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KAC FUNCTIONALS OF DIFFUSION PROCESSES APPROXIMATING CRITICAL BRANCHING PROCESSES

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Abstract. A critical Galton-Watson process with immigration, initiated by $[xn]$ objects, tends in the sense of finite-dimensional distributions, as $n \longrightarrow \infty$, to a diffusion process $\{X_x(t),\ t \geqslant 0\}$ on $[0, \infty)$ with drift parameter $\mu(v) = \gamma > 0$ and diffusion parameter $\sigma^2(v) = 2v$, if additionally conditioned on having the size $[yn]$ at time n to a diffusion bridge

$$\{X_{x,y}(t),\ 0 \leqslant t \leqslant 1\} \quad \text{with} \quad X_{x,y}(0) = x \quad \text{and} \quad X_{x,y}(1) = y.$$

In this paper we study Kac functionals of these processes obtaining diffusion counterparts of limit theorems for the total progeny of critical Galton-Watson processes.

1. Introduction

Let X_n be the number of particles at time n in a critical Galton-Watson process with immigration and f and h be the probability generating functions of its offspring and immigration distributions respectively. Denote the variance of the offspring distribution by 2α and let $\gamma = \alpha^{-1}h'(1-)$. Imposing some further assumptions it was shown [7] that, in the sense of finite-dimensional distributions,

$$(1) \qquad \left\{\frac{X_{[nt]}}{\alpha n};\ 0 \leqslant t \leqslant 1 \,|\, X_0 = [\alpha x n + \mathrm{o}(n)],\ X_n = [\alpha y n + \mathrm{o}(n)]\right\},$$

converges, as $n \longrightarrow \infty$, to a nonhomogeneous diffusion process $\{X_{x,y}(t),\ 0 \leqslant t \leqslant 1\}$ on $[0, \infty)$ with initial state $X_{x,y}(0) = x$, final state $X_{x,y}(1) = y$, infinitesimal variance $\sigma^2(v, s) = 2v$ and a rather complicated drift parameter (see [7]). Its n-dimensional density function is given in Lemma 4. In Section 3 we find an explicit expression for the Laplace transform

$$(2) \qquad K(x, y, \lambda, t) = E(\exp\{-\lambda \textstyle\int_0^t X_{x,y}(s)\, ds\}), \qquad 0 \leqslant t \leqslant 1,$$

of the diffusion bridge $\{X_{x,y}(t),\ 0 \leqslant t \leqslant 1\}$.

In Section 4 we study Kac functionals of the form

$$(3) \qquad K_g(x, \lambda, t) = E[\exp\{-\lambda \textstyle\int_0^t X_x(s)\, ds\} g(X_x(t))], \qquad t \geqslant 0,$$

where $\{X_x(t), t \geqslant 0\}$ denotes the weak limit $(n \longrightarrow \infty)$ of

$$\left\{\frac{X_{[nt]}}{\alpha n};\ t \geqslant 0 \,|\, X_0 = [\alpha x n + \mathrm{o}(n)]\right\},$$

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and where g may be a power function (exponent $> -\gamma$) or the exponential function. The multivariate density function of $\{X_x(t),\ t \geqslant 0\}$ is stated in Lemma 5 and may be used to identify the process as a time homogeneous diffusion process on $[0, \infty)$, with initial state $X_x(0) = x$, drift parameter $\mu(v) = \gamma$, diffusion parameter $\sigma^2(v) = 2v$ and no boundary conditions adjoined. (See also [3]).

At this point we mention that the distribution of $X_x(t)$ may be obtained from that of $X_{x,y}(t)$ by randomizing the quantity y (see Remark 2), more precisely, treating it as a random variable with the density function f_x of $X_x(1)$, given by

$$f_x(v) = \left(\frac{v}{x}\right)^{(\gamma-1)/2} \exp\{-x - v\} I_{\gamma-1}(2\sqrt{xv}), \qquad v \geqslant 0, \tag{4}$$

where $I_{\gamma-1}$ is the modified Bessel function. Whence integrating (2) with respect to $f_x(y)$ should lead to (3) with $g \equiv 1$. In this spirit is Corollary 1 of Section 3.

