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THE SHIFT OPERATOR OF A NON-STATIONARY SEQUENCE IN HILBERT SPACE*

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1. Introduction

In this work we investigate conditions under which a sequence in Hilbert space has a shift operator and study the spectral structure of such shift operators. In section 2 we find conditions on a strongly harmonizable sequence for it to have a shift operator (see M. M. Rao [7] for definition and properties of weakly and strongly harmonizable sequences). In section 3 we show that a strongly harmonizable sequence is uniformly bounded linearly stationary (u.b.l.s.) if and only if it has a shift operator which is similar to a unitary operator (see Niemi [6] and Tjøstheim & Thomas [8] for definition and properties of u.b.l.s. processes). In section 4 we state the linear prediction problem for a harmonizable sequence having a normal shift operator. This problem seems to be very difficult except for the particular case when the shift operator is self adjoint. This leads us to investigate how common it is for a shift operator to be normal. Thus in section 5 we show that a strongly harmonizable sequence with a normal shift operator is necessarily stationary and in section 6 we obtain the same result for a discrete strict sense process having a shift operator. These results show that it is unnatural to assume that the shift operator of a non-stationary sequence is normal and suggest that in general shift operators may be *spectral operators of scalar type* (as defined in Dunford & Schwartz [4]) with spectral measure concentrated on the unit circle. This however is only a conjecture.

2. Dilations and shifts of harmonizable sequences

Let $\{x_n\}$ be a sequence in a Hilbert space H with inner product $(\ ,\ )$ and norm $\|\ \|$, where n ranges through the set Z of all integers. $\{x_n\}$ is strongly harmonizable if for some complex valued regular Borel measure μ on T^2 (where T denotes the unit interval [0, 1] with its end points identified with each other),

$$(x_m, x_n) = \iint_{T^2} e^{2\pi i(ms-nt)}\mu(ds, dt)$$

for every m, n, in Z. μ will be called the covariance spectral measure of $\{x_n\}$. The sequence $\{x_n\}$ is stationary if and only if it is strongly harmonizable and its covariance spectral measure is concentrated on the diagonal $\Delta = \{(s, t) \in T^2 : s = t\}$ of T^2. We say that $\{x_n\}$ has a shift operator group $\{S^k\}$, where k ranges over Z, if there exists an invertible linear operator S on $l.s.\{x_n\}$ (the

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linear subspace of H generated by $\{x_n\}$) such that $Sx_n = x_{n+1}$ for every integer n. Such operator S is called the shift operator of $\{x_n\}$. It is well known (see for example Doob [3]) that a stationary sequence always has, a shift operator which is actually unitary. If $\{x_n\}$ is stationary with spectral covariance measure μ then the measure ν defined on the Borel sets of T by $\nu(A) = \mu(A \times T)$ is positive and

$$(x_m, x_n) = \int_T e^{2\pi i(m-n)s} \nu(ds)$$

for every m, n in Z. ν will be called the spectral measure of the stationary sequence $\{x_n\}$.

THEOREM (2.1). *Let $\{x_n\} \subset H$ be a strongly harmonizable sequence with covariance spectral measure μ. Suppose $\{x_n\}$ has a shift operator group $\{S^k\}$. Then there exists a stationary sequence $\{y_n\}$ in some Hilbert space H', with spectral measure ν, and a linear map $D: l.s.\{x_n\} \to l.s.\{y_n\}$ with an inverse D^{-1} which is actually a contraction, such that:*
(i) *$Dx_n = y_n$ for every n in Z,*
(ii) *for every $f \in L^2(\nu)$, $f \otimes \bar{f}$ is integrable with respect to μ and*

$$\int_{T^2} f \otimes \bar{f}\, d\mu \leq \int_T |f|^2\, d\nu,$$

(iii) *if for some trigonometric polynomial p,*

$$\int_{T^2} p \otimes \bar{p}\, d\mu = 0 \quad \textit{then} \quad \int_T |p|^2\, d\nu = 0.$$

