Boletín de la Sociedad Matemática Mexicana Vol. 28, No. 2, 1983

A CLASS OF INVALID ASSERTIONS CONCERNING FUNCTION HILBERT SPACES*

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1. PROPOSITION. *Let* (i) $\mathscr{X}$ *be an infinite-dimensional Hilbert space over* $\mathbb{F}$[#] *under the inner product* $(\cdot, \cdot)$, (ii) $0 \neq x_0 \in \mathscr{X}$, *and* $f_0(\cdot) = (\cdot, x_0)$ *on* $\mathscr{X}$. *Then* $\exists$ *an inner product* $((\cdot, \cdot))$ *for* $\mathscr{X}$ *under which* $\mathscr{X}$ *remains a Hilbert space, but the functional* f_0 *on* $\mathscr{X}$ *to* $\mathbb{F}$ *becomes discontinuous.*

The proof will be given later. First, we apply the proposition to function Hilbert spaces:

2. *Definition.* Let Λ be a non-void set and write

$$\forall f \in \mathbb{F}^{\Lambda} \ \& \ \forall \lambda \in \Lambda, \qquad \mathscr{E}_{\lambda}(f) =_d f(\lambda).$$

We say that $\mathscr{F}$ is a Λ, $\mathbb{F}$ *function Hilbert space,* iff (i) $\mathscr{F} \subseteq \mathbb{F}^{\Lambda}$, (ii) $\mathscr{F}$ is a Hilbert space over $\mathbb{F}$, and (iii) $\forall \lambda \in \Lambda$, $\mathscr{E}_{\lambda}$ is a continuous linear functional on $\mathscr{F}$ to $\mathbb{F}$.

3. COROLLARY. *Let* (i) Λ *be an infinite set,* (ii) $\mathscr{F}$ *be an infinite dimensional linear manifold in the vector space* $\mathbb{F}^{\Lambda}$, (iii) $(\cdot, \cdot)_{\mathscr{F}}$ *be an inner product for* $\mathscr{F}$, *under which* $\mathscr{F}$ *is a* Λ, $\mathbb{F}$ *function Hilbert space over* $\mathbb{F}$. *Then* $\exists$ *an inner product* $((\cdot, \cdot))$ *for* $\mathscr{F}$ *under which* $\mathscr{F}$ *remains a Hilbert space but ceases to be a* Λ, $\mathbb{F}$ *function Hilbert space.*

Proof. By (ii), the Hilbert space $\mathscr{F}$ of (iii) is infinite dimensional. Hence certainly $\exists\ \phi \in \mathscr{F}$ such that ϕ is not the zero-function, i.e., $\exists \lambda_0 \in \Lambda \ni \phi(\lambda_0) \neq 0$. By (iii), $\mathscr{E}_{\lambda_0}$ is a continuous linear functional on $\mathscr{F}$ to $\mathbb{F}$. Hence by the Riesz Theorem, $\exists \psi_0 \in \mathscr{F}$ such that

$$\mathscr{E}_{\lambda_0}(\cdot) = (\cdot, \psi_0)_{\mathscr{F}}.$$

Now since $\mathscr{E}_{\lambda_0}(\phi) = \phi(\lambda_0) \neq 0$, therefore $\psi_0 \neq 0$. Hence by Proposition 1, $\exists$ an inner product $((\cdot, \cdot))$ for $\mathscr{F}$ under which $\mathscr{F}$ remains a Hilbert space, but $\mathscr{E}_{\lambda_0}$ is no longer continuous on $\mathscr{F}$. Thus $\mathscr{F}$, with inner product $((\cdot, \cdot))$, is not a function Hilbert space. □

This corollary yields an interesting metamathematical theorem, to enunciate which it is convenient to introduce a metamathematical notation:

4. *Metanotation.* For a non-void set Λ and a linear manifold $\mathscr{M}$ of the function vector space $\mathbb{F}^{\Lambda}$, let "Prop($\mathscr{M}$)" abbreviate the mathematical proposition: $\forall\ \mathscr{F} \subseteq \mathscr{M}$, ($\mathscr{F}$ is a Hilbert space $\Rightarrow$ $\mathscr{F}$ is a Λ, $\mathbb{F}$ function Hilbert space).

* Presented at the "Workshop on the prediction theory on non-stationary processes and related topics", held at Centro de Investigación en Matemáticas (CIMAT), Guanajuato, México, June 20–26, 1982.