It is obvious that the distribution of the process $X_{x,y}(t)$, just as that of $X_x(t)$, depends on the parameter γ, the case $\gamma = 2$ playing a special rôle in other contexts: If in (1) $\{X_n\}$ is a (sufficiently nice) critical Galton-Watson process with variance 2α (i.e. $E(X_1^2 \mid X_0 = 1) = 2\alpha + 1$), then (1) is found [7] to be approximated by $\{X_{x,y}(t),\ 0 \leqslant t \leqslant 1\}$ with parameter $\gamma = 2$. Moreover the distribution of the diffusion process gotten by Lamperti & Ney [5] as an approximation of the critical Galton-Watson process with finite survival time (i.e. by conditioning on its not being extinct at time n and on having initial size $[\alpha xn]$) may be obtained from that of $X_{x,y}(t)$ with $\gamma = 2$ by randomization of y. For more details see Section 5 where we deal with the Laplace transform of the integral of the Lamperti-Ney diffusion. Finally, $\{X_x(t), t \geqslant 0\}$ with drift parameter $\gamma = 2$, approximates the critical Galton-Watson process with infinite survival time, the so-called Q-process [5].

Pakes ([9], [10]) has proven limit theorems for the total progeny of the critical Galton-Watson process (with and without immigration) and the critical Q-process. Their diffusion analogues will be presented in Section 5, the relations of the preceding paragraph connecting the different processes being relevant.

The observation that in the case $\gamma = n/2$, $n = 1, 2, \cdots$, $X_0(t)$ equals in distribution $\frac{1}{2}(B_1^2(t) + \cdots + B_n^2(t))$, where $\{B_k(t),\ t \geqslant 0\}$ are independent standard Brownian motions, is used in Section 6 to derive a generalization of Kac's formula giving the distribution of the integral of the squared Brownian motion process.

For a detailed survey of the work done by several authors on integral functionals of stochastic processes in various domains of applications the reader may consult Puri [11].

2. Some preliminary results

The proofs of the following three lemmas are straightforward and will be omitted.

LEMMA (1). *Let a, b and c be constants. If the sequence $\{c_n\}$ satisfies the recurrence relation*

$$c_n = (ab^2 + 2b)c_{n-1} - b^2 c_{n-2}, \qquad n = 3, 4, \cdots$$

with

$$c_1 = abc + b + c$$

$$c_2 = (ab^2 + 2b)c_1 - cb$$

then

$$c_n = cb^{n-1} \sum_{j=0}^{n} \binom{n+j}{2j} (ab)^j + b^n \sum_{j=0}^{n-1} \binom{n+j}{2j+1} (ab)^j, \tag{5}$$

$$n = 1, 2, \cdots$$

In the case $b = c$ the solution reduces to

$$c_n = b^n \sum_{j=0}^{n} \binom{n+j+1}{2j+1} (ab)^j, \qquad n = 1, 2, \cdots . \tag{6}$$

LEMMA (2). *Let $a', b' > 0$ and c_n be as in* (5) *with*

$$a = \frac{a'}{n}, \qquad b = \frac{b'}{n}$$

and $c \geqslant 0$. Set $w = \sqrt{a'b'}$. Then

(i) $$\lim_{n\to\infty} \frac{c_n}{b^{n-1}} = c \cdot \cosh(w) + \sqrt{b'/a'}\sinh(w).$$

(ii) $$\lim_{n\to\infty} \left(\frac{c_{n-1}}{c_n} - \frac{1}{b}\right) = -\frac{\cosh(w) + c\sqrt{a'/b'}\sinh(w)}{c \cdot \cosh(w) + \sqrt{b'/a'} \cdot \sinh(w)}.$$

An obvious consequence of this result is stated separately as

LEMMA (3). *Let a', b', a, b, w be as in the preceding lemma. If c_n has the form* (6) *then*

(i) $$\lim_{n\to\infty} \frac{c_n}{b^{n-1}} = \sqrt{b'/a'}\sinh(w)$$

(ii) $$\lim_{n\to\infty} \left(\frac{c_{n-1}}{c_n} - \frac{1}{b}\right) = -\sqrt{a'/b'} \cdot \coth(w)$$

In what follows it is of fundamental importance to know the multidimensional distribution functions of the processes $X_{x,y}(t)$ and $X_x(t)$. To state the formulae we introduce some notation. From now on let

$$0 = t_0 < t_1 < \cdots < t_n, \qquad d_n = t_n - t_{n-1}, \qquad n = 1, 2, \cdots$$

and define for nonnegative $x = x_0, x_1, \cdots, x_n$

$$(7) \quad g_x(x_1, \cdots, x_n) = \prod_{k=1}^{n} \frac{1}{d_k} I_{\gamma-1}\left(\frac{2}{d_k}\sqrt{x_{k-1}x_k}\right) \cdot \exp\left\{-\frac{1}{d_k}(x_{k-1}+x_k)\right\},$$

where I_ρ is the modified Bessel function of order ρ, given by

$$I_\rho(2v) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{\Gamma(k+1+\rho)} v^{2k+\rho}.$$