Proof. Define $\nu(A) = |\mu|(A \times T)$ for every Borel subset A of T, where $|\mu|$ denotes the total variation of μ. Then, as shown in Cramér [2] and Abreu [1], for every f in $L^2(\nu)$, $f \otimes \bar{f}$ is integrable with respect to μ and

$$\int_{T^2} f \otimes \bar{f}\, d\mu \leq \int_T |f|^2\, d\nu.$$

This establishes (ii). Since the shift operator S exists, it follows that whenever $\Sigma a_n x_n = 0$ (finite sum) then also $\Sigma a_n x_{n+k} = 0$ for every k in Z and hence also $(\Sigma a_n x_{n+k}, x_m) = 0$ for every k, m in Z. Let $\chi^n(s) = e^{2\pi ins}$ for n in Z and s in T. Then, using the properties of μ we see that whenever $p = \Sigma a_n \chi^n$ is a trigonometric polynomial such that $\int_{T^2} p \otimes \bar{p}\, d\mu = 0$ then

$$\int_{T^2} \Sigma a_n \chi^{n+k} \otimes \chi^{-m}\, d\mu = 0$$

for every k, m in Z, i.e.

$$\int_{T^2} (\chi^k \otimes \chi^{-m})(\Sigma a_n \chi^n \otimes 1)\, d\mu = 0$$

for every k, m in Z. This implies that $\Sigma a_n \chi^n \otimes 1\, d\mu = 0$ as a measure on T^2 and so its total variation $|\Sigma a_n \chi^n| \otimes 1\, d|\mu| = 0$. Integrating over a set of the form $A \times T$ we obtain

$$\int_A |\Sigma a_n \chi^n|\, d\nu = 0$$

and this is true for every Borel subset A of T. Thus $\Sigma a_n \chi^n = 0$ a.e. with respect to ν and therefore

$$\int_T |p|^2 \, d\nu = \int_T |\Sigma a_n \chi^n|^2 \, d\nu = 0.$$

This proves (iii). Finally define $y_n = \chi^n$ in $L^2(\nu)$. Then, whenever $\Sigma a_n x_n = 0$ then $\Sigma a_n y_n = 0$ also, according to what we just proved above. Therefore we can define $D: l.s.\{x_n\} \to l.s.\{y_n\}$ by $D(\Sigma a_n x_n) = \Sigma a_n y_n$. It is clear then that $\{y_n\}$ is a stationary sequence in $H' = L^2(\nu)$ with spectral measure ν. Since $\| \Sigma a_n x_n \|^2 \le \int_T |\Sigma a_n \chi^n|^2 \, d\nu = \| \Sigma a_n y_n \|^2$, it follows that whenever $Dx = 0$, $x = 0$ and hence $D^{-1}: l.s.\{y_n\} \to l.s.\{x_n\}$ exists and it is a contraction, i.e., $\| D^{-1} y \| \le \| y \|$ for every y in $l.s.\{y_n\}$. This completes the proof of Theorem 2.1.

The next two theorems are inverses to (i) of Theorem 2.1 and to (ii) and (iii) of Theorem 2.1, respectively.

THEOREM (2.2). *If $\{x_n\}$ is a sequence in H and if there exists a linear one to one operator $D: l.s.\{x_n\} \to H'$ for some Hilbert space H' such that $\{Dx_n\}$ is stationary, then $\{x_n\}$ possesses a shift operator group.*

Proof. If $\Sigma a_n x_n = 0$ then $\Sigma a_n D x_n = 0$, and since $\{Dx_n\}$ is stationary it follows that $\Sigma a_n D x_{n+k} = 0$ for every k in Z; since D is one to one we conclude that $\Sigma a_n x_{n+k} = 0$ for every k in Z. Thus $\{x_n\}$ has a shift operator group. Q.E.D.

THEOREM (2.3). *Let $\{x_n\}$ be a strongly harmonizable sequence with covariance spectral measure μ and suppose there exists a positive measure ν with the following properties:*

(a) *For every f in $L^2(\nu)$, $f \otimes \bar{f}$ is integrable with respect to μ and*

$$\int_{T^2} f \otimes \bar{f} \, d\mu \le \int_T |f|^2 \, d\nu.$$

(b) *If p is a trigonometric polynomial such that $\int_{T^2} p \otimes \bar{p} \, d\mu = 0$ then $\int_T |p|^2 \, d\nu = 0$.*

Then $\{x_n\}$ has a shift operator group.

