Boletín de la Sociedad Matemática Mexicana Vol. 28, No. 2, 1983

A SIMPLE PROOF OF THE CAMERON-MARTIN THEOREM MAKING USE OF SCHWARTZ REPRODUCING KERNELS*

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Introduction

The abstract version of the theorem of Cameron-Martin states that, given a Gaussian Radon measure m on a quasi-complete locally convex space E, the translate m_a of m through a vector $a \in E$ is equivalent to m or singular with respect to m, according to whether a does or does not belong to the Hilbert subspace $\mathscr{H}$ of E whose reproducing operator is the covariance operator of m. In the case where a belongs to $\mathscr{H}$ the theorem specifies the density of m_a with respect to m [1], [6]. It is the purpose of this paper to give a simple proof of this theorem by making use of the machinery of continuously embedded Hilbert spaces and their reproducing operators as developed by L. Schwartz [10]. In fact we shall need only a very small part of Schwartz's theory: the correspondence between Hilbert subspaces and their reproducing operators, and the reproducing operator of an image space. By considering appropriate images we show that the proof of the absolute continuity of translates may be reduced to the case where E is two dimensional. For background information we refer to H. H. Kuo [6] and to the survey article by D. Kölzow [5], in which the reader will find an extensive bibliography on generalized Wiener spaces.

The paper, which is self contained, is organized as follows:

§1 Treats the necessary facts on Hilbert subspaces and their reproducing operators, following L. Schwartz.

§2 Treats the correspondence between a Gauss measure and the associated Hilbert subspace.

§3 Gives the proof of the abstract Cameron-Martin theorem.

1. Embedded Hilbert spaces

Throughout this paper we denote by E a quasi-complete locally convex Hausdorff space over $\mathbb{R}$.

A Hilbert space embedded in E, briefly: a Hilbert subspace of E, is a linear subspace.

$$\mathscr{H} \underset{j}{\hookrightarrow} E$$

equipped with an inner product making it into a Hilbert space, such that the inclusion map j is continuous. The latter condition is equivalent to the fact that the unit ball of $\mathscr{H}$ is a bounded subset of E.

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* Presented at the "Workshop on the prediction theory on non-stationary processes and related topics,", held at Centro de Investigación en Matemáticas (CIMAT), Guanajuato, México, June 20–26, 1982.

We now define the reproducing operator of $\mathscr{H} \hookrightarrow E$. Thanks to the Riesz-Fréchet representation theorem $\mathscr{H}$ is canonically isomorphic to its dual $\mathscr{H}'$, and we identity $\mathscr{H}$ and $\mathscr{H}'$. Thus the transpose of j, which we denote by ${}^tj$, becomes a linear map

$$E' \underset{{}^tj}{\longrightarrow} \mathscr{H}$$

The composition $H = j\,{}^tj$, which is a linear operator from E' to E is the reproducing operator of $\mathscr{H}$. Since we have

(1) $$(x \mid {}^tj\xi) = \langle jx, \xi \rangle \quad \forall x \in \mathscr{H} \quad \forall \xi \in E'$$

the operator H is characterized by the relations

(2) $$({}^tj\eta \mid {}^tj\xi) = \langle H\eta, \xi \rangle$$

Where $(\,|\,)$ and $\langle , \rangle$ are the inner product in H and the canonical bilinear form on $E \times E'$, respectively.
It follows that H is symmetric and positive, i.e.

(3) $$\langle H\xi, \eta \rangle = \langle H\eta, \xi \rangle \qquad \forall \eta, \xi \in E'$$

(4) $$\langle H\eta, \eta \rangle \geq 0 \qquad \forall \eta \in E'.$$

Let us denote by Hilb(E) the set of Hilbert subspaces of E, and let Γ denote the set of linear operators $H{:}E' \to E$ having the properties (3) and (4). Then one has the following fundamental result, due to L. Schwartz [10]:

PROPOSITION (1). *Let E be a quasi-complete locally convex space.*
a) The Hilbert subspace $\mathscr{H}$ is uniquely determined by its reproducing operator H.
b) Conversely, every operator $H \in \Gamma$ is the reproducing operator of a Hilbert subspace of E.
Briefly: the correspondence $\mathscr{H} \leftrightarrow H$ is a bijection between Hilb(E) and Γ.