LEMMA (4). ([7]). *Let* $(x_0 =)\ x, y > 0$, $t_n < 1$ *and set* $d_{n+1} = 1 - t_n$. *The joint density function of* $(X_{x,y}(t_1), \cdots, X_{x,y}(t_n))$ *is*

$$(8) \quad f_{x,y}(x_1, \cdots, x_n) = [d_{n+1} I_{\gamma-1}(2\sqrt{xy})]^{-1} \cdot I_{\gamma-1}\left(\frac{2}{d_{n+1}}\sqrt{yx_n}\right) \cdot \exp\left\{x + y - \frac{y + x_n}{d_{n+1}}\right\} \cdot g_x(x_1, \cdots, x_n).$$

Remark 1. We do not give the explicit form of the densities $f_{0,y}$, $f_{x,0}$ and $f_{0,0}$ but mention that they may be obtained as limits of (8), sending x resp. y resp. x and y to zero.

LEMMA (5). *Let* $x_0 = x \geqslant 0$. *The random vector* $(X_x(t_1), \cdots, X_x(t_n))$ *has density function*

$$(9) \quad f_x(x_1, \cdots, x_n) = \left[\frac{x_n}{x}\right]^{(\gamma-1)/2} \cdot g_x(x_1, \cdots, x_n), \qquad x > 0.$$

Furthermore

$$f_0(x_1, \cdots, x_n) = \lim_{x\downarrow 0} f_x(x_1, \cdots, x_n).$$

PROOF. As $m \to \infty$ apply the local limit theorem for the Galton-Watson process with immigration [6] to

$$P\left(\frac{X_{[mt_1]}}{\alpha m} < x_1, \cdots, \frac{X_{[mt_n]}}{\alpha m} < x_n \;\middle|\; X_0 = [\alpha x m + o(n)]\right)$$

to get (9) immediately.

Remark 2. Consulting e.g. [8] it is easy to verify that

$$f_x(x_1, \cdots, x_n) = \int_0^\infty f_{x,y}(x_1, \cdots, x_n) f_x(y)\, dy,$$

where f_x is the density of X_x (1) stated in (4). Hence the distributions of the processes $X_x(t)$ and $X_{x,y}(t)$ are connected by randomization.

3. The diffusion bridge $X_{x,y}(t)$.

Our approach to find the integral functional (2) is the most direct one. As suggested in [1] we approximate the integral under consideration by the

Riemann sum

$$\frac{t}{n} \sum_{k=1}^{n} X_{x,y}\left(\frac{kt}{n}\right)$$

and obtain (2) as the limit of the multidimensional Laplace transform

$$L_{n,x,y}(\theta_1, \cdots, \theta_n; t_1, \cdots, t_n) = E(\exp\{-\sum_{k=1}^{n} \theta_k X_{x,y}(t_k)\}) \tag{10}$$

of the $X(kt/n)$ with $\theta_k = \lambda t/n$ and $t_k = kt/n$, $k = 1, \cdots, n$.

We concentrate on the case $x > 0$, $y > 0$. The density function given in Lemma 4 admits writing (10) as an iterated integral. Carrying out the integrations in the order $x_1, x_2, \cdots, x_n$ (the necessary integral formulae may be found e.g. in [8]) leads to

$$\begin{aligned} L_{n,x,y}(\theta_1, \cdots, \theta_n; t_1, \cdots, t_n) &= \frac{e_n}{s_n} \cdot I_{\gamma-1}\left(2\frac{e_n}{s_n}\sqrt{xy}\right)[I_{\gamma-1}(2\sqrt{xy})]^{-1} \\ &\cdot \exp\left\{y\left(1 - \frac{1}{d_{n+1}} + \frac{d_n s_{n-1}}{d_{n+1}s_n}\right) + x + x\psi_n(d_1, \cdots, d_{n+1})\right\} \end{aligned} \tag{11}$$

with $e_n = d_2 \cdots d_n$,

$$\psi_n(d_1, \cdots, d_{n+1}) = -\frac{1}{d_1} + \frac{d_2}{d_1 s_1} + \frac{d_2 d_3}{s_1 s_2} + \sum_{r=1}^{n-1} \frac{d_{r+1}d_{r+2}}{s_r s_{r+1}} \cdot \prod_{j=2}^{r} d_j^2, \tag{12}$$

and the s_n satisfying the recurrence relation

$$s_k = (\theta_k d_k d_{k+1} + d_k + d_{k+1})s_{k-1} - d_{k-1}d_{k+1}s_{k-2}, \qquad k = 3, 4, \cdots, n, \tag{13}$$