Proof. Suppose $\Sigma a_n x_n = 0$. Then $\int_{T^2} \Sigma a_n \chi^n \otimes \overline{\Sigma a_n \chi^n} \, d\mu = 0$ and because of (b) we have also $\int_T |\Sigma a_n \chi^n|^2 \, d\nu = 0$. Hence $\int_T |\Sigma a_n \chi^{n+k}|^2 \, d\nu = 0$ and because of (a) we conclude that $\Sigma a_n x_{n+k} = 0$ for every k in Z. Thus the shift operator group exists. Q.E.D.

Combining theorems 2.1, 2.2 and 2.3 we can summarize the results of this section for strongly harmonizable sequences as follows.

THEOREM (2.4). *Let $\{x_n\}$ be a strongly harmonizable sequence in a Hilbert space H, with covariance spectral measure μ. Then the three statements below are equivalent.*

(1) *$\{x_n\}$ has a shift operator group.*

(2) *There exists a positive Borel measure ν on T such that*
(a) *for every f in $L^2(\nu)$, $f \otimes \bar{f}$ is integrable with respect to μ and*

$$\int_{T^2} f \otimes \bar{f}\, d\mu \leq \int_T |f|^2\, d\nu$$

(b) *if p is a trigonometric polynomial such that*

$$\int_{T^2} p \otimes \bar{p}\, d\mu = 0 \quad \text{then} \quad \int_T |p|^2\, d\nu = 0.$$

(3) *There exists a Hilbert space H' and a linear one to one map $D: l.s.\{x_n\} \to H'$ such that $\{Dx_n\}$ is stationary.*

Definition (2.5). The linear map of 2.4 (3) is called a stationary dilation transformation of $\{x_n\}$. When D is such that the spectral measure of $\{Dx_n\}$ is given by $\nu(A) = |\mu|(A \times T)$ then D is called a canonical stationary dilation transformation of $\{x_n\}$.

Remark (2.6). From the proof of Theorem 2.1 it follows that whenever a strongly harmonizable sequence $\{x_n\}$ possesses a stationary dilation transformation then it also possesses a canonical stationary dilation transformation. Notice that the equivalence between (1) and (3) of Theorem 2.4 means that a harmonizable sequence has a shift operator group if and only if it possesses a stationary dilation transformation.

3. Spectral representation for shifts of harmonizable sequences

In this section we establish some spectral properties of the shift operators of strongly harmonizable sequences based on the results of section 2.

PROPOSITION (3.1). *Let $\{x_n\}$ be a strongly harmonizable sequence in H with a shift operator group $\{S^k\}$. Let $H(x)$ denote the closure in H of $l.s.\{x_n\}$. There exists a unitary operator group $\{U^k\}$ on some Hilbert space H' and a linear transformation $D: H(x) \to H'$ densely defined and with a bounded inverse D^{-1} which is actually a contraction, such that $S^k = D^{-1}U^kD$ on $l.s.\{x_n\}$ for every k in Z.*

Proof. Let D be a canonical stationary dilation transformation of $\{x_n\}$ (see 2.5) and let $\{U^k\}$ be the shift operator group of $\{Dx_n\}$.

PROPOSITION (3.2). *Let $\{x_n\}$ be a strongly harmonizable sequence in H with a bounded stationary dilation transformation D. Then $\{x_n\}$ has a bounded shift operator group $\{S^k\}$ and*

$$S^k = \int_{T^2} e^{2\pi iks} F(ds)$$

on $l.s.\{x_n\}$ for every k in Z, where F is a projection valued set function defined on the Borel subsets of T, which is countably additive in the strong operator topology and $F(A \cap B) = F(A)F(B)$ for every pair A, B of Borel subsets of T.