We shall only need, and give, the proof of a): First remark that the image ${}^tj(E')$ is a dense subspace of $\mathscr{H}$, no vector $x \neq 0$ being orthogonal to it by (1). Thus, for all $x \in \mathscr{H}$, we have $\| x \| = \sup_{\| y \| \leq 1} |(x \mid y)|$ where y is constrained to belong to ${}^tj(E')$. By (1) and (2) this means we have

(5) $$\| x \| = \sup_{\langle H\eta, \eta \rangle^{1/2} \leq 1} |\langle x, \eta \rangle| \qquad \forall x \in \mathscr{H}$$

(where for convenience we have left out j). Conversely, if $x \in E$ is such that the right hand side of (5) is finite, x belongs to $\mathscr{H}$. In fact, there then exists a linear form L on ${}^tj(E')$, continuous with respect to the topology induced by $\mathscr{H}$, such that $L({}^tj\eta) = \langle x, \eta \rangle$. By the Riesz-Fréchet theorem there exists $x' \in \mathscr{H}$ such that $L({}^tj\eta) = (x' \mid {}^tj\eta)$ for all $\eta \in E$. But then $\langle x, \eta \rangle = \langle x', \eta \rangle$ for all η, and so $x = x' \in \mathscr{H}$.
Thus both the subspace $\mathscr{H}$ and the norm on it are described by (5) in terms of H.

Naturally the correspondence $\mathscr{H} \leftrightarrow H$ can also be described in terms of the corresponding symmetric bilinear form on $E' \times E'$. We shall denote this one by the same letter H and refer to it as the *reproducing kernel* of $\mathscr{H}$. Thus we put

$$H(\eta, \xi) = \langle H\eta, \xi \rangle \tag{6}$$

Note that a symmetric bilinear form on $E' \times E'$ can be obtained from a symmetric linear operator $H: E' \to E$ via formula (6) if and only if it is separately continuous with respect to the weak $*$-topology $\sigma(E', E)$.

Examples

1. If E happens itself to be a real Hilbert space, identified with its dual, the positive symmetric operators $H: E' \to E$ are just the usual positive self adjoint operators of the Hilbert space theory. The Hilbert subspace $\mathscr{H} \hookrightarrow E$ is then a closed subspace with the inner product inherited from E, if and only if the reproducing operator H is a projection: $H = H^2$, in which case $\mathscr{H} = \mathrm{Im}(H)$. In the case of an arbitrary Hilbert subspace $\mathscr{H} \hookrightarrow E$ we have instead $\mathscr{H} = \mathrm{Im}(H^{1/2})$ (cf. example 3).

2. $E = \mathscr{C}_{\mathbb{R}}(T)$, the space of continuous real functions on a locally compact space T. Then E' is, by the theorem of Riesz-Markov, identified with the space $\mathscr{M}_c(T)$ of real Radon measures with compact support. If $\mathscr{H} \hookrightarrow \mathscr{C}_{\mathbb{R}}(T)$, and K is the classical Aronszajn-Bergman reproducing kernel defined by

$$(f \mid K(t, \cdot)) = f(t),$$

the reproducing operator $H: \mathscr{M}_c(T) \to \mathscr{C}_{\mathbb{R}}(T)$ is the integral operator defined by

$$H\mu(s) = \int K(t, s)\, d\mu(t)$$

and we have

$$H(\mu, \nu) = \int \int K(t, s)\, d\mu(t)\, d\nu(s).$$

The function K being $\mu \otimes \nu$-integrable, because, being locally bounded and separately continuous, it is actually universally measurable [4] p. 239.