where

$$s_1 = \theta_1 d_1 d_2 + d_1 + d_2$$

and

$$s_2 = (\theta_2 d_2 d_3 + d_2 + d_3)s_1 - d_1 d_3.$$

Reversing the order of integration in (10) yields

$$\begin{aligned} L_{n,x,y}(\theta_1, \cdots, \theta_n; t_1, \cdots, t_n) &= \frac{e_n}{r_n} I_{\gamma-1}\left(2\frac{e_n}{r_n}\sqrt{xy}\right)[I_{\gamma-1}(2\sqrt{xy})]^{-1} \\ &\cdot \exp\left\{x\left(1 - \frac{1}{d_1} + \frac{d_2 r_{n-1}}{d_1 r_n}\right) + \xi_n(y, d_1, \cdots, d_{n+1})\right\} \end{aligned} \tag{14}$$

where $\xi_n(y, d_1, \cdots, d_{n+1})$ stands for an expression (equally complicated looking as that for ψ_n in (12)) not depending on x and r_n being recursively determined by

$$r_k = (\theta_1 d_1 d_2 + d_1 + d_2)r_{k-1} - d_1 d_3 r_{k-2}, \qquad k = 3, 4, \cdots, n,$$

with

$$r_1 = \theta_n d_n d_{n+1} + d_n + d_{n+1}$$

and

$$r_2 = (\theta_{n-1} d_{n-1} d_n + d_{n-1} + d_n) r_1 - d_{n-1} d_{n+1},$$

giving us the possibility to avoid a direct analysis of the asymptotic behaviour of ψ_n in (12). To see this set

$$\theta_1 = \cdots = \theta_n = \theta = \frac{\lambda t}{n}, \qquad d_1 = \cdots = d_n = d = \frac{t}{n}, \qquad d_{n+1} = 1 - t,$$

$$t \leq 1,$$

and observe that $\{r_n\}$ forms a sequence of the type appearing in Lemma 1 with $a = \theta$, $b = d$ and $c = 1 - t$.
Hence Lemma 2 (i) gives

$$\text{(15)} \qquad \lim_{n\to\infty} \frac{e_n}{r_n} = d(\lambda, t),$$

where we put

$$\text{(16)} \qquad d(\lambda, t) = \left[(1 - t)\cosh(t\sqrt{\lambda}) + \frac{1}{\sqrt{\lambda}} \sinh(t\sqrt{\lambda})\right]^{-1}$$

On the other hand, inspection of (13) shows that Lemma 1 with $a = \theta$, $b = c = d$ applies to $s_1, \cdots, s_{n-1}$, yielding

$$\text{(17)} \qquad s_k = d^k \, \Sigma_{j=0}^{k} \binom{k + j + 1}{2j + 1} (\theta d)^j, \qquad k = 1, 2, \cdots, n - 1.$$

Substituting in (13) we obtain

$$\text{(18)} \quad s_n = d^n (1 + \theta(1 - t)) \, \Sigma_{j=0}^{n-1} \binom{n + j}{2j + 1} (\theta d)^j + d^{n-1}(1 - t) \, \Sigma_{j=0}^{n-1} \binom{n + j - 1}{2j} (\theta d)^j$$

to conclude that

$$\text{(19)} \qquad \lim_{n\to\infty} \frac{e_n}{r_n} = \lim_{n\to\infty} \frac{e_n}{s_n}.$$

But this together with (11) and (14) implies that

$$\text{(20)} \qquad \lim_{n\to\infty} \psi_n(d_1, \cdots, d_{n+1}) = \lim_{n\to\infty} \left(-\frac{1}{d_1} + \frac{d_2 r_{n-1}}{d_1 r_n}\right).$$

The limit on the right-hand side of this relation is dealt with in Lemma 2 (ii) with $a = \theta$, $b = d$ and $c = 1 - t$.

Putting

$$e(\lambda, t) = \cosh(t\sqrt{\lambda}) + \sqrt{\lambda} \cdot (1 - t)\sinh(t\sqrt{\lambda})$$

and recalling (16) we can formulate

LEMMA (6). *With the foregoing notations*

$$\lim_{n\to\infty}\psi_n(d_1, \cdots, d_{n+1}) = \begin{cases} -\sqrt{\lambda}\coth(\sqrt{\lambda}) & \text{if } t = 1 \\ -d(\lambda, t)e(\lambda, t) & \text{if } 0 < t < 1. \end{cases}$$

In the case $xy > 0$ the following theorem follows from (11), (15), (17), (18), (19) and Lemma 6. The cases $xy = 0$ may be obtained by passing to the respective limits (see Remark 1).

