbf{C}$, c.f. Rozanov [20]. So it is clear that X_n is harmonizable $\Leftrightarrow$ X_n is generalized harmonizable and the support of $\eta \subseteq \Gamma$.

THEOREM (1.12). *Let X_n be generalized harmonizable, i.e. $X_n = \int_K z^n d\eta(z)$. Then $(X_n, X_m) = \iint_{K \times K} z_1^n \bar{z}_2^m \, dF(z_1, z_2)$, where $F(\Delta_1 \times \Delta_2) = (\eta(\Delta_1), \eta(\Delta_2))$ for Δ_1, Δ_2 Borel sets, and $F(\cdot)$ satisfies the following conditions*:

(a) *F is finitely additive on the algebra generated by the rectangles $\Delta_1 \times \Delta_2$ with Δ_1, Δ_2 Borel subsets of K,*

(b) *F is positive definite in the sense that for any set of complex numbers α_1, $\alpha_2, \cdots, \alpha_n$, $\sum_{i,j=1}^n \alpha_i \bar{\alpha}_j F(\Delta_i \times \Delta_j) \geq 0$,*

(c) *F is continuous from above, i.e. for any monotonic decreasing sequences of sets Δ_n, $\Delta_n \supseteq \Delta_{n+1}$, $\bigcap_{n=1}^\infty \Delta_n = \emptyset$; Δ_n', $\Delta_n' \supseteq \Delta_{n+1}'$, $\bigcap_{n=1}^\infty \Delta_n' = \emptyset$ we have $F(\Delta_n, \Delta_m') \to 0$ as $n, m \to \infty$.*

(d) *F is of bounded semi-variation, i.e.,*

$$\sup | \textstyle\sum_{k=1}^n \sum_{j=1}^n \alpha_K \bar{\beta}_j F(\Delta_k \times \Delta_1') | < \infty,$$

when the sup *is taken over all finite families of disjoint measurable sets $\Delta_1, \cdots, \Delta_n$, $\Delta_k \subseteq K$ and $\Delta_1', \cdots, \Delta_n'$, $\Delta_j' \subseteq K$, and complex numbers $\alpha_1, \cdots, \alpha_n$, $|\alpha_k| \leq 1$ and $\beta_1, \cdots, \beta_n$, $|\beta_j| \leq 1$.*

Proof. The proof of assertions (a)–(c) can be carried out in a similar fashion to the proof given in [20]. (Condition (d) is not included in Theorem 1.2 [20]).

(d) follows because $\eta(\cdot)$ is countably additive and hence η is of bounded semi-variation (c.f. [8] p. 320). Therefore

$$\sup_{i,j} | \alpha_i \bar{\beta}_j F(\Delta_i \times \Delta_j') | = \sup_{i,j} \in (\textstyle\sum_i \alpha_i \eta(\Delta_i), \sum_j \beta_j \eta(\Delta_j')) | \leq \sup_i | \sum_i \alpha_i \eta(\Delta_i) \sup_j | \sum \beta_j \eta(\Delta_j') | \leq | \text{semi-variation of } \eta |^2 < \infty.$$

Actually when $K = \Gamma$, (a), (b), (c) $\Rightarrow$ (d). This for instance follows from the discussion above and Theorem 1.1 [20] or from Theorem 2.10 [13].

THEOREM (1.13). *Let $S = Z$ or Z^+ and X: $S \to H$. Assume $(X_n, X_m) = \iint_{K\times K} z_1^n \bar{z}_2^m \, dF(z_1, z_2)$, where F satisfies the conditions (a) – (c) listed in Theorem* 1.12. *then there exists an H-valued measure η on K such that $X_n = \int_K z^n \, d\eta(z)$, that is X_n is generalized harmonizable.*

Proof. As mentioned earlier (c.f. [20] Theorem 1.1, [13] Theorem 2.10) the conditions (a) – (c) guarantee the existence of a separable Hilbert space H_1 and an H_1-valued measure $\tilde{\eta}$ such that $F(\Delta_1 \times \Delta_2) = (\tilde{\eta}(\Delta_1), \tilde{\eta}(\Delta_2))$. We may assume additionally that $H_1 = V\tilde{\eta}(\sigma)$, σ ranges over measurable subsets of K. Let $\tilde{X}_n = \int_k z^n \, d\tilde{\eta}(z)$ and let $H_0 = V_{n\in S}\tilde{X}_n$. It is easy to see that $(\tilde{X}_n, \tilde{X}_m) = \iint_{K\times K} z_1^n \bar{z}_2^m \, dF(z_1, z_2) = (X_n, X_m)$. Consequently there exists a unitary map U of H_0 onto $V_{n\in S} X_n \subseteq H$ such that $U\tilde{X}_n = X_n$. Clearly $H_0 \subseteq H_1$. Let P be the orthogonal projection of H_1 onto H_0. Evidently $P\tilde{X}_n = \tilde{X}_n$. Thus $X_n = U\tilde{X}_n = UP\tilde{X}_n = \int_K z^n \, d(UP\tilde{\eta}(z)) = \int_K z^n \, d\eta(z)$, with $\eta(\Delta) = UP\tilde{\eta}(\Delta)$. This finishes the proof.

Remark (1.14). Condition (c) may be replaced by the weaker condition (c'): $F(\cdot, \cdot)$ is separately continuous in each of its arguments, that is for each fixed A and each fixed B the complex-valued measures $F(A, \cdot)$ and $F(\cdot, B)$ are countably additive. This fact was brought to our attention recently by H. Niemi, c.f. [18] p. 241. (In this regard useful information can be gained from [7] p. 7 and p. 28).

Remark (1.15). (a) For a given F satisfying (a) – (b) one can find a process X_n such that $(X_n, X_m) = \iint_{K\times K} z_1^n \bar{z}_2^m \, dF(z_1, z_2)$ using reproducing kernel Hilbert space method, c.f. [13]. If additionally F has property (c) then the process X_n is generalized harmonizable.

(b) We observe that generally $(\tilde{\eta}(\Delta_1), \tilde{\eta}(\Delta_2))_{H_1} = F(\Delta_1 \times \Delta_2)$, but η need not have this property, because

$$(\eta(\Delta_1), \eta(\Delta_2)) = (P\tilde{\eta}(\Delta_1), (P\tilde{\eta}(\Delta_2)) = (P\tilde{\eta}(\Delta_1), \tilde{\eta}(\Delta_2)).$$

(c) Using Theorem 1.13 we can define $\tilde{F}(\Delta_1 X \Delta_2) = (\eta(\Delta_1), \eta(\Delta_2))$ which has the property $(X_n, X_m) = \iint_{K\times K} z_1^n \bar{z}_2^m \, dF(z_1, z_2) = \iint_{K\times K} z_1^n \bar{z}_2^m \, d\tilde{F}(z_1, z_2)$, but generally $\tilde{F}$ need not be equal to F.