We now consider the *image of a Hilbert subspace* under a continuous linear map. Let F be a second locally convex space and let

$$u: E \to F$$

be a continuous linear map. If $\mathscr{H}$ is a Hilbert subspace of E one defines a Hilbert space structure on the image space $u(\mathscr{H})$ as follows: Let $\mathscr{N} = \{x \in H: ux = 0\}$. This is a closed subspace of $\mathscr{H}$, and the restriction of u to the orthogonal complement $\mathscr{H} \ominus \mathscr{N}$ is a linear bijection. The inner product on $u(\mathscr{H})$ is now chosen in such a way as to make this an isometry. Then $u(\mathscr{H})$ is a Hilbert space and its unit ball, the image under u of the unit ball of $\mathscr{H}$, is a bounded subset of F. Thus $u(\mathscr{H})$ is a Hilbert subspace of F.

PROPOSITION (2). *The reproducing operator of $u(\mathscr{H})$ is the operator $uH{}^tu$, ${}^tu$ being the transpose of u. The reproducing kernel of $u(\mathscr{H})$ is the bilinear form*

$$(\eta, \xi) \to H({}^tu\eta, {}^tu\xi).$$

Proof: Let $v: \mathscr{H} \to u(\mathscr{H})$ be the restriction of u. Then v being a partial isometry $v{}^tv$ is the identity map of $u(\mathscr{H})$. If we denote by i the inclusion map

$$i: u(\mathscr{H}) \hookrightarrow F$$

we have $iv = uj$, hence ${}^tv{}^ti = {}^tj{}^tu$ and $uH{}^tu = uj{}^tj{}^tu = iv{}^tv{}^ti = i{}^ti$, which is the reproducing operator of $u(\mathscr{H})$. The last assertion is obvious from the relation $\langle uH{}^tu\eta, \xi\rangle = \langle H{}^tu\eta, {}^tu\xi\rangle$.

In particular, given any Hilbert space $\mathscr{K}$ and a continuous linear map $u: \mathscr{K} \to E$, we obtain a Hilbert subspace $\mathscr{H} = u(\mathscr{K}) \hookrightarrow E$, whose reproducing operator is $u{}^tu$, and whose reproducing kernel is

$$H(\eta, \xi) = ({}^tu\eta \mid {}^tu\xi)_{\mathscr{K}} \tag{7}$$

Example **3**. Let E be a Hilbert space and let $\mathscr{H} \hookrightarrow E$ be a Hilbert subspace of E with reproducing operator $H: E \to E$. Then the image space $H^{1/2}(E)$ has a reproducing operator $H^{1/2}\,{}^tH^{1/2} = H^{1/2}\,H^{1/2} = H$, i.e. it is equal to $\mathscr{H}$.

2. Gauss measures

Let m be a bounded positive Radon measure in E, i.e. a finite non negative measure defined on the Borel subsets of E, inner regular with respect to the compact subsets of E. The inner regularity implies among other things that m is uniquely determined by its Fourier transform

$$\mathscr{F}(m)(\eta) = \int [\exp i\langle x, \eta\rangle]\, dm(x), \qquad \eta \in E', \text{ where } \exp A = e^A.$$

PROPOSITION (3). *The following three properties of m are equivalent*:

1) *For every $\eta \in E'$ the image of m under the map η is a centered Gauss measure on $\mathbb{R}$ (possibly δ).*
2) *There exists a symmetric bilinear form $H: E' \times E' \to \mathbb{R}$ such that*

$$\int [\exp i\langle x, \eta\rangle]\, dm(x) = \exp[-\tfrac{1}{2}H(\eta, \eta)] \qquad \forall \eta \in E', \tag{8}$$

3) *There exists a Hilbert subspace*

$$\mathscr{H} \underset{j}{\hookrightarrow} E$$

such that, as cylinder set measure, m is the image under j of the canonical normal cylinder set measure on $\mathscr{H}$.

Moreover, if m satisfies these conditions, the space $\mathscr{H}$ is uniquely determined by m, and has as its reproducing kernel the bilinear map H.

The measure m is then called a centered Gauss measure on E, $\mathscr{H}$ is called the reproducing kernel Hilbert space associated with m and H is called the covariance kernel of m.

COROLLARY. *If m is a Gauss measure and $\mathscr{H}$ is the associated Hilbert space, the image $u(m)$ of m under a continuous linear map u, is a Gauss measure, and the associated Hilbert space is $u(\mathscr{H})$.*

This is an immediate consequence of formula (8) and proposition 2 (cf. [9]).