THEOREM (1). *The Kac functional* (2) *of the diffusion bridge* $\{X_{x,y}(t), 0 \leqslant t \leqslant 1\}$ *is*

$$K(x, y, \lambda, t) = \begin{cases} d(\lambda, t)[I_{\gamma-1}(2\sqrt{xy})]^{-1}I_{\gamma-1}(2d(\lambda, t)\sqrt{xy}) \\ \quad \cdot \exp\left\{x[1 - d(\lambda, t)e(\lambda, t)] + \dfrac{y}{1-t} \cdot\left[\dfrac{d(\lambda, t)}{\sqrt{\lambda}}\sinh(t\sqrt{\lambda}) - t\right]\right\} \\ \qquad \text{if } t < 1, \quad xy > 0, \\ \sqrt{\lambda} \cdot I_{\gamma-1}\left(\dfrac{2\sqrt{xy\lambda}}{\sinh(\sqrt{\lambda})}\right)[\sinh(\sqrt{\lambda})I_{\lambda-1}(2\sqrt{xy})]^{-1} \\ \quad \cdot \exp\{(x + y)[1 - \sqrt{\lambda}\coth(\sqrt{\lambda})]\} \\ \qquad \text{if } t = 1, \quad xy > 0, \\ [d(\lambda, t)]^{\gamma} \cdot \exp\left\{\dfrac{y}{1-t}\left[\dfrac{d(\lambda, t)}{\sqrt{\lambda}}\sinh(t\sqrt{\lambda}) - t\right]\right\} \\ \qquad \text{if } t < 1, \quad x = 0, y > 0, \\ \left[\dfrac{\sqrt{\lambda}}{\sinh(\sqrt{\lambda})}\right]^{\gamma}\exp\{y[1 - \sqrt{\lambda}\coth(\sqrt{\lambda})]\} \\ \qquad \text{if } t = 1, \quad x = 0, \quad y > 0, \\ [d(\lambda, t)]^{\gamma}\exp\{x[1 - d(\lambda, t)\cdot e(\lambda, t)]\} \\ \qquad \text{if } t \leqslant 1, \quad x > 0, \quad y = 0, \\ [d(\lambda, t)]^{\gamma} \qquad \text{if } t \leqslant 1, \quad x = y = 0. \end{cases}$$

COROLLARY (1) *Let* f_x *be as defined in* (4). *Then*

$$\int_0^\infty K(x, y, \lambda, 1)f_x(y)\,dy = (\operatorname{sech}\sqrt{\lambda})^{\gamma}\exp\{-x\sqrt{\lambda}\tanh\sqrt{\lambda}\}, \qquad x \geqslant 0.$$

Proof. Consult e.g [8] and use the fact that for all x, $1 + \sinh^2(x) = \cosh^2(x)$.

Taking into account what we said in Section 1 about the randomization relation connecting $X_{x,y}(t)$ and $X_x(t)$ it is obvious that we will meet again with the expression of Corollary 1 in the next section when studying Kac functionals of the form (3).

4. The diffusion $X_x(t)$.

In this section we investigate the Kac functional (3) of the diffusion process $\{X_x(t),\ t \geqslant 0\}$. Adopting the same method presented in the preceding section we study the limit of

$$L_{n,x,g}(\theta_1, \cdots, \theta_n; t_1, \cdots, t_n) = E[\exp\{-\ \Sigma_{k=1}^{n}\ \theta_k X_x(t_k)\} \cdot g(X_x(t))],$$

confining ourselves to the case

$$\theta_k = \frac{\lambda t}{n},\ t_k = \frac{kt}{n},\ t > 0, \qquad k = 1, \cdots, n.$$

Denoting the resulting Laplace transform $L_{n,x,g}(\lambda, t)$, setting $\theta = \dfrac{\lambda t}{n}$, $d = \dfrac{t}{n}$ and with the aid of Lemma 5 and [8] carrying out $n - 1$ integrations leads to

$$\begin{aligned} L_{n,x,g}(\lambda, t) &= (\sqrt{x})^{1-\gamma}\,\frac{d^{n-2}}{s_{n-1}}\cdot\exp\{x\psi_{n-1}(d, \cdots, d)\} \\ &\cdot \textstyle\int_0^\infty \exp\left\{-v\left(\theta + \dfrac{1}{d} - \dfrac{s_{n-2}}{s_{n-1}}\right)\right\} I_{\gamma-1}\left(2\,\dfrac{d^{n-2}}{s_{n-1}}\,\sqrt{xv}\right)(\sqrt{v})^{\gamma-1}\cdot g(v)\,dv, \end{aligned} \tag{21}$$

with ψ_{n-1} the function introduced in (12) and s_{n-1}, s_{n-2} obeying the recursion rule (13).