COROLLARY (1.16). *If $H_0 = H_1 (P = I)$, then η has the property*

$(\eta(\Delta_1), \eta(\Delta_2)) = F(\Delta_1 \times \Delta_2)$ and F is uniquely determined (η is uniquely determined up to unitary equivalence).

In the following lemma we provide conditions under which $H_0 = H_1$.

LEMMA (1.17). *If $S = Z$ and $\{z^n, n \in Z\}$ is linearly dense in $C(K)$ = space of continuous functions on K or if $S = Z$ or Z^+, and $P(K)$ = uniform closure of polynomials in $C(K) = C(K)$, then $H_0 = H_1$.*

Proof. Under these conditions the characteristic functions of measurable sets can be approximated by finite sums $\sum_n \alpha_n z^n$, and consequently the values of $\tilde{\eta}$ can be approximated by sums $\sum_n \alpha_n \tilde{X}_n$ which finishes the proof.

COROLLARY (1.18). *If $\{z^n, n \in Z\}$ is linearly dense in $C(K)$ or if $S = Z$ or Z^+ and $P(K) = C(K)$ then η has the property $(\eta(\Delta_1), \eta(\Delta_2)) = F(\Delta_1 \times \Delta_2)$ and F is uniquely determined.*

LEMMA (1.19). *If $K = \Gamma_r$ = the circle of radius r with center at 0, then $\{z^n, n \in Z\}$ is linearly dense in $C(K)$.*

Proof. We note that $z^n = r^{2n} \dfrac{1}{\bar{z}^n}$. Use the fact that $\{z^n, n \in Z\} \cup \{\bar{z}^n, n \in Z\}$ is linearly dense in $C(\Gamma_r)$.

Remark (1.20). In Rozanov's work [20], $K = \Gamma_1$ which we denote by Γ = unit circle and $\{z^n, n \in Z\}$ is linearly dense in $C(\Gamma)$. Hence $H_0 = H_1$, η has the property $(\eta(\Delta_1), \eta(\Delta_2)) = F(\Delta_1 X \Delta_2)$ and F is uniquely determined.

THEOREM (1.21). *Let $S = Z$ or Z^+. Then X_n is generalized harmonizable and η is orthogonally scattered $\Leftrightarrow$ X_n is subnormal.*

Proof ($\Rightarrow$) $X_n = \int_K z^n \, d\eta(z)$. By Theorem 1.12 we have

$$(X_n, X_m) = \iint_{K \times K} z_1{}^n \bar{z}_2{}^m \, dF(z_1, z_2), \; F(\Delta_1 X \Delta_2) = (\eta(\Delta_1), \eta(\Delta_2)).$$

Because η is orthogonally scattered F is countably additive and is concentrated on K_1 = diagonal of $K \times K$ which can be identified with K. This induces a measure μ on K with $(X_n, X_m) = \int_K z^n \bar{z}^m \, d\mu(z)$, K is compact and μ has finite measure on K. Now use Theorem 1.7. ($\Leftarrow$) Since $(X_n, X_m) = \int_K z^n \bar{z}^m \, d\mu(z)$, by reversing the argument given in the proof of ($\Rightarrow$) we can write $(X_n, X_m) = \iint_{K \times K} z_1{}^n \, \bar{z}_2{}^m \, dF(z_1, z_2)$ and F is concentrated on the diagonal of $K \times K$. By Theorem 1.13 this shows that $X_n = \int_K z^n \, d\eta(z)$ and η is orthogonally scattered, that is X_n is a generalized harmonizable sequence and η is orthogonally scattered.

This theorem immediately gives the following corollary.

COROLLARY (1.22). (a) *Let $S = Z$. Then X_n is harmonizable with support of $\eta \subseteq \Gamma$ and η is orthogonally scattered $\Leftrightarrow$ X_n is stationary.*

(b) *Let $S = Z^+$. Then X_n is harmonizable with support $\eta \subseteq \Gamma$ and η is* *orthogonally scattered* $\Leftrightarrow X_n$ *is conservative* (*meaning* $(X_{n+k}, X_{m+k}) = (X_n, X_m)$ *for all* $n, m, k \in Z^+$, c.f. [15]).

The next Theorem is on normal dilation of generalized harmonizable sequences.

THEOREM (1.23). (Subnormal dilation) *Let* $S = Z$ *or* Z^+. *Then* X_n *is a generalized harmonizable sequence taking values in* $H \Leftrightarrow$ *there exists a Hilbert space* $\tilde{H} \supseteq H$ *and a subnormal sequence* Y_n: $S \to H$ *such that* $X_n = PY_n$, $n \in S$, *where* $P =$ *orthogonal projection of* $\tilde{H}$ *onto* H.

Proof. $(\Rightarrow)$ $X_n = \int_K x^n \, d\eta(z)$. By Rosenberg's work [19] Theorem 3.9, we get that η is 2-majorizable, i.e. There exists a nonnegative measure μ on Borel subsets of K such that $| \sum_{i=1}^n a_i \eta(\Delta_i) |^2 \le \sum_{i,j=1}^n a_i \bar{a}_j \mu(\Delta_i \cap \Delta_j)$ for all $n \ge 1$ and for all Borel subsets $\Delta_1, \ldots, \Delta_n$ and all scalars $a_1, \ldots, a_n$. By Theorem 2.9 of Rosenberg's work [19] we get that η has a quasi-isometric dilation η_1, i.e. $\eta(\Delta) = J^* \eta_1(\Delta)$, where η_1 is an H_1-valued measure such that $(\eta_1(\Delta_1), \eta_1(\Delta_2)) = \mu(\Delta_1 \wedge \Delta_2)$, and J: $H \to H_1$ is an isometry. Let $\tilde{H} = H \oplus (H_1 \ominus JH)$. Define the map U: $\tilde{H} \to H_1$ as follows:

For $X = X_1 + X_2 (X_1 \in H$ and $X_2 \in H_1 \ominus JH)$ put $UX = JX_1 \oplus X_2$. It is easy to see that U is a unitary map from $\tilde{H}$ into H_1. Define $\tilde{\eta}(\Delta) = U^{-1} \eta_1(\Delta)$ and let P be the orthogonal projection of $\tilde{H}$ onto H. Clearly $\tilde{\eta}$ is an $\tilde{H}$-valued measure. For $Y \in \tilde{H}$,

$$\begin{aligned}(P\tilde{\eta}(\Delta), Y)_{\tilde{H}} &= (\tilde{\eta}(\Delta), PY)_{\tilde{H}} = (U^{-1}\eta_1(\Delta), PY)_{\tilde{H}} \\ &= (\eta_1(\Delta), UPY)_{H_1} = (\eta_1(\Delta), JPY)_{H_1} = (J^*\eta_1(\Delta), PY)_H \\ &= (\eta(\Delta), PY)_H = (\eta(\Delta), PY)_{\tilde{H}} = (P\eta(\Delta), Y)_{\tilde{H}} = (\eta(\Delta), Y)_{\tilde{H}}.\end{aligned}$$

Consequently $\eta(\Delta) = P\tilde{\eta}(\Delta)$ and $(\tilde{\eta}(\Delta_1), \tilde{\eta}(\Delta_2))_{\tilde{H}} = (\eta_1(\Delta_1), \eta_2(\Delta_2))_{H_1} = \mu(\Delta_1 \cap \Delta_2)$. Obviously $\tilde{\eta}$ is orthogonally scattered. Define the process

$$Y_n = \int_K z^n \, d\tilde{\eta}(z).$$

Then by Theorem 1.21 we get that Y_n is subnormal. Notice that $PY_n = \int_K z^n \, dP\tilde{\eta}(z) = \int_K z^n \, d\eta = X_n$. This completes the proof of $(\Rightarrow)$. $(\Leftarrow)$ the proof of this part follows from the spectral representation of normal operators.