For the sake of completeness, and because we shall need certain elements from it, we give a proof of proposition 3:

1) $\Leftrightarrow$ 2) According to 1) we have

$$\int e^{its}\, d\eta(m)(s) = \exp(-\tfrac{1}{2}t^2\sigma^2(\eta)) \qquad \forall t \in \mathbb{R}, \tag{9}$$

for some number $\sigma^2(\eta) \geq 0$. This implies that

$$\int \langle x, \eta\rangle^2\, dm(x) = \int s^2\, d\eta(m)(s) = \sigma^2(\eta) < +\infty$$

i.e., $\langle \cdot, \eta\rangle$ belongs to $\mathscr{L}^2(m)$. One then defines a positive bilinear form $H{:}E' \times E' \to \mathbb{R}$ by the formula

$$H(\eta, \xi) = \int \langle x, \eta\rangle\langle x, \xi\rangle\, dm(x) \tag{10}$$

It has the property $H(\eta, \eta) = \sigma^2(\eta)$, and so putting $t = 1$ in (9) we obtain (8). Conversely (8) obviously implies (9) with $\sigma^2(\eta) = H(\eta, \eta)$.

2) $\Rightarrow$ 3) Since m is inner regular, and since by the Krein-Smulian theorem the closed convex hull of a weakly compact subset is weakly compact, we have

$$\sup_{C\in\mathscr{C}} m(C) = 1 \tag{11}$$

where $\mathscr{C}$ is the class of weakly compact convex subsets of E. We now give two proofs of the fact that H is the reproducing kernel of a Hilbert subspace of E:

i) It follows from (11) that $\mathscr{F}(m)$ is the uniform limit of $\mathscr{F}(1_C m)$ as C runs through the directed set $\mathscr{C}$. Consequently $\mathscr{F}(m)$ is continuous with respect to the topology of uniform convergence on the sets C belonging to $\mathscr{C}$, i.e. the Mackey topology τ. Hence the quadratic form $\eta \to H(\eta, \eta)$ is continuous with respect to τ, and so is the bilinear form $(\eta, \xi) \to H(\eta, \xi)$, thanks to the polarisation formula. Now τ being compatible with the duality E, E', H is weak * separately continuous, i.e. H is associated with an auto-correlation operator $E' \to E$ as in formula (6). It follows from proposition 1 that H is the reproducing kernel of a Hilbert subspace $\mathscr{H} \hookrightarrow E$.

However, basically because we already know that $L^2(m)$ is complete, the following argument of Dudley-Feldman-Le Cam [3] is more direct. It requires no completeness of E but only the condition (11):

ii) Consider the map v from E' to $L^2(m)$ which associates with η the equivalence class in L^2 of the map $\eta{:}x \to \langle x, \eta\rangle$. Let $\mathscr{K}$ be the closure of its image. If η tends to zero with respect to the Mackey topology τ, it follows from (11) that η tends to zero in m-measure, and since the distribution $\eta(m)$ is centered Gaussian this implies that η tends to 0 in $\mathscr{L}^2(m)$, Thus $v{:}E_\tau' \to \mathscr{K}$

is continuous, and τ being compatible with the duality E', E it has a continuous transpose $u: \mathscr{H} \to E$. Now let $\mathscr{A} = u(\mathscr{H})$. Then we have

$$H(\eta, \xi) = \int \langle x, \eta\rangle\langle x, \xi\rangle \, dm(x) = (v(\eta) \mid v(\xi)) = ({}^tu(\eta) \mid {}^tu(\xi))$$

which is the reproducing kernel of $\mathscr{A}$. (Note moreover that, v having a dense range, the transpose u is one-to-one, hence unitary).

3) $\Rightarrow$ 2) If n is the canonical normal cylinder measure on $\mathscr{A}$ its Fourier transform if $\mathscr{F}(n)(y) = \exp(-\frac{1}{2}\| y \|^2)$. Hence the Fourier transform of its image $m = j(n)$ is $\exp(-\frac{1}{2}\| {}^tj\eta \|^2) = \exp(-\frac{1}{2}H(\eta, \eta))$ by (2).