Proof. Let E be the resolution of the identity for the unitary shift U of $\{Dx_n\}$ and define $F(A) = D^{-1}E(A)D$ for every Borel subset A of T.

Notice that the values of F in Proposition 3.2 are projection operators on H but they are not in general orthogonal projections. In the language of Dunford and Schwartz [4] we can say that $\{S^k\}$ is a group of spectral operators of scalar type with its spectrum contained in the unit circle of the complex plane. From propositions 3.1 and 3.2 we can easily deduce the following result which is already well-known (see for example [8]).

COROLLARY (3.3). *A strongly harmonizable sequence* $\{x_n\}$ *is* u.b.l.s. *if and only if it has a shift operator group* $\{S^k\}$ *such that* $S^k = D^{-1}U^kD$ *for some unitary operator* U *and some operator* D *such that both* D *and* D^{-1} *are bounded.*

4. Prediction theory for sequences with normal shift operators

Getoor [5] studied continuous time domain processes with normal shift operator groups. He did not, however, treat the prediction problem. In this section we will state the prediction problem for sequences having a normal shift operator group.

Suppose $\{x_n\}$ is a sequence in H with a shift operator group $\{S^k\}$ such that S is a normal operator. Let E be the resolution of the identity for S. Then

$$x_k = S^k x_0 = \int_C z^k E(dz) x_0$$

for every integer k. Let $m(A) = \| E(A)x_0 \|^2$ for every Borel subset A of C. Then m is a positive countably additive Borel measure on C and

$$(x_k, x_n) = (S^k x_0, S^n x_0) = \int_C z^k \overline{z^n} m(dz)$$

for every k, n in Z.

The prediction problem for $\{x_n\}$ consists in estimating

$$\sigma_{n,k}{}^2 = \inf \| x_{n+k} - \sum_j a_j x_{n-j} \|^2 = \inf \int_C | z^k - \sum_j a_j z^{-j} |^2 | z^n |^2 \, m(dz)$$

where the inf is taken over all sequences $\{a_j \in C : j \geq 0\}$ with only finitely many non-zero terms. This is in general a very difficult problem which as far as we know is still unsolved. One particular case that can be easily solved is the following. Suppose S is self adjoint and both S and S^{-1} are bounded. Then the support of m is bounded, bounded away from zero and contained in the real line. The functions of the form

$$\sum_{j=0}^{N} a_j x^{-j} \qquad \text{with } N \geq 0$$

defined on the support K of m, constitute an algebra of continuous functions that contains constants and separates points. Therefore from the Stone Weierstrass theorem it follows that $\sigma_{n,k} = 0$ for every n, k in Z. Thus in this case the sequence $\{x_n\}$ is always deterministic.

In order to justify the effort to solve the prediction problem for sequences with a normal shift operator we would have to exhibit a large class of such sequences. However, as we will show in the remaining sections of this paper, it is unnatural to assume that the shift operator of a sequence in a Hilbert space is normal. This does not mean that normal shift operators are worthless but that it may be more important to study the prediction problem for sequences with shift operators that are spectral operators of scalar type with their spectrum contained in the unit circle.

5. Normal shifts of harmonizable sequences are unitary

The purpose of this section is to prove that the shift operator of a non-stationary strongly harmonizable sequence $\{x_n\}$ in H cannot have a normal extension to the closure $H(x)$ of *l.s.*$\{x_n\}$. This will be an immediate consequence of the following result.

PROPOSITION (5.1). *Suppose N is an invertible (possibly unbounded) normal operator on the Hilbert space H. Suppose there exist: a Hilbert space H', a unitary operator U on H' and a linear (possibly unbounded) map $D: H \to H'$ with a bounded inverse D^{-1}, such that $N^k = D^{-1}U^kD$ on a dense subspace L of H, for every integer k. Then N is actually unitary.*