COROLLARY (1.24). (Normal dilation). *Let* $S = Z$. *Then* X_n *is generalized harmonizable* $\Leftrightarrow$ *there exists a Hilbert space* $\tilde{H} \supseteq H$ *and a normal sequence* Y_n: $S \to \tilde{H}$ (Y_n *need not be minimal*) *such that* $Y_n = PX_n$, $P =$ *orthogonal projection of* $\tilde{H}$ *onto* H.

Proof. Any subnormal process can be extended to a normal one. This gives the proof.

Note (1.25). It is recently shown by M. Rosenberg [19] that any countably additive Hilbert space valued measure on a σ-algebra of subsets of a set Ω has an orthogonally scattered dilation. This dilation result of Rosenberg is an extension of similar results for the case Ω is the real line or the unit circle or more generally a locally compact Hausdorff space, c.f. [14], [16], [5]. The dilation theorems for Hilbert space valued measures are based on the fact that any Hilbert space valued measure is 2-majorizable. The latter has its foundation in a profound lemma of Grothendieck [11] as demonstrated in the work of Rosenberg [19].

The starting point of interest in orthogonally scattered dilation of Hilbert space valued measure was the proof given by Abreu for stationary dilation of Hilbert space valued harmonizable processes with bounded spectral distribution [2]. The first step in Abreu's proof contains the construction of a 2-majorant for the spectral distribution without appealing to the Grothendieck lemma. This is accomplished because the spectral distribution is of bounded variation. Being unaware of the Grothendieck lemma, Miamee-Salehi obtained a 2-majorization result from several lemmas proved in their paper [14]. Because of the unboundedness of the variation of the spectral distribution they were not able to give a constructive method for the 2-majorant, but were only able to obtain an existence type result by appealing to the Hahn-Banach Theorem. Similarly the dilation result of Rosenberg [19] is of existence nature based on the Hahn-Banach Theorem.

Remark (1.26). If we additionally assume in Theorem 1.23 that the measure $F(\Delta_1 \times \Delta_2) = (\eta(\Delta_1), \eta(\Delta_2))$ is of bounded variation, then repeating, step by step, Abreu's proof [2] we can obtain the result of Theorem 1.23 where the 2-majorizing measure μ and the control measure $\tilde{\eta}$ is obtained in a constructed way. Namely if F is the spectral distribution of X_n, then the spectral distribution measure of Y_n is given by $\mu(dz) = (\|F\|(dz, B) + \|F\|(B, dz))/2$, where $\|F\|$ denotes the total variation of F.

Remark (1.27). Since each generalized harmonizable process X_n is of the form PY_n, where Y_n is subnormal and P is an orthogonal projection, we have that

$$\|X_n - \textstyle\sum_{k=1}^{s} a_k X_{n-k}\| \leq \|Y_n - \sum_{k=1}^{s} a_k Y_{n-k}\|$$

and hence

$$\inf \|X_n - \textstyle\sum_{k=1}^{s} a_k X_{n-k}\| \leq \inf \|Y_n - \sum_{k=1}^{s} a_k Y_{n-k}\|, \tag{1}$$

where the inf is over finite sequences of complex numbers $a_1, \cdots, a_s$. The right hand side of (1) can be written as

$$\inf \|Y_n - \textstyle\sum_{k=1}^{s} a_k Y_{n-k}\|^2 = \inf \int_K |z^n - \sum_{k=1}^{s} a_k z^{n-k}|^2 \, d\mu(z). \tag{2}$$

In case the right hand side of (2) is zero for $n = 0$ we conclude that Y_n and hence X_n is deterministic. For the case that X_n is harmonizable and F is of bounded variation μ can be taken to be $(\|F\|(dz, \Gamma) + \|F\|(\Gamma, dz))/2$, c.f.

Abreu [2], and for this case the celeberated Kolmogorov-Szégo Theorem gives a necessary and sufficient condition in terms of the log of the density of this measure. Similary when X_n is a generalized harmonizable sequence and of bounded variation the measure, μ is also given by the expression $(\|F\|(dz, K) + \|F\|(K, dz))/2$. However generally there is no nice condition such as log integrability of $\frac{dF_y}{dm}$, m = Lebesgue measure on K, for the determination of Y_n. In the following we discuss some cases where the log condition may or may not hold. The log formula is not true in some easy cases as is shown in the following example:

Example (1.28). Let K = the closed unit disc $= \{z \in C: |z| \leq 1\}$, and let m be the planar Lebesgue measure on K. Define a function w on K and a measure F as follows

$$w(z) = \begin{cases} 0 & \text{if } |z| < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq |z| \leq 1, \end{cases} dF = wdm$$

We can show

$$\inf_{P(0)=0} \int |1 - P(z)|^2 \, dF(z) \neq e^{\int \log w dm}.$$

Because the function w is zero on a set of positive measure, hence log w is not summable. Consequently the right side $e^{\int \log w dm} = 0$. On the other hand, the functions z^n, $n > 0$ form an orthogonal sequence in $L^2(F)$. In fact

$$(z^n, z^m)_{L_2(F)} = \int_{\bar{D}} \int z^n \bar{z}^m w(z) \, dm(z) = \iint_{1/2 \leq |z| \leq 1} z^n \bar{z}^m \, dm(z)$$

$$= \int_{1/2}^{1} \int_0^{2\pi} r^{n+m+1} e^{i(n-m)\theta} \, d\theta \, dr = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2n+2}\left[1 - \left(\frac{1}{2}\right)^{2n+2}\right] & \text{if } n = m. \end{cases}$$

Thus for any polynomial $P(z) - \sum_{k=1}^{n} a_k z^k$ we have

$$\int |1 - P(z)|^2 \, dF(z) = \|1 - P\|^2 = \|1 - \textstyle\sum_{k=1}^{n} a_k z^k\|^2$$
$$= \|1\|^2 + \textstyle\sum_{k=1}^{n} \|a_k z^k\|^2 \geq \|1\|^2 = \frac{3}{8} > 0.$$

This shows that the left side $\inf_{P(0)=0} \int |1 - P(z)|^2 \, dF(z) \geq \|1\|^2 > 0$. Consequently the left side is not equal to the right side.