The integral can be evaluated in closed form for functions g such as

(i) $g(v) = \exp\{-\rho v\}, \qquad \rho \geqslant 0$

(ii) $g(v) = v^s, \qquad s > -\gamma.$

We first treat the *case* (*i*) which is of particular interest as (3) then gives the joint distribution of $X_x(t)$ and $\int_0^t X_x(w)\,dw$, $t > 0$. Performing the integration in (21) and setting

$$q_n = (1 + (\theta + \rho)d)s_{n-1} - ds_{n-2}$$

one verifies that

$$L_{n,x,g}(\lambda, t) = \left[\frac{d^{n-1}}{q_n}\right]^\gamma \exp\left\{x\left[\psi_{n-1}(d, \cdots, d) + \frac{d^{2n-3}}{s_{n-1}q_n}\right]\right\}.$$

The sequence $\{s_n\}$ and its associated sequence $\{r_n\}$ in (20) being of the form (6) we conclude from Lemma 3 and (20)

THEOREM (2). *For all $x, t \geqslant 0$*

$$E(\exp\{-\rho X_x(t) - \lambda \textstyle\int_0^t X_x(w)\, dw\}) = \tag{22}$$

$$\left[\frac{\rho}{\sqrt{\lambda}} \sinh(t\sqrt{\lambda}) + \cosh(t\sqrt{\lambda})\right]^{-\gamma} \cdot \exp\left\{-x \frac{\lambda \sinh(t\sqrt{\lambda}) + \rho\sqrt{\lambda} \cosh(t\sqrt{\lambda})}{\rho \sinh(t\sqrt{\lambda}) + \sqrt{\lambda} \cosh(t\sqrt{\lambda})}\right\}.$$

Remark 3. **a)** In the case $\lambda = 0$ (22) reduces to

$$(1 + \rho t)^{-\gamma} \exp\left\{-x \frac{\rho}{1 + \rho t}\right\}$$

which for $t = 1$ is the Laplace transform of f_x in (4).

b) If $\rho = 0$, $t = 1$ (22) coincides with the expression gotten in Corollary 1.

Assume now that *g is of the form (ii)*. From (21) we obtain

$$L_{n,x,g}(t, \lambda) = \frac{\Gamma(s + \gamma)}{\Gamma(\gamma)} \left[\frac{d^{n-2}}{s_{n-1}}\right]^{\gamma} \cdot \left[\frac{ds_{n-1}}{p_n}\right]^{\gamma+s} \exp\{x\psi_{n-1}(d, \cdots, d)\}$$

$$\cdot {}_1F_1\left(\gamma + s; \gamma; \frac{xd^{2n-3}}{p_n s_{n-1}}\right),$$

where we put $p_n = (1 + \theta d)s_{n-1} - ds_{n-2}$ and ${}_1F_1$ denotes the confluent hypergeometric function. Just as above we get with the aid of Lemma 3 and (20).

THEOREM (3). *For all $x, t \geqslant 0$ and $s > -\gamma$*

$$E[\exp\{-\lambda \textstyle\int_0^t X_x(v)\, dv\}[X_x(t)]^s] = \frac{\Gamma(s + \gamma)}{\Gamma(\gamma)} [\operatorname{sech}(t\sqrt{\lambda})]^{\gamma}$$

$$\cdot\left[\frac{1}{\sqrt{\lambda}} \tanh(t\sqrt{\lambda})\right]^s \exp\{-x\sqrt{\lambda} \coth(t\sqrt{\lambda})\}\, {}_1F_1\left(\gamma + s; \gamma; \frac{2x\sqrt{\lambda}}{\sinh(2t\sqrt{\lambda})}\right).$$

Remark 4. It is known that the confluent hypergeometric function may be written in terms of the Laguerre polynomials (see e.g. [8])

$$ {}_1F_1(-n; \beta; v) = n!\, \frac{\Gamma(\beta)}{\Gamma(n + \beta)} \cdot L_n^{\beta-1}(v). \tag{23}$$

Being interested in moments of $X_x(t)$ we may set $\lambda = 0$ in Theorem 3 to obtain for integral $s > -\gamma$

$$E[X_x(t)]^s = \frac{\Gamma(s + \gamma)}{\Gamma(\gamma)} t^s \exp\left\{-\frac{x}{t}\right\} {}_1F_1\left(\gamma + s; \gamma; \frac{x}{t}\right)$$

$$= \frac{\Gamma(s + \gamma)}{\Gamma(\gamma)} t^s {}_1F_1\left(-s; \gamma; -\frac{x}{t}\right) \qquad \text{(see [8])}$$

$$= s!\, t^s L_s^{\gamma-1}\left(-\frac{x}{t}\right) \qquad \text{due to (23).}$$

This last expression reduces e.g. for $s = 1$ to

$$EX_x(t) = x + \gamma t.$$

Remark 5. It is well known (see e.g. [3]) that the Kac functional satisfies a partial differential equation which reads in our case

$$\frac{\partial}{\partial t} K_g(x, \lambda, t) = -\lambda x K_g(x, \lambda, t) + \gamma \frac{\partial}{\partial x} K_g(x, \lambda, t) + x \frac{\partial^2}{\partial x^2} K_g(x, \lambda, t), \tag{24}$$

with the initial condition $K_g(x, \lambda, 0) = g(x)$. Hence Theorems 2 and 3 state explicit solutions of (24).