[#] $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

5. METATHEOREM. *Let* (i) Λ *be an infinite set,* (ii) $\mathscr{M}$ *be a infinite dimensional linear submanifold of* $\mathbb{F}^{\Lambda}$, (iii) $\exists\ \mathscr{F}_0 \subseteq \mathscr{M}$ *such that* $\mathscr{F}_0$ *is a Hilbert space over* $\mathbb{F}$. *Then "*Prop($\mathscr{M}$)*" is false.*

Note. When (iii) fails, the antecedent of the $\Rightarrow$ in "Prop($\mathscr{M}$)" is false for any $\mathscr{F}_0 \subseteq \mathscr{M}$. Hence "Prop($\mathscr{M}$)" becomes true, but only vacuously.

Metaproof. Let $(\cdot, \cdot)$ be the inner product of the Hilbert space $\mathscr{F}_0 \subseteq \mathscr{M}$.

If this $\mathscr{F}_0$ is not a Λ, $\mathbb{F}$ function-Hilbert space, then for $\mathscr{F} = \mathscr{F}_0$, the premise of "Prop($\mathscr{M}$)" is true but the conclusion of "Prop($\mathscr{M}$)" is false. Hence "Prop($\mathscr{M}$)" is false.

Next let $\mathscr{F}_0$ be a Λ, $\mathbb{F}$ function Hilbert space. Then by Corollary 1, $\exists$ an inner product $((\cdot, \cdot))$ for $\mathscr{F}$ such that

(1) $(\mathscr{F}_0, ((\cdot, \cdot)))$ is a Hilbert space,

(2) $(\mathscr{F}_0, ((\cdot, \cdot)))$ is not a Λ, $\mathbb{F}$ function Hilbert space.

By (1), the premise of "Prop($\mathscr{M}$)", with $\mathscr{F} = \mathscr{F}_0$, is true. By (2), the conclusion of "Prop($\mathscr{M}$)" with $\mathscr{F} = \mathscr{F}_0$ is false. Hence again "Prop($\mathscr{M}$)" is false. □

6. *Example.* Let $C[a, b]$ be the class of all $\mathbb{F}$-valued continuous functions on the closed interval $[a, b]$. The "Prop($C[a, b]$)" *abbreviates the assertion*:

$$\begin{cases} \text{Every Hilbert space of } \mathbb{F}\text{-valued continuous functions} \\ \text{on } [a, b] \text{ is an } [a, b], \mathbb{F} \text{ function Hilbert space.} \end{cases}$$

Now $\exists\ \mathscr{F}_0 \subseteq C[a, b]$ such that $\mathscr{F}_0$ is a Hilbert space. For instance,

$$\mathscr{F}_0 =_d \{f : f \in C_1(a, b],\ f' \in L_2[a, b]\ \&\ f(a) = 0\}$$

with the inner product

$$\forall f, g \in \mathscr{F}_0, \qquad (f, g) =_d \int_a^b f'(\lambda)\overline{g'(\lambda)}\, d\lambda.$$

Hence by Metatheorem 5, "Prop($C[a, b]$)" is false. Thus, the corollary in Higgins [2, p. 54], attributed to Professor J. I. Richards, is invalid.

Likewise "Prop.{Hol(D)}" is invalid, where Hol(D) is the set of holomorphic functions on a domain D in $\mathbb{C}$. The lesson, in general, is that no degree of smoothness of the members of a Hilbert space $\mathscr{F}$ of functions can compel the evaluation functionals to become continuous on $\mathscr{F}$.

To turn to the proof of Proposition 1, we need two lemmas.

LEMMA 1. *Let* (i) $\mathscr{H}$ *be an infinite dimensional Hilbert space over* $\mathbb{F}$, (ii) g *be a discontinuous linear functional on* $\mathscr{H}$ *to* $\mathbb{F}$, *and* (iii) $\mathscr{N}_g$ *be the null-space of* g. *Then*[#] (*a*) $\exists u_0 \in \mathscr{H} \ni \mathscr{N}_g + \langle u_0 \rangle = \mathscr{H}\ \&\ \mathscr{N}_g \cap \langle u_0 \rangle = \{0\}$, (*b*) $\mathscr{N}_g$ *is unclosed and everywhere dense in* $\mathscr{H}$.

[#] For $A \subseteq \mathscr{H}$, $\langle A \rangle$ is the linear manifold spanned by A in $\mathscr{H}$.

Proof. (a) Let $\mathscr{M}$ be a linear manifold complementary to $\mathscr{N}_g$. Then g is a one-one linear operator on $\mathscr{M}$ onto $\mathbb{F}$. Hence dim $\mathscr{M} = 1$, and therefore $\mathscr{M} = \langle u_0 \rangle$ for some $u_0 \in \mathscr{H}$. Thus (a).

(b) This follows at once from parts (ii), (iii) of Prop. 5.4 in Kelley et al [3, p. 37].## □

LEMMA 2. *Let* (i) $\mathscr{X}$, $\mathscr{Y}$ *be vector spaces over* $\mathbb{F}$ *of the same Hamel dimension,* (ii) A, B *be Hanel bases for* $\mathscr{X}$, $\mathscr{Y}$ *respectively,* (iii) T_0 *be a one-one function on* A *onto* B. *Then* T_0 *has a one-one linear extension* T *on* $\mathscr{X}$ *onto* $\mathscr{Y}$.