A sufficient condition under which the log formula holds can be formulated in the function algebra language as follows {c.f. [4]}. Let A be a function algebra on X(X compact space) if m is the unique representing measure for a multiplicative functional, i.e. $\varphi(f) = \int_X f \, dm$, and if $\varphi(f) = \int_X f \, d\mu$ then $\mu =$

m, then

$$\inf_{f\in A_0} \int_X |1 - f|^2 w \, dm = \exp \int_X \log w \, dm$$

where $A_0 = \{f \in A: \varphi(f) = 0\}$.

Example below shows an application of the above condition.

Example (1.29). Let S be a compact set in $\mathbb{C}$ with connected complement and let $K = \partial S$ (topological boundary of S). Suppose $0 \in S$. Let $P(S)$ = the uniform closure of polynomials on the set S. Let the functional φ on $P(S)$ be defined as follows. $\varphi(f) = f(0)$, and let μ be a representing measure for φ on K. By Walsh's theorem [9] the algebra $P(S)$ is a Dirichlet algebra on $K = \partial S$. This implies that μ is unique [9]. If now F is any measure on K absolutely continuous with respect to μ then

$$\inf_{f(0)=0} \int_K |1 - f| \, dF = \exp \int_K \log \frac{dF}{d\mu} \, d\mu.$$

Actually the sufficient condition for the uniqueness of the representing measure which insures the validity of the log condition holds for a bigger class than the Dirichlet algebra, namely the logmodular algebra [9]. We make the observation that in our Example 1.28 the representing measure for the evaluation functional at zero is not unique. In fact in addition to the planar Lebesgue measure, the measure with weight 1 concentrated at zero also represents the same functional. With the notation of Example 1.29 we can give an example which behaves like Example 1.28 for which the representing measure is concentrated on the set $\{z \in \mathbb{C}: |z| = 1 \text{ or } \| z | = \frac{1}{2}\}$.

Example (1.30). Let $S = \{z: \frac{1}{2} \leq |z| \leq 1\}$ and $K = \partial S$. With the setting of Example 1.29 we let m be a representing measure for the evaluation functional at zero such that $m(|z| = \frac{1}{2}) \neq 0 \neq m(|z| = 1)$. Let $dF = w \, dm$, where $w = 0$ on $\{|z| = \frac{1}{2}\}$ and 1 on $\{|z| = 1\}$. As in Example 1.28 the log condition for w does not hold (of course the representing measure is not unique in this case).

Using the notion of "spectrum" we will show in §3 that with a Cesàro type approximation in formula (1) of Remark 1.27 the $\leq$ sign can be replaced by the $=$ sign. This is done only for a special class of generalized harmonizable sequences, namely those whose control measure μ is concentrated on a countable number of concentric circles. Formula (1) of Theorem 2.3 in conjunction with Theorem 1.7 establishes a bridge between a mean weighted approximation for a process and the approximation of a subnormal sequence with spectral measure related to the spectrum of the given process. This includes generalized harmonizable sequences whose spectral measures are concentrated on a countable number of concentric circles, particularly harmonizable processes. Therefore formula (1) of Theorem 2.3 extends the result of stationary approximation for harmonizable sequences and it can be viewed as a subnormal approximation for some classes of generalized harmonizable sequences.

§2. Spectrum

The question of stationary approximation of a harmonizable process is discussed by Rozanov [20] and recently by Abreu in a forthcoming article [3]. The crucial point of their analysis is the study of the concept of spectrum for harmonizable processes whose spectral distributions are of bounded variation. For harmonizable sequences the spectral measure η sits on $\Gamma = \{z: |z| = 1\}$ and the spectrum as defined by Rozanov is shown to be $F(d\lambda, d\lambda)$ which is concentrated on the diagonal $\lambda = \mu$ of $[0, 2\pi] \times [0, 2\pi]$ = the support of $F(d\lambda, d\mu) = (\eta(d\lambda), \eta(d\mu))$. For our generalized harmonizable sequence direct use of the Rozanov definition of spectrum, using Theorem 1.12, would entail the following:

(1) Given that the set $K \backslash \{|z| \leq 1\}$ has positive measure $F(d\lambda, d\mu)$ then the Rozanov spectrum may not exist (even when F is of bounded variation).

(2) If $K \subseteq \{z \mid |z| \leq 1\}$ with F being of bounded variation, the Rozanov spectrum would exist and may be positive only on the set $\Gamma = \{z: |z| = 1\}$

(3) *If* $K \subseteq \{z: |z| < 1\}$, then the Rozanov spectrum is zero everywhere.

So we need some modification in the definition of the Rozanov spectrum in order that it could differentiate processes with say support of $\mu \subseteq \{z: |z| < 1\}$. In this article this modification is done following the work of Abreu [3] for processes whose spectral measure η is concentrated on a countable number of concentric circles with centers at 0, since in general for generalized harmonizable processes new difficulties may arise. It is not clear to us how one would define a fruitful notion of spectrum for any generalized harmonizable sequence. This problem needs further study. With this introduction we now proceed to give our definition of spectrum.

Suppose we have a generalized harmonizable process $X(n) = \int_K z^n \, d\eta(z)$ where $K = \cup_{k=0}^{\infty} \Gamma_k$, Γ_k is a circle with center at 0 and radius r_k. Consider the integrals $\int_{\Gamma_k} z^n \, d\eta(z)$. If we define η_k as the restriction of η to Γ_k this integral can be written as $\int_{\Gamma_k} z^n \, d\eta_k(z)$. After change of variables we obtain $\int_{\Gamma_k} z^n \, d\eta(z) = \int_{\Gamma_k} z^n \, d\eta_k(z) = r_k^n \int_0^{2\pi} e^{in\theta} \, d\tilde{\eta}_k(\theta)$, where $\tilde{\eta}_k(\Delta) = \eta_k(r_k e^{i\Delta})$. Define $X_k(n) = \int_0^{2\pi} e^{in\theta} \, d\tilde{\eta}_k(\theta)$. For each k, the process $X_k(n)$ is harmonizable in the sense of Rozanov. It is also easy to see that $X(n) = \sum_{k=0}^{\infty} r_k^n X_k(n)$. This suggests that for any process X of the form $X(n) \sum_{k=0}^{\infty} r_k^n X_k(n)$ we may consider the function

$$B_\rho(p, q) = \sum_{k=0}^{\infty} r_k^p r_k^q \left[\lim_{N\to\infty} \frac{1}{\rho(N)} \sum_{n=-N}^{N} (X_k(n + p), X_k(n + q)) \right]$$