Remark: The argument (i) also yields the following: let m be a not necessarily centered Gauss measure on E, i.e. a Radon probability such the image $\eta(m)$ is Gaussian for all $\eta \in E'$. Then m has a mean value in E.

Proof: The Fourier transform of m is of the form $\mathscr{F}(m)(\eta) = \exp(-\frac{1}{2}H(\eta, \eta) + i\ell(\eta))$], where H is as before and $\ell: E' \to \mathbb{R}$ is linear. Now by (11) this is again continuous with respect to the Mackey topology τ. It follows that $e^{i\ell}$ is continuous with respect to τ. By the following lemma this implies that ℓ is continuous with respect to τ, hence of the form $\ell(\eta) = \langle a, \eta\rangle$ with $a \in E$, which implies $a = \int x \, dm(x)$.

Here again one may relax the hypotheses both on E and on m: all that is required is (11). However, even in the case of a separable Banach space (where all bounded Borel measures are Radon) the result does not seem to be widely known (cf. [6] p. 153).

LEMMA. *Let E be a topological vector space and let $\ell: E \to \mathbb{R}$ be a linear form. Then, if $e^{i\ell}$ is continuous, so is ℓ.*

Proof. By hypothesis $\cos \ell(\eta) \to 1$ as $\eta \to 0$. Let ϵ be given with $0 < \epsilon < \pi$, and let $\delta > 0$ be such that $| 1 - \cos x | \leq \delta$ implies that the distance of x to $2\pi\mathbb{Z}$ is at most ϵ. Then if V is a starred neighborhood of 0 such that for $\eta \in V$, $| 1 - \cos \ell(\eta) | \leq \delta$, it follows that $\eta \in V$ implies $| \ell(\eta) | \leq \epsilon$.

Before starting and proving the main theorem we shall need one more notation: from relations (2) and (10) it follows that we have $\| {}^tj\eta \|^2 = \int \langle x, \eta\rangle^2 \, dm(x)$ for all $\eta \in E'$. The space ${}^tj(E')$ being dense in $\mathscr{A}$ we therefore have:

PROPOSITION (4). *There exists a linear isometry*

$$\phi: \mathscr{A} \to L^2(m)$$

such that, if $a = {}^tj\eta$, $\Phi(a)$ equals the equivalence class in $L^2(m)$ of the function $\eta: x \to \langle x, \eta\rangle$.

This is usually expressed by saying that the elements of $\mathscr{A}$ may be considered as random variables. (ϕ is of course just the inverse of the unitary map $u: \mathscr{H} \to \mathscr{A}$ considered above).

3. The main theorem

Let m be a Gauss measure on a quasi-complete locally convex space E, and let m_a denote its translate through the vector $a \in E$, defined by

$$m_a(A) = m(A - a).$$

We note $\mathscr{A}$ the Hilbert subspace of E whose reproducing kernel is the covariance kernel of m.

THEOREM. *The translate m_a is either absolutely continuous with respect to m or singular with respect to m, according to whether a belongs to $\mathscr{A}$ or does not belong to $\mathscr{A}$. If a belongs to $\mathscr{A}$ we have*

$$m_a = [\exp(-\tfrac{1}{2} \| a \|^2 + \phi(a))]m.$$

In particular (Feldman-Hajek), *if E is a Hilbert space $m_a \sim m \Leftrightarrow a \in H^{1/2}(E)$.*

Proof. We first show that for $a \notin \mathscr{A}$, the measures m and m_a are mutually singular. By (5) and the argument following it we see that $\sup_{H(\eta,\eta)^{1/2}\leq 1} | \langle a, \eta \rangle | = +\infty$, so there exists a sequence $(\eta_i)_{i\geq 1}$ such that $H(\eta_i, \eta_i) \leq 1$ and $\sup_i | \langle a, \eta_i \rangle | = +\infty$. Let $(\alpha_i)_{i\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ such that $\sum_{i=1}^{\infty} \alpha_i | \langle a, \eta_i \rangle | = +\infty$. for $x \in E$ let $f(x) = \sum_{i=1}^{\infty} \alpha_i | \langle x, \eta_i \rangle |$. Then

$$\int f(x)\, dm(x) = \sum_{i=1}^{\infty} \alpha_i \int | \langle x, \eta_i \rangle |\, dm(x) \leq \sum_{i=1}^{\infty} \alpha_i \int \langle x, \eta_i \rangle^2\, dm(x)^{1/2}$$

$$= \sum_{i=1}^{\infty} \alpha_i H(\eta_i, \eta_i)^{1/2} \leq \sum_{i=1}^{\infty} \alpha_i < +\infty.$$