Proof. Let E be the resolution of the identity for N. Suppose there exists a set A contained in the complement of the unit disk in C such that $E(A) \neq 0$. Then there exist a positive number ϵ and a set A' contained in A such that $E(A') \neq 0$ and for every z in A', $|z| \geq 1 + \epsilon$. Let x be a unit vector in H such that $E(A')x = x$. Let y in L be such that $\|x - y\| \leq \frac{1}{2}$. Then $\|E(A')y\| \geq \frac{1}{2}$. Therefore, for every positive integer k

$$\frac{(1+\epsilon)^{2k}}{4} \leq (1+\epsilon)^{2k}\|E(A')y\|^2 \leq \int_{A'} |z^k|^2 \|E(dz)y\|^2$$

$$\leq \int_C |z^k|^2 \|E(dz)y\|^2 = \|N^k y\|^2 \leq \|D^{-1}\|^2 \|Dy\|^2.$$

But this is impossible. Thus the support of E is contained in the unit disk. Now suppose there exists a set B contained in the interior of the unit disk such that $E(B) \neq 0$. Then there exist $\epsilon > 0$ and a subset B' of B such that $\epsilon < 1$, $E(B') \neq 0$ and for every z in B', $|z| \leq 1 - \epsilon$. Let x be a unit vector in H such that $E(B')x = x$. Let y be an element of L such that $\|y - x\| \leq \frac{1}{2}$. Then $\|E(B')y\| \geq \frac{1}{2}$. Hence, for every positive integer k,

$$\frac{(1-\epsilon)^{-2k}}{4} \leq (1-\epsilon)^{-2k}\|E(B')y\|^2 \leq \int_{B'} |z^{-k}|^2 \|E(dz)y\|^2$$

$$\leq \int_C |z^{-k}|^2 \|E(dz)y\|^2 = \|N^{-k} y\|^2 \leq \|D^{-1}\|^2 \|Dy\|^2.$$

This is impossible. Therefore the support of E is contained in the unit circle and thus N is unitary Q.E.D.

THEOREM (5.2). *Let $\{x_n\}$ be a strongly harmonizable sequence with a shift operator group $\{S^k\}$. Suppose S has a normal extension in $H(x)$ = c.l.s.$\{x_n\}$. Then S is unitary and $\{x_n\}$ is stationary.*

Proof. According to proposition 3.1 the normal extension N of S has the properties stated in proposition 5.1 Thus N is unitary, i.e. S can be extended to be a unitary operator and therefore $\{x_n\}$ is stationary. Q.E.D.

6. Normal shifts of strict sense processes are unitary

This final section is dedicated to show that whenever a stochastic process in the strict sense has a normal shift operator then the process is stationary and the shift operator is unitary.

PROPOSITION (6.1). *Let* (Ω, Σ, p) *be a probability space and suppose* $T: \Omega \to \Omega$ *is a measurable map. A necessary and sufficient condition for the map* $f \to f \circ T$ *to be well defined on a subspace of* $L_2(p)$ *containing the simple functions is that the measure* $p \circ T^{-1}$ *be absolutely continuous with respect to* p.

Proof. Suppose the map $f \to f \circ T$ is well defined on a subspace of $L_2(p)$ which contains the simple functions. Let A be a measurable set such that $p(A) = 0$. Then $\chi_A = 0$ in $L_2(p)$ and therefore $\chi_{T^{-1}(A)} = \chi_A \circ T = 0$ in $L_2(p)$. Therefore $p(T^{-1}(A)) = 0$. Thus $p \circ T^{-1}$ is absolutely continuous with respect to p. Conversely, suppose $p \circ T^{-1}$ is absolutely continuous with respect to p. Let h be the Radon-Nikodym derivative of $p \circ T^{-1}$ with respect to p. Define D to be the subspace of $L_2(p)$ consisting of those $f \in L_2(p)$ such that $\int_\Omega |f|^2 h \, dp < \infty$. It is clear that D contains the simple functions and for every f in D,

$$\int_\Omega |f \circ T|^2 \, dp = \int_\Omega |f|^2 \, d(p \circ T^{-1}) = \int_\Omega |f|^2 h \, dp < \infty.$$

Finally, if $f = 0$ a.e. with respect to p then $f \circ T = 0$ a.e. with respect to p. This proves that $f \to f \circ T$ is well defined as a linear map from D into $L_2(p)$. Q.E.D.