for $\quad n, p, q \in Z$

$$B_\rho(p, q) = \sum_{K=0}^{\infty} r_k^p r_k^q \left[\lim_{N\to\infty} \frac{1}{\rho(N)} \sum_{n=0}^{N} (X_k(n + p), X_k(n + q)) \right]$$

for $\quad n, p, q, \in Z^+$,

if the limit exists and is finite, where $\rho(N)$ is a positive nondecreasing function of N such that $\text{limit}_{N\to\infty} \dfrac{\rho(N+1)}{\rho(N)} = 1$. Of special interest to us are the two cases where $\rho(N) = 2N+1$ or $\rho(N) = 1$. The choice of $\rho(N) = 2N+1$ is due to Rozanov [20] and the usefulness of $\rho(N) = 1$ was recently observed by Abreu [3] since for square summable sequences the choice of $\rho(N) = 2N+1$ gives rise to zero spectrum and no useful conclusion can be drawn, whereas $\rho(N) = 1$ leads to a useful criterium of approximating a sequence by a moving average from its past.

Definition (2.1). If $B_\rho(p, q)$ is well-defined and if there exists a finite measure G_ρ on K such that

$$B_\rho(p, q) = \int_K z^p \bar{z}^q \, dG_\rho(z),$$

then we say that X has a ρ-spectrum and we call G_ρ its ρ-spectrum.

THEOREM (2.2) *Suppose $X(n) = \int_K z^n \, d\eta$, $n \in Z$ or Z^+, where $K = \cup_{K=0}^{\infty} \Gamma_K$ as described above. If $F(\Delta_1, \Delta_2) = (\eta(\Delta_1), \eta(\Delta_2))$ is of bounded variation then $B(p, q)$ corresponding to $\rho(N) = 2N+1$ is well-defined and $B(p, q) = \int_K z^p \bar{z}^q \, dG(z)$ where $G(\Delta) = F(\Delta, \Delta)$, i.e. X has a ρ-spectrum and its ρ-spectrum $=$ $F(\Delta, \Delta)$.*

Proof. An easy computation shows that

$$\frac{1}{2N+1} \sum_{n=-N}^{N} e^{in(\theta_1-\theta_2)} \quad \text{and} \quad \frac{1}{N+1} \sum_{n=0}^{N} e^{in(\theta_1-\theta_2)} \quad \text{tends to} \begin{cases} 1, & \theta_1 = \theta_2, \\ 0, & \theta_1 \neq \theta_2 \end{cases}$$

where the convergence is pointwise and boundedly. We complete the proof for the Z only, since the proof of Z^+ needs only a minor modification. Let $F_k(\Delta_1, \Delta_2)$ be the restriction of F to $\Gamma_k \times \Gamma_k$ and $\tilde{F}_k(\Delta_1, \Delta_2) = F_k(r_k e^{i\theta_1\Delta_1}, r_k e^{i\theta_2\Delta_2})$. F_k is a measure of bounded variation on $[0, 2] \times [0, 2]$. Consequently we get

$$r_k^p r_k^q \frac{1}{2N+1} \sum_{n=-N}^{N} (X_k(n+p), X_k(n+q))$$

$$= r_k^p r_k^q \frac{1}{2N+1} \sum_{n=-N}^{N} \int_0^{2\pi} \int_0^{2\pi} e^{i(n+p)\theta_1} e^{-i(n+q)\theta_2} \, d\tilde{F}_k(\theta_1, \theta_2)$$

$$= r_k^p r_k^q \int_0^{2\pi} \int_0^{2\pi} \frac{1}{2N+1} \sum_{n=-N}^{N} e^{in(\theta_1-\theta_2)} e^{ip\theta_1} e^{-iq\theta_2} \, d\tilde{F}_k(\theta_1, \theta_2)$$

which as $N \to \infty$ tends to

$$r_k^p r_k^q \int_0^{2\pi} e^{i(p-q)\theta} \, d\tilde{F}_k(\theta, \theta) = \int_0^{2\pi} r_k^p e^{ip\theta} r_k^q e^{-iq\theta} \, d\tilde{F}_k(\theta, \theta) = \int_{\Gamma_k} z^p \bar{z}^q \, dG_k(z),$$

where $d\tilde{F}_k(\theta, \theta)$ is the restriction of the measure $d\tilde{F}(\theta_1, \theta_2)$ to the diagonal. (We have used the fact that F is of bounded variation to apply the Lebesgue convergence theorem).

Now

$$B(p, q) = \sum_{k=0}^{\infty} \int_{\Gamma_k} z^p \bar{z}^q \, dG_k(z) = \int_k z^p \bar{z}^q \, dG(z),$$

which finishes the proof.

(We note that under the assumptions of this theorem $B(p, q)$ is a positive definite function).

THEOREM (2.3) *Let* $X_n = \sum_{k=0}^{\infty} r_k{}^n X_k(n)$, *and* (*a*) $n \in Z$ *or* (*b*) $n \in Z^+$. *Suppose the function* $B_\rho(p, q)$ *exists as defined earlier for* $\rho(N) = 2N + 1$. *Assume that* X *has a* ρ*-spectrum and that its* ρ*-spectrum* G *is concentrated on a compact set* $K = \bigcup_{k=0}^{\infty} \Gamma_{r_k}$, $\beta_\rho(p, q) = \int_K z^p \bar{z}^q \, dG$. *Then*

$$\inf \int_K |1 - \sum_{p=1}^{s} a_p z^p|^2 \, dG =$$

$$(1)\quad \begin{cases} \inf \sum_{k=0}^{\infty} \lim_{N\to\infty} \dfrac{1}{2N+1} \sum_{n=-N}^{N} \| X_k(n) - \sum_{p=1}^{s} a_p r_k{}^p X_k(n+p) \|^2 & \textit{in case } (a) \\ \inf \sum_{k=0}^{\infty} \lim_{N\to\infty} \dfrac{1}{N+1} \sum_{n=0}^{N} \| X_k(n) - \sum_{p=1}^{s} a_p r_k{}^p X_k(n+p) \|^2 & \textit{in case } (b) \end{cases}$$

where inf *is taken over the set of all finite sequences* $a_1, \cdots, a_s, s \in Z^+$. *Formula* (1) *gives prediction from the future values. For* Z *a similar formula for prediction from the past values can be obtained simply by changing* z^p *and* $X(n+p)$ *to* z^{-p} *and* $X(n-p)$ *respectively.*

Proof. We give the proof only for Z since the other one is very similar. We note that for any $a_1, \cdots, a_s, s \in Z^+$ we have the following chain of equalities:

$$\sum_{k=0}^{\infty} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \| X_k(n) - \sum_{p=1}^{s} a_p r_k{}^p X_k(n+p) \|^2$$

$$= \sum_{k=0}^{\infty} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n-=N}^{N} [(X_k(n), X_k(n))$$

$$- \sum_{p=1}^{s} \bar{a}_p r_k{}^p (X_k(n), X_k(n+p))$$

$$- \sum_{p=1}^{s} a_p r_k{}^p (X_k(n+p), r_k(n)$$

$$+ \sum_{p=1}^{s} \sum_{q=1}^{s} \bar{a}_p r_k{}^p a_q r_k{}^q (X_k(n+q), X_k(n+p))]$$