Thus m is concentrated on the linear subspace $V = \{x : f(x) < +\infty\}$, m_a is concentrated on $V + a$, but $V \cap (V + a) = \phi$ since a does not belong to V.

Next we show that if $a \in \mathscr{A}$, $m_a = [\exp(-\tfrac{1}{2} \| a \|^2 + \phi(a)]m$, i.e.

$$\int f(x + a)\, dm(x) = \int f(x)[\exp(-\tfrac{1}{2} \| a \|^2 + \phi(a)(x))]\, dm(x) \tag{12}$$

for all continuous bounded functions f.

We first prove this when $\dim E = 2$. We then equip E with a Hilbert space structure, equal to that of $\mathscr{A}$ if $\dim \mathscr{A} = 2$, and such that $\mathscr{A}$ is a subspace with the norm induced by that of E if $\dim \mathscr{A} < 2$. Accordingly we identify E and E'. Thus there are three cases:

i) $\mathscr{A} = E$, $dm(x) = c[\exp(-\tfrac{1}{2} \| x \|^2)]\, dx$
ii) $\mathscr{A} = \mathbb{R} \times \{0\} \subset E = \mathbb{R}^2$, $dm(x) = c(\exp \tfrac{1}{2} x_1^2)\, dx_1 \otimes \delta$
iii) $\mathscr{A} = (0)$ $\quad m = \delta$.

In case i) we have

$$\int f(x + a)\, dm(x) = c \int f(x) \exp(-\tfrac{1}{2} \| x - a \|^2)\, dx$$

$$= \int f(x)[\exp(-\tfrac{1}{2} \| a \|^2 + (x \mid a)]\, dm(x)$$

Now j being the identity we have, if $a = {}^t j \eta$, $(x \mid a) = \langle x, \eta \rangle = \phi(a)(x)$, for all x.

In case ii) $a = (a_1, 0)$ and

$$\int f(x + a)\, dm(x) = c \int f(x_1 + a_1, 0)[\exp(-\tfrac{1}{2}x_1^2)]\, dx_1$$

$$= c \int f(x_1, 0)[\exp(-\tfrac{1}{2}(x_1 - a_1)^2)]\, dx_1 = \int f(x)[\exp(-\tfrac{1}{2}\| a \|^2 + x_1 a_1)]\, dm(x).$$

Now ${}^t j$ is the orthogonal projection. Hence if $a = {}^t j\eta$ we have $\phi(a)(x) = \langle x, \eta\rangle = x_1 a_1$ for m-almost all x (for all x if we choose $\eta = a$). Thus we again have (12).

In the last case $a = 0$ and there is nothing to check.

Now to prove that $m_a = [\exp(-\frac{1}{2}\| a \|^2 + \phi(a)]m$ in the case of arbitrary E we observe that it is sufficient to prove the equality of the Fourier transforms, i.e. to check (12) with $f(x) = \exp i\langle x, \xi\rangle$:

$$(13) \quad \int (\exp i\langle x + a, \xi\rangle)\, dm(x) = \int \exp[i\langle x, \xi\rangle - \tfrac{1}{2}\| a \|^2 + \phi(a)(x)]\, dm(x).$$

To prove this we first consider the case where $a = {}^t j\eta$ for some $\eta \in E'$. Then (13) becomes

$$(14) \quad \int (\exp i\langle x + H\eta, \xi\rangle)\, dm(x) = \int [\exp(i\langle x, \xi\rangle - \tfrac{1}{2}\langle H\eta, \eta\rangle + \langle x, \eta\rangle)]\, dm(x)$$