Proof. Since A is linearly independent, therefore, cf. Day [1, p. 5, Lma. 1], T_0 has a linear extension on $\langle A \rangle$ onto $\langle B \rangle$, i.e. on $\mathscr{X}$ onto $\mathscr{Y}$. Also since T_0 is one-one and B is linearly independent, T has to be one-one. □

Proof of Proposition 1.

By (i), $\alpha = \dim \mathscr{A} \geq \aleph_0$, and by (ii), $\dim\langle x_0 \rangle = 1$; hence

(1) $$\dim\langle x_0 \rangle^{\perp} = \dim \mathscr{A} = \alpha.$$

Let

(2) $$A =_d \text{a Hamel basis for } \{x_0\}^{\perp}.$$

Then obviously

(3) $$A^* =_d A \cup \{x_0\} = \text{a Hamel basis for } \mathscr{A}.$$

Now let $\mathscr{H}$ be a Hilbert space over $\mathbb{F}$ of dimension α and let g be a *discontinuous* linear functional on $\mathscr{H}$ to $\mathbb{F}$.† Then by Lemma 1(b) & (a),

(4) $$\mathscr{N}_g \subsetneq \text{cls } \mathscr{N}_g = \mathscr{H},$$

and $\exists\, u_0 \in \mathscr{H}$ such that

(5) $$\mathscr{N}_g + \langle u_0 \rangle = \mathscr{H} \ \&\ \mathscr{N}_g \cap \langle u_0 \rangle = 0.$$

Now let

(6) $$B = \text{a Hamel basis for } \mathscr{N}_g.$$

Then by (5)

(7) $$B^* = B \cup \{u_0\} = \text{a Hamel basis for } \mathscr{H}.$$

Now since $\mathscr{H}$ and $\{x_0\}^{\perp}$ have the same (ortho-normal) dimension, viz. α, therefore they have the same Hamel dimension, say β. Thus, cf. (7) and (3), card $B^* = \beta =$ card A^*. Also obviously card $B =$ card $B^* =$ card $A^* =$ card A. Hence $\exists$ a one-one function T_0 on A^* onto B^* such that

(8) $$T_0(A) = B \ \&\ T_0(x_0) = u_0.$$

We thank the Referee for this information, and other suggestions.

† The existence of such g is ensured by a well-known argument using Hamel basis.

But by Lemma 2, T_0 has a one-one linear extension T on $\langle A^* \rangle$ onto $\langle B^* \rangle$, i.e. by (3) and (7), on $\mathscr{A}$ onto $\mathscr{H}$. Also from (2), (8) and (6) we conclude that

(9) $$T(\{x_0\}^\perp) = T(\langle A \rangle) = \langle T(A) \rangle = \langle B \rangle = \mathscr{N}_g.$$

Now we define $((\cdot, \cdot))$ on $\mathscr{A} \times \mathscr{A}$ by

(10) $$\forall x, y \in \mathscr{A}, \qquad ((x, y)) =_d (T(x), T(y))_{\mathscr{H}}.$$

This just transplants, linearly and biunivocally, the inner product for $\mathscr{H}$ to $\mathscr{A}$. Since $\mathscr{H}$ is a Hilbert space under $(\cdot, \cdot)_{\mathscr{H}}$, it follows that $\mathscr{A}$ is a Hilbert space under $((\cdot, \cdot))$.

It only remains to show that f_0 is discontinuous under the topology induced by the new inner product. For this, it suffices to show that

(I) $$\exists (y_n)_1^\infty \text{ in } \mathscr{A} \ni \quad \text{as } n \to \infty,\ y_n \to x_0 \text{ but } f_0(y_n) \not\to f_0(x_0).$$

Proof of (I). Since $u_0 \in \mathscr{H} =_d \text{cls}\, \mathscr{N}_g$, cf. (4), therefore $\exists (v_n)_1^\infty$ in $\mathscr{N}_g \ni v_n \to u_0$. Let $y_n =_d T^{-1}(v_n)$. Then by (10) and (8),

$$\| y_n - x_0 \| = | T(y_n) - T(x_0) |_{\mathscr{H}} = | v_n - u_0 |_{\mathscr{H}} \to 0.$$

But since $v_n \in \mathscr{N}_g$, therefore by (9), $y_n \in \{x_0\}^\perp$, therefore $f_0(y_n) =_d (y_n, x_0) = 0$, whereas $f_0(x_0) =_d | x_0 |^2 \neq 0$. Thus (I) is established. □

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