$= B(0, 0) - \sum_{p=1}^{\infty} \bar{a}_p B(0, p) - \sum_{p=1}^{s} a_p B(p, 0)$
$+ \sum_{p=1}^{s} \sum_{q=1}^{s} \bar{a}_p a_q B(q, p)$

$$= \int_K 1\, dG - \int_K \sum_{p=1}^{s} \bar{a}_p \bar{z}^p\, dG - \int_K \sum_{p=1}^{s} a_p z^p\, dG$$

$$+ \int_K \sum_{p=1}^{s} \sum_{q=1}^{s} \bar{a}_p a^q z^q \bar{z}^p\, dG$$

$= \int_K |1 - \sum_{p=1}^{s} a_p z^p|^2\, dG$. Taking inf on both sides completes the proof.

Combining Theorems 2.2 and 2.3 we have

THEOREM (2.4). *Suppose $X_n = \int_K z^n\, d\eta$, $n \in Z$ or Z^+, where $K = \bigcup_{k=0}^{\infty} \Gamma_{r_k}$, Γ_{r_k} is the circle of radius r_k with center at 0. (X_n is generalized harmonizable). Then the approximation formula* (1) *in Theorem* 2.3 *holds.*

§3. Weighted square summable sequences.

In the paper [17] Niemi obtained a stationary dilation for square summable processes. We can obtain subnormal dilation of Niemi type for more general class of processes. We have the following theorem.

THEOREM (3.1). *Let $S = Z$ or Z^+ and let $X: S \to H$ be a minimal H-valued sequence such that for some r, $r > 0$, $\sum_{n \in S} r^{2n} \| X_n \|^2 < \infty$. Then there exists a separable Hilbert space $K \supseteq H$ and a K-valued subnormal process Y_n, $n \in S$ such that*

(1) *$X_n = PY_n$, $n \in S$; where P is the orthogonal projection of K into H and*

(2) *$Y_n \cong \int_{\Gamma_r} z^n\, dE(z)h$, $(Y_n, Y_m) = \int_{\Gamma_r} z^n \bar{z}^m |h(z)|^2\, dm(z)$; where $H(z) = \sum_n \bar{z}^n X_n$* in *$L^2(H, m)$, m = Lebesgue measure and $\cong$ indicates unitary equivalence, $E(\Delta)$ = multiplication by χ_Δ in $L^2(H, m)$.*

Proof. Consider the space $\tilde{K} = L^2(H, \Gamma_r, m)$, where Γ_r is the circle with radius r and center at 0, and m is the Lebesgue measure on Γ_r. One can see that the series $\sum_{n \in S} \bar{n}^n X_n$ converges in $\tilde{K}$. Let $h(z) = \sum_{n \in S} \bar{z}^n X_n$. Define the $\tilde{K}$-valued processes $\tilde{Y}_n$ and $\tilde{X}_n$ by the formulas $\tilde{Y}_n(z) = z^n h(z)$ and $\tilde{X}_n(z) = X_n$.

It is easy to see that $\tilde{P}\tilde{Y}_n = \tilde{X}_n$, where $\tilde{P}$ is the orthogonal projection of $\tilde{K}$ onto $\tilde{H}$ = the subspace spanned by $\tilde{X}_n$, $n \in S$, in $\tilde{K}$. Because $(\tilde{Y}_n, \tilde{Y}_m)_{\tilde{K}} = \int_{\Gamma_r} z^n \bar{z}^m |h(z)|^2\, dm(z)$, by Theorem 1.7 we conclude that $\tilde{Y}_n$ is a subnormal sequence. Identifying $\tilde{H}$ with H the proof is finished. A more detailed proof is as follows. Define the map $\tilde{\Phi}: \tilde{H} \to H$ by formula $\tilde{\Phi}(\sum \alpha_n \tilde{X}_n) = \sum \alpha_n X_n$, the sum is finite. As in the proof of Theorem 1.7. This $\tilde{\Phi}$ has a unique unitary extension on $\tilde{H}$ onto H. Let us call this extension Φ. Similar to the proof of Theorem 1.23, let $K = H \otimes (\tilde{K} \ominus \tilde{H})$ and $U: \tilde{K} \to K$ be given by the formula $U(X_1 + X_2) = \Phi X_1 + X_2$, $X_1 \in H$ and $X_2 \in \tilde{K} \ominus \tilde{H}$. U is unitary. Now define

$Y_n = U\tilde{Y}_n$. Then Y_n is a subnormal sequence taking values in K and $PU = U\tilde{P}$, where P is the orthogonal projection of K onto H. Consequently $PY_n = PU\tilde{Y}_n = U\tilde{P}\tilde{Y}_n = U\tilde{X}_n = U^{-1}\Phi^{-1}X_n = \Phi\Phi^{-1}X_n = X_n$ which finishes the proof.

Using Niemi's method it is possible to obtain a dilation for processes of the form $X_n = \int_K z^n\psi(z)\, d\mu(z)$, where μ is an arbitrary σ-finite measure and ψ is a Hilbert space valued function such that $\int_k (\psi(z), \psi(z))\, d\mu(z) < \infty$. Without any loss of generality we may assume that μ is normalized so that $\mu(K) = 1$.

THEOREM (3.2). *Suppose μ is a normalized measure on the compact set $B \subseteq$ **C** and suppose ψ: $B \to H$, a Hilbert space, is a function such that $\int_B (\psi(z), \psi(z))\, d\mu(z) < \infty$. For the process $X_n = \int_B z^n\psi(z)\, d\mu(z)$, $n \in S$ ($S = Z$ or Z^+) we have that there exists a separable Hilbert space $K \supseteq H$ and a subnormal process Y_n: $S \to K$ such that*

(1) $PY_n = X_n$, $n \in S$, where P is the orthogonal projection of K onto H and

(2) $Y_n \cong \int_{\Gamma_r} z^n\, dE(z)h$, $(Y_n, Y_m) = \int_{\Gamma_r} z^n\bar{z}^m |h(z)|^2\, d\mu(z)$, where $h(z) = \sum_n \bar{z}^n X_n$ in $L^2(H, \mu)$ and $\cong$ indicates unitary equivalence, $E(\Delta)$ = multiplication by χ_Δ in $L^2(H, \mu)$.