5. The total progeny of a Galton-Watson process

Let $\{X_n\}$ be the critical Galton-Watson process with immigration as described in Section 1. Assume $X_0 = 0$. Pakes [10] studied the behaviour of

$$Y_n = \sum_{k=0}^{n} X_k,$$

the total number of individuals that have existed up to and including the n-th generation. He has shown that

$$\lim_{n\to\infty} E\left(\exp\left\{-\frac{\lambda}{\alpha n^2} \cdot Y_n\right\}\right) = [\operatorname{sech}(\sqrt{\lambda})]^{\gamma}. \tag{25}$$

As observed in Remark 3*b*) Theorem 2 states that

$$E(\exp\{-\lambda \textstyle\int_0^1 X_0(v)\, dv\}) = [\operatorname{sech}(\sqrt{\lambda})]^{\gamma},$$

hence constituting a diffusion counterpart of (25).

In Section 1 we pointed out that a critical Galton-Watson process $\{Z_n\}$ with $Z_0 = 1$, Var $Z_1 = 2\alpha$ and infinite survival time, the so-called (critical) Q-process converges weakly to a limiting diffusion with $\mu(v) = 2$, $\sigma^2(v) = 2v$ i.e. to $\{X_0(t), t \geq 0\}$ with parameter $\gamma = 2$ [5]. Whence we would expect that

$$\lim_{n\to\infty} E\left(\exp\left\{-\frac{\lambda}{\alpha n^2} \sum_{k=0}^{n} Z_k\right\} \middle| Z_\infty > 0\right) = [\operatorname{sech}(\sqrt{\lambda})]^2$$

which is indeed the case [9].

Also in [9] Pakes has proved the limit relation

$$\lim_{n\to\infty} E\left(\exp\left\{-\frac{\lambda}{\alpha n^2} \sum_{k=0}^{n} Z_k\right\} \middle| Z_n > 0\right) = \frac{2\sqrt{\lambda}}{\sinh(2\sqrt{\lambda})}, \tag{26}$$

whose diffusion analogue we produce in the following manner.

In (1) replace X_n by Z_n and recall what we said in Section 1: In the sense of finite-dimensional distributions (1) is approximated by $\{X_{x,y}(t), 0 \leq t \leq 1\}$

with $\gamma = 2$. On the other hand as shown by Lamperti & Ney [5],

$$\left\{\frac{Z_{[nt]}}{\alpha n};\ 0 \leq t \leq 1 \mid Z_0 = [\alpha x n + o(n)],\ Z_n > 0\right\} \tag{27}$$

tends weakly to a nonhomogeneous diffusion $\{Y_x(t),\ 0 \leq t \leq 1\}$ whose distribution may be derived from that of $\{X_{x,y}(t),\ 0 \leq t \leq 1\}$ with $\gamma = 2$ by randomization [7], treating y as random variable with density function ξ_x on $(0, \infty)$, defined by

$$\xi_x(z) = \begin{cases} \exp\{-z\} & \text{if } x = 0 \\ \sqrt{\dfrac{x}{z}}\, I_1(2\sqrt{xz})\cdot\dfrac{\exp\{-x - z\}}{1 - \exp\{-x\}} & \text{if } x > 0. \end{cases}$$

So (26) should be equal to

$$\int_0^\infty E(\exp\{-\lambda \textstyle\int_0^1 X_{0,\,y}(v)\, dv\})\,\xi_0(y)\, dy. \tag{28}$$

Indeed, with the aid of Theorem 1 we may write (28) as

$$\left[\frac{\sqrt{\lambda}}{\sinh(\sqrt{\lambda})}\right]^2 \int_0^\infty \exp\{-y\sqrt{\lambda}\coth(\sqrt{\lambda})\}\, dy$$

$$= \left[\frac{\sqrt{\lambda}}{\sinh(\sqrt{\lambda})}\right]^2 [\sqrt{\lambda}\coth(\sqrt{\lambda})]^{-1} = \sqrt{\lambda}[\sinh(\sqrt{\lambda})\cdot\cosh(\sqrt{\lambda})]^{-1}$$

$$= \frac{2\sqrt{\lambda}}{\sinh(\sqrt{\lambda})},$$

affirming the conjecture.