Consider the map $u: E \to \mathbb{R}^2$ defined by $u(x) = (\langle x, \eta\rangle, \langle x, \xi\rangle)$. Then the linear forms η and ξ factor via u: if $\eta'(t, s) = t$ and

$\xi'(t, s) = s$, we have

$$\eta = \eta' \circ u = {}^t u(\eta'), \qquad \xi = \xi' \circ u = {}^t u(\xi').$$

Now observe that (14) is precisely equivalent to the analogous identity with E replaced by $F = \mathbb{R}^2$, m replaced by $u(m)$ and $\mathscr{A}$ replaced by $u(\mathscr{A})$. In fact writing $y = u(x)$ we have

$$\langle x, \xi\rangle = \langle y, \xi'\rangle,$$
$$\langle H\eta, \xi\rangle = \langle uH^t u\eta', \xi'\rangle, \qquad \text{and}$$
$$\langle x, \eta\rangle = \langle y, \eta'\rangle,$$

so (14) becomes

$$\int [\exp i\langle y + uH^t u\cdot\eta', \xi'\rangle]\, du(m)(y) = \int [\exp(i\langle y, \xi'\rangle - \tfrac{1}{2}\langle uH^t u\eta', \eta'\rangle + \langle y, \eta'\rangle)]\, du(m)(y)$$

which however we have proved already.

Thus (12) or (13) has been proved in the particular case where $a = {}^t j\eta$ for some $\eta \in E'$.

It remains to pass the limit in (12) when $a = \lim_{n\to\infty} a_n$ in $\mathscr{A}$, where $a_n \in {}^t j(E')$. The function f being continuous we have $\int f(x + a)\, dm(x) = \lim_{n\to\infty}$

$\int f(x+a_n)\,dm(x)$ by Lebesgue's dominated convergence theorem, a_n converging a fortiori to a in E. Since $\phi(a)=\lim_{n\to\infty}\phi(a_n)$ in $L^2(m)$, ϕ being isometric, we may assume, replacing (a_n) by a subsequence if necessary, that $\phi(a_n)(x)\to\phi(a)(x)m-$ *a.e.* The functions $\exp(-\frac{1}{2}\|a_n\|^2+\phi(a_n)(x))$ converge pointwise almost everywhere but are not obviously dominated by any integrable function. However taking $f=1$ we see that their integrals equal 1, i.e.

$$\int [\exp \phi(a_n)(x)]\, dm(x) = \exp \tfrac{1}{2}\|a_n\|^2$$

whence

$$\int [\exp \phi(a_n)(x)]^2\, dm(x) = \int [\exp 2\phi(a_n)(x)]\, dm(x) = \exp 2\|a_n\|^2.$$

and so these functions are bounded in $L^2(m)$. Hence by the Cauchy-Schwarz inequality we see that

$$\lim_{m(A)\to 0} \int_A [\exp \phi(a_n)(x)]\, dm(x) = 0$$

uniformly with respect to n, and so by Egoroff's theorem it follows that

$$\lim_{n\to\infty} \int f(x)[\exp(-\tfrac{1}{2}\|a_n\|^2 + \phi(a_n)(x)]\, dm(x)$$

$$= \int f(x)[\exp(-\tfrac{1}{2}\|a\|^2 + \phi(a)(x)]\, dm(x),$$

and we have (12) for a given $a\in\mathscr{A}$.

Remark. The analogue of this theorem within the framework of abstract Gauss processes (equivalently: Gauss probabilities on the Kolmogorov σ-algebra of $\mathbb{R}^T$) is due to E. Parzen [8] (see also [6] as well as [9] for further bibliographical data). Since E is a subspace of $\mathbb{R}^{E^*}$ and the trace on E of the Kolmogorov σ-algebra of $\mathbb{R}^{E^*}$ is contained in the Borel σ-algebra of E, the above theorem can be deduced from Parzen's theorem. On the other hand, the above proof applies equally well to the case of a Gauss measure on the Kolmogorov σ-algebra of $\mathbb{R}^T$, thus reducing the general case directly to $\mathbb{R}^2$.

I should like to thank H. von Weizsäcker who, when I discovered a gap in my reasoning, supplied the equiintegrability argument above.

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