Proof. Without any loss of generality we may assume that X_n is minimal. Let $\tilde{K} = L^2(H, \mu)$. Define $\tilde{Y}$: $S \to \tilde{K}$ as follows:

$$\tilde{Y}_n(z) = z^n\psi(z),\ z \in B.$$

By the assumption of square summability of ψ, $\tilde{Y}_n$ is well defined. It is easy to see that $(\tilde{Y}_n, \tilde{Y}_m)_{\tilde{K}} = \int_B z^n\bar{z}^m \|\psi(z)\|^2\, d\mu(z)$. By Theorem 1.7, $\tilde{Y}_n$ is subnormal. Define now $\tilde{X}$: $S \to \tilde{K}$ by the formula $\tilde{X}_n(z) = X_n$, $n \in S$, and let $\tilde{H} = V_{n\in S}\tilde{X}_n$. By minimality of X_n we get that $\tilde{H}$ is the space of all constant functions in $\tilde{K}$. This implies that $\tilde{P}\tilde{Y}_n = \tilde{X}_n$, because $(\tilde{Y}_n, \tilde{X}_K)_{\tilde{K}} = (z^n\psi(z), \int_B z^k\psi(z)\, d\mu(z))_{\tilde{K}}$ $= \int_B (z_1{}^n\psi(z_1), \int_B z_2{}^k\psi(z_2))_H\, d\mu(z_1) = \int\int_{B\times B} z_1{}^n\bar{z}_2{}^k(\psi(z_1), \psi(z_2))\, d\mu(z_1)\, d\mu(z_2)$ $= (\int_B z^n\psi(z)\, d\mu(z), \int_B z^k\psi(z)\, d\mu(z))_H = (X_n, X_k)_H$. But also $(\tilde{X}_n, \tilde{X}_k)_{\tilde{K}} = \int_B (X_n, X_k)_H\, d\mu = (X_n, X_k)_H \int_B d\mu = (X_n, X_k)_H$, because $\mu(B) = 1$.

This implies that $(\tilde{Y}_n - \tilde{X}_n, \tilde{X}_k)_{\tilde{K}} = 0$ for all $k \in S$, and hence $\tilde{X}_n = \tilde{P}\tilde{Y}_n$. Define the map $\tilde{\Phi}(\sum_2 \alpha_i\tilde{X}_i) = \sum_i \alpha_i X_i$(sum over finite sets). As in the last theorem this $\tilde{\Phi}$ has a unique unitary extension Φ on $\tilde{H}$ onto H. Define the unitary map U from $\tilde{\mathbb{K}}$ onto $K = H \oplus (\tilde{K} \ominus \tilde{H})$ as follows $U(X_1 + X_2) = \Phi(X_1) + X_2$; $X_1 \in \tilde{H}$ and $X_2 \in \tilde{K} \ominus \tilde{H}$. Exactly the same proof as in last theorem shows that $U\tilde{P} = PU$, P = orthogonal projection of K onto H. Consequently if we define $Y_n = U\tilde{Y}_n$ we get that $PY_n = PU\tilde{Y}_n = U\tilde{P}\tilde{Y}_n = U\tilde{X}_n = U\Phi^{-1}X_n = \Phi\Phi^{-1}X_n = X_n$. Evidently Y_n is subnormal. This finishes the proof.

Remark (3.3). As we pointed out earlier in Remark 1.27, in view of Remark 1.26, we can write

$$\inf \| X_n - \textstyle\sum_{k=1}^{s} a_k X_{n-k} \| \le \inf \int_B |z^n - \sum_{k=1}^{s} a_k z^{n-k}|^2\, d\mu(z),$$

with $\mu(dz) = (\|F\|(dz, B) + \|F\|(B, dz))/2$ and $dF(z_1, z_2) =$

$(\psi(z_1), \psi(z_2))_H \, d\mu(z_1) \, d\mu(z_2)$. Because $\int_B (\psi(z), \psi(z) \, d\mu(z) < \infty$, $\mu(B) = 1$, it follows that $dF(z_1, z_2)$ is of bounded variation and $d\|F\|(z_1, z_2) = |(\psi(z_1), \psi(z_2)_H| \, d\mu(z_1) \, d\mu(z_2)$. As before, a condition for determinism of Y_n, and hence for X_n, in terms of the density of μ, is hard to obtain.

Applying the above discussion to Theorem 3.1 we can see that again $\mu(dz)$ is given by the same formula with $d\mu$ replaced by the Lebesgue measure and $\psi(z) = \sum_n \bar{x}^n X_n$, $z \in \Gamma_r$. For this case, by application of Example 1.2a we obtain the log condition in terms of $|\psi(z)|$, for the determinism of Y_n and hence that of X_n. Actually Theorem 3.4 gives a slightly stronger result of this type.

Using the same technique as in Theorems 3.1 and 3.2 we can obtain a weighted approximation for extrapolation similar to our Theorem 2.3 With $\rho = 2N + 1$ and $B(p, q)$ defined as before, Theorem 2.3 can be applied. However, in view of the condition $\sum_n r^{2n} \| X_n \|^2 < \infty$, the ρ-spectrum G is the zero measure, given no new information. For this case as was observed by Abreu the right choice for ρ is $\rho \equiv 1$. When we do not deal with the unit circle this choice of $\rho \equiv 1$ necessitates a slight change in the definition of the function B which gives the spectrum. Of course for the unit circle both definitions are the same and are consistent with Abreu's definition of ρ-spectrum.

THEOREM (3.4). *Let $S = Z$ and $X: S \to H$. Assume there exists $r > 0$ such that $\sum_{n \in S} r^{2n} \| X_n \|^2 < \infty$. Define*

$$(1) \qquad b(k) = r^{2k} \sum_{n \in S} r^{2n}(X_n, X_{n+k}), \; k \in S.$$

$$h(z) = \sum_{n \in S} \bar{z}^n X_n.$$

Then we have:

$$(2) \quad b(k) = \int_{\Gamma_r} \bar{z}^k |h(z)|^2 \, dm(z), \quad |h(z)|^2 \, dm(z) \text{ is the } \rho\text{-spectrum for } \rho \equiv 1.$$

$$(3) \quad \sum_{n \in S} r^{2n} \| X_n - \sum_{k=1}^{\infty} a_k X_{n-k} \|^2 = \int_{\Gamma_r} |1 - \sum_{k=1}^{\infty} a_k \bar{z}^k|^2 \, \| h(z) \|^2 \, dm(z),$$

where a_k, $k \geq 1$ is a bounded sequence of scalars.

$$(4) \quad \inf_{a_1, \ldots, a_s} \sum_{n \in S} r^{2n} \| X_n - \sum_{k=1}^{s} a_k X_{n-k} \|^2 = \inf_{a_1, \ldots, a_s} \int_{\Gamma_r} |1 - \sum_{k=1}^{s} a_k \bar{z}^k|^2 \, \| h(z) \|^2 \, dm(z).$$

hence X_n is deterministic $\Leftrightarrow$ right hand of (4) is zero, that is 1 is in the subspace spanned by $\sum_{k=1}^{s} a_k \bar{z}^k$ in the Hilbert space of square integrable functions with respect to the measure $\| h(z) \|^2 \, dm(z)$ on Γ_r.