Similarly one can obtain the Laplace transform of the integral of the Lamperti-Ney process $\{Y_x(t),\ 0 \leq t \leq 1\}$ from the expressions of Theorem 1, integrating with respect to $\xi_x(y)$. Confining ourselves to the case $t = 1$ we can state

THEOREM (4). *Let* $\{Y_x(s),\ 0 \leq s \leq 1\}$ *be the Lamperti-Ney diffusion process approximating* (27). *Then*

$$E(\exp\{-\lambda \textstyle\int_0^1 Y_x(s)\, ds\})$$

$$= \begin{cases} \dfrac{2}{1 - \exp\{-x\}} \exp\{-x\sqrt{\lambda}\coth(2\sqrt{\lambda})\}\sinh\left(\dfrac{x\sqrt{\lambda}}{\sinh(2\sqrt{\lambda})}\right) & \textit{if } x > 0 \\ \dfrac{2\sqrt{\lambda}}{\sinh(2\sqrt{\lambda})} & \textit{if } x = 0. \end{cases}$$

Proof. Use e.g. formula 15.25 (multiplied by the missing factor $\exp\{-a^2/4p\}$) in [8] and the relation [2]

$$\gamma(1, 2x) = 2e^{-x}\sinh(x)$$

for the incomplete gamma function $\gamma(\cdot, \cdot)$.

6. A connection with Brownian motion

Let n be a natural number, $B(t) = (B_1(t), \cdots, B_n(t))$ be a standard n-dimensional Brownian motion process and set

$$W(t) = \tfrac{1}{2}(B_1^2(t) + \cdots + B_n^2(t)).$$

Assume $\gamma = n/2$. It is easy to see (use e.g. Theorem 2.1, p. 173 in [3]) that then $\{X_0(t), t \geq 0\}$ and $\{W(t), t \geq 0\}$ are equal in distribution. Clearly, this observation (together with Theorems 2 and 3) provides the possibility to deduce explicit expressions for Kac functionals of $W(t)$.

We give just one example assuming $n = 1$, whence $\gamma = \frac{1}{2}$. Suppose that $B(t)$ is started in $x \geq 0$. As this induces the initial condition $W(0) = x^2/2$ for $W(t) = B^2(t)/2$ we obtain from Theorem 2,

$$E(\exp\{-\lambda \textstyle\int_0^t B^2(v)\, dv\} \mid B(0) = x)$$

$$= E(\exp\{-2\lambda \textstyle\int_0^t X_{x^2/2}(v)\, dv\})$$

$$= \sqrt{\operatorname{sech}(t\sqrt{2\lambda})} \cdot \exp\{-\frac{x^2}{2} \cdot \sqrt{2\lambda} \cdot \tanh(t\sqrt{2\lambda})\}$$

which reduces for $t = 1$, $x = 0$ to Kac's formula [4].

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REFERENCES

[1] W. J. BÜHLER, AND B. MELLEIN, *Integrals of Birth and Death Processes*; J. Math. Biology **10** (1980) 205–207.

[2] H. EXTON, Handbook of Hypergeometric Integrals; Ellis Horwood Limited, Chichester, 1978.

[3] S. KARLIN, AND H. M. TAYLOR, A second course in stochastic processes, Academic Press, New York, 1981.

[4] H.-H. KUO, *Gaussian Measures in Banach Spaces*; Lecture Notes in Mathematics **463**, Springer-Verlag, Berlin, 1975.

[5] J. LAMPERTI, AND P. NEY, *Conditioned branching processes and their limiting diffusions*; Theory Prob. Appl. **13** (1968) 128–139.

[6] B. MELLEIN, *Local limit theorems for the critical Galton-Watson process with immigration*; Rev. Col. Mat. **XVI**, **1** (1982) 31–56.

[7] ———, *Diffusion limits of conditioned critical Galton-Watson processes,* Rev. Col. Mat. **XVI**, **3** (1982) 125–140.

[8] F. OBERHETTINGER, AND L. BADII, Tables of Laplace transforms, Springer-Verlag, Berlin, 1973.

[9] A. G. PAKES, *Some limit theorems for the total progeny of a branching process*; Adv. Appl. Prob. **3** (1971) 176–192.

[10] ———, *Further results on the critical Galton-Watson process with immigration*; J. Austral. Math. Soc. **13** (1972) 277–290.

[11] P. S. PURI, *A method for studying the integral functionals of stochastic processes with applications. I. Markov Chain case*; J. Appl. Prob. **8** (1971) 331–343.