Definition (6.2). A (discrete time) stochastic process in the strict sense is a probability space (Ω, Σ, p) together with a map $T: \Omega \to \Omega$ which is one to one, onto and such that T and T^{-1} are both measurable. The process is stationary if $p \circ T^{-1} = 0$.

Definition (6.3). A stochastic process in the strict sense $(\Omega, \Sigma, p; T)$ is said to have a shift operator if the maps $f \to f \circ T$ and $f \to f \circ T^{-1}$ are well defined on certain subspaces of $L_2(p)$ which contain the simple functions. The shift operator of the process $(\Omega, \Sigma, p; T)$ is defined by $Sf = f \circ T$ for every $f \in D = \{f \in L_2(p): \int_\Omega |f|^2 h \, dp < \infty\}$, where h is the Radon-Nikodym derivative of $p \circ T^{-1}$ with respect to p.

Remarks. (6.4). The shift operator S of a stochastic process has an inverse S^{-1} with domain $D' = \{f \in L_2(p): \int_\Omega |f|^2 h^{-1} \circ T \, dp < \infty\}$. Indeed, it is easy to see that $p \circ T$ is absolutely continuous with respect to p and its Radon-Nikodym derivative with respect to p is $h^{-1} \circ T$. Notice that if the process is stationary it has a shift operator which is actually unitary.

LEMMA (6.5). *Let S be the shift operator of a strict sense stochastic process* $(\Omega, \Sigma, p; T)$. *Then* $S^* = M_h S^{-1}$ *where* S^* *denotes the adjoint of S and* M_h *is the operator of multiplication by h, the Radon-Nikodym derivative of* $p \circ T^{-1}$ *with respect to p.*

Proof. Let f be in the domain of $M_h S^{-1}$ and g in the domain D of S. Then

$$(M_h S^{-1} f, g) = \int_\Omega h f \circ T^{-1} \bar{g}\, dp = \int_\Omega f \circ T^{-1} \bar{g}\, d(p \circ T^{-1})$$

$$= \int_\Omega f \overline{g \circ T}\, dp = \int_\Omega f \overline{Sg}\, dp = (f, Sg).$$

Since D is dense in $L_2(p)$ it follows that f is in the domain of S^* and $S^* f = M_h S^{-1} f$. Conversely, let f be in the domain of S^* and g in D. Then $(S^* f, g) = (f, Sg)$, and from the string of equalities above we conclude that $(S^* f, g) = (M_h S^{-1} f, g)$. Hence f is in the domain of $M_h S^{-1}$ and $S^* F = M_h S^{-1} f$. This completes the proof of the lemma.

THEOREM (6.6). *Let* $(\Omega, \Sigma, p; T)$ *be a strict sense process with a shift operator S. If S is normal then T is measure preserving, S is unitary and the process is stationary.*

Proof. From lemma 6.5 we have $S^* S = M_h$ *and* $SS^* = S M_h S^{-1} = M_{h \circ T}$. Since S is normal we conclude that $M_{h \circ T} = M_h$. For every positive integer n define $A_n = \{\omega \in \Omega : |h(\omega)| \leq n\}$. Then χ_{A_n} is in the domain of M_h and $h \chi_{A_n} = (h \circ T) \chi_{A_n}$. This means that $h = h \circ T$ a.e. on A_n, but since the union of all the A_n's is Ω we conclude that $h = h \circ T$ a.e. on Ω. We know that $\int_\Omega h\, dp = 1$. For every positive integer n

$$\int_\Omega h^n\, dp = \int_\Omega (h \circ T)^n\, dp = \int_\Omega h^n d(p \circ T^{-1}) = \int_\Omega h^{n+1}\, dp.$$

Hence $\int_\Omega h^n\, dp = 1$ for every positive integer n. This implies that h is bounded by 1, but since $\int_\Omega h\, dp = 1$ it follows that $h = 1$ a.e. Therefore $p \circ T^{-1} = p$. Q.E.D.

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