Proof. Without any loss of generality we assume m is the normalized Lebesgue measure on Γ_r. We observe that

$$\int_{\Gamma_r} z^n h(z) \, dm(z) = \sum_{k \in S} \int_{\Gamma_r} z^n \bar{z}^k X_k \, dm(z) = \sum_{k \in S} X_k \frac{1}{2\pi r} \int_0^2 r^{n+k+1} e^{i(n-k)m}$$

$$= \sum_{k \in S} X_k r^{n+k} \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-k)\theta} \, d\theta = r^{2n} X_n.$$

Now

$$(X_{n+p}, X_{n+k}) = (X_{n+p}, r^{-2(n+k)} \int_{\Gamma_r} z^{n+k} h(z)\, dm(z))$$

$$= r^{-2(n+k)} \int_{\Gamma_r} \bar{z}^{n+k}(X_{n+p}, h(z))\, dm(z)$$

$$= r^{-2(n+k+p)} \int_{\Gamma_r} \bar{z}^k z^P (z^{n+p} X_{n+p}, h(z))\, dm(z).$$

Consequently

$$\sum_{n\in S} r^{2n}(X_n + p, X_{n+k}) = r^{-2(k+p)} \sum_{n\in S} \int_{\Gamma_r} \bar{x}^k z^p (z^{n+p}, X_{n+p}, h(z))\, dm(z)$$

$$(5) \qquad = r^{-2(k+p)} \int_{\Gamma_r} \bar{z}^k z^p (\textstyle\sum_{n\in S} z^{n+p} X_{n+p}, h(z))\, dm(z)$$

$$= r^{-2(k+p)} \int \bar{z}^k z^p \parallel h(z) \parallel^2 dm(z).$$

Therefore

$$b(k) = r^{2k} \sum_{n\in S} r^{2n}(X_n, X_{n+k}) = \int_{\Gamma_r} \bar{z}^k \parallel h(z) \parallel^2 dm(z),$$

giving (2).

Again, calculations similar to the ones used in the proof of (2), yield

$$\sum_{n\in S} r^{2n} \parallel X_n - \sum_{k=1}^{\infty} a_k X_{n-k} \parallel^2 = \int_{\Gamma_r} | 1 - \sum_{k=1}^{\infty} a_k \bar{z}^k |^2 \parallel h(z) \parallel^2 dm(z),$$

completing the proof of (3).

(4) follows from (3).

Remark (3.5). (a) Our Theorem 3.4 could have been deduced from the corresponding result of Abreu [3] upon transforming Γ_r onto Γ and keeping track of the arithmetics involved. A direct approach as given here gives a better insight to the problem.

(b) In case $\parallel X_n \parallel = 0$ for $n \leq -1$ in Theorem 3.4, then of course (2)–(4) hold and the prediction becomes a finite segment prediction.

(c) Comparing relation (2) of Theorem 3.1 and relation (2) of Theorem 3.4 one can see that the spectral distribution of the subnormal dilation in Theorem 3.1 and the spectrum of X_n are identical. We also note that the log condition on $| h(z) |^2$ guarantees the determinism of X_n through formula (3). We also point out that Theorem 3.4 may be viewed as a subnormal approximation to a weighted square summable sequence in the same spirit as of stationary approximation of square summable sequences, c.f. [3].

MICHIGAN STATE UNIVERSITY, EAST LANSING, MI., U.S.A.
JAGIELLONIAN UNIVERSITY, CRACOW, POLAND.

REFERENCES

[1] M. B. ABRAHAMSE, *Commuting subnormal operators.* Illinois J. Math. **22** (1978), 171–176.
[2] J. L. ABREU, *A note on harmonizable and stationary sequences.* Bol. Soc. Mat. Mexicana **15**

(1970), 48–51.

[3] J. L. ABREU (1983). *Stationary prediction of nonstationary sequences in Hilbert space.* Prediction theory and harmonic analysis, the Pesi Masani volume 1–12, edited by V. Mandrekar and H. Salehi, North-Holland, Amsterdam.

[4] A. BROWDER (1969). Introduction to function algebras. W. A. Benjamin, New York.

[5] S. D. CHATTERJI (1982). *Orthogonally scattered dilations of Hilbert space valued set functions.* Lecture Notes in Math., **945**, Measure theory 270–281.

[6] H. CRAMÉR, *On some classes of non-stationary stochastic processes.* Proc. IV Berkeley Symp. on Math. Stat. Prob. **2** (1962), 57–78.

[7] J. DISTEL AND J. J. UHL (1977). *Vector measures.* Mathematical Surveys 15, Amer. Math. Soc., Providence, R.I.

[8] N. DUNFORD AND J. T. SCHWARTZ (1953). Linear operators I, Wiley, New York.

[9] T. W. GAMELIN (1969). Uniform algebras, Prentice-Hall, Inc., Englewood Cliffs, N.J.

[10] R. K. GETOOR, *The shift operator for non-stationary processes.* Duke Math. J. **23** (1956), 175–187.

[11] A. GROTHENDIECK, *Résumé de la théorie métrique de produits tensoriels topologiques.* Bol. Soc. Mat. Sao Paulo **8** (1956), 1–79.

[12] T. ITO, *On the commutative family of subnormal operators.* J. Fac. Sci. Hokkaido Univ. Series 1, **14.1** (1958), 1–15.

[13] P. MASANI, Dilations as propagators of Hilbertian varieties. SIAM J. Math. Anal. **9** (1978), 414–456.

[14] A. G. MIAMEE AND H. SALEHI, Harmonizability. V-boundedness and stationary dilation of stochastic processes. Indiana Univ. Math. J. **27** (1978), 37–50.

[15] W. MLAK (1977). *Dissipative random processes.* Lecture Notes in Math. **656,** Probability theory on vector spaces, 139–169.

[16] H. NIEMI, *On orthogonally scattered dilation of bounded vector measures.* Ann. Acad. Sci. Fenn. Ser. A. I **3** (1977), 43–52.

[17] H. NIEMI, *A class of deterministic non-stationary sequences.* J. London Math Soc. (2) **17** (1978), 187–192.

[18] H. NIEMI (1982). *Diagonal measure of a positive definite bimeasure.* Lecture Notes in Math. **945,** Measure theory, 237–246.

[19] M. ROSENBERG, *Quasi-isometric dilation of operator-valued measures and Grothendieck's inequality.* Pacific J. Math., **103** (1982), 135–161.

[20] YU. A. ROZANOV, *Spectral analysis of abstract functions.* Theory Probab. Appl. **4** (1959), 271–287.

[21] M. SLOCINSKI, *Normal extensions of commutative subnormal operators.* Studia Math. **54** (1976), 259–266.