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MAPPING THREE-MANIFOLDS INTO THE PLANE I

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Introduction

This paper is the beginning of our study of stable maps of compact three manifolds into the plane. If $f: M \to \mathbf{R}^2$ is such a map, we consider an auxiliary space W_f defined by identifying points in M if they belong to the same connected component of the fibre of f.

In §1, we study W_f in detail, showing that it is a two dimensional complex with very simple singularities.

In §2, we apply the preceding to the problem of the existence of an immersion F of M in $\mathbf{R}^4$ over f, but using unnecessarily strong restrictions.

Throughout we assume that M is orientable.

1.1. Stable Maps

In this paragraph we describe the stable maps from three dimensional manifolds into the plane. Throughout, we will denote by M, a compact orientable 3-manifold without boundary.

Let $C(M, \mathbf{R}^2)$ be the smooth maps of M into $\mathbf{R}^2$. For $f \in C(M, \mathbf{R}^2)$, the singular set of f, $S(f) = \{x \in M \mid \text{rank } Tf(x) < 2\}$. The stable maps, $\mathscr{S}(M, \mathbf{R}^2) \subseteq C(M, \mathbf{R}^2)$ are those whose multijet extensions satisfy the usual transversality conditions [Mather, G^2].

We give an equivalent description of $\mathscr{S}(M, \mathbf{R}^2)$.

Definition. $f \in \mathscr{S}(M, \mathbf{R}^2)$, if near each point $p \in S(f)$, and in some local coordinates centered at p and $f(p)$, f is one of the following:

$(L_0)(u, x, y) \to (u, x^2 + y^2)$, definite fold point or fold point.

$(L_1)(u, x, y) \to (u, x^2 - y^2)$, indefinite fold point or saddle point.

$(L_2)(u, x, y) \to (u, y^2 + ux - x^3/3)$, cusp point.

In addition, the following global conditions are satisfied:

(G_1) If p is a cusp point, then $f^{-1}(f(p)) \cap S(f) = \{p\}$.
(G_2) $f \mid S(f) - \{\text{cusp}\}$ is an immersion with normal crossings.
(In particular $f \mid S(f)$ has no triple points)

We let $S_0 = \{\text{definite fold points}\}$, $S_1 = \{\text{indefinite fold points}\}$, $C = \{\text{cusps}\}$. As an immediate consequence of these conditions, we see that for $f \in \mathscr{S}(M,$

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$\mathbf{R}^2$), $S(f)$ is a finite disjoint union of embedded circles in M, and C is a finite set of points. On any component of $S(f)$, the points of C separate arcs belonging to S_0 from those of S_1. Hence on every component of $S(f)$ there is an even number of cusps. A component of $S(f) - C$ is in either S_0 or S_1 and is called definite or indefinite, accordingly.

From now on we will write $\mathcal{S}$ for $\mathcal{S}(M, \mathbf{R}^2)$.

1.2. The Space W_f and the Local Structure at Simple Points

Let $f \in \mathcal{S}$. Define W_f as the quotient of M obtained by identifying two points of M if they are in the same connected component of the same fibre of f. Let $q:M \to W = W_f$ be the quotient map and let $\bar{f}:W \to \mathbf{R}^2$ be defined by $f = \bar{f}oq$.

Proposition (1). *If* $x \in S(f)$, *then* $q^{-1}(q(x)) \cap S(f) = \{x\}$ *if* $x \in S_0 \cup C$ *and if* $x \in S_1$, $q^{-1}(q(x)) = \{x\}$ *or* $\{x, x'\}$.

Proof: Condition (G_1) guarantees that $q^{-1}(q(x)) \cap S(f) = \{x\}$ if $x \in C$ and condition (G_2), guarantees that $f \mid S(f)$ has at most double points. From (L_0), the local form of f about a point of S_0, we see that locally the f preimage of a point $(V, Y) \in \mathbf{R}$ is just: $\{(V, x; y) \mid x^2 + y^2 = Y\}$. So the preimage of the origin is just the origin which is a connected component of the fibre. □

Definition. A point $x \in S(f)$ for which $q^{-1}(q(x)) = \{x\}$ is called a simple singular point.

By the preceding proposition, if $x \in S(f)$ is not simple, then $x \in S_1$.

Remarks

(1) Under these assumptions, it is clear that we are interested in exactly six normal forms: three types of singular points and *possibilities* for three crossings for pairs of singular points. As a matter of fact, the local structure for at least four of these is immediately clear, by using elementary knowledge of Morse singularities—using normal forms for pairs of singularities as well as for single ones. ([W]).

These structures will be analysed in this section.

(2) Since f is a submersion on $M - S(f)$ and a local diffeomorphism at a saddle or cusp point, hence locally open, it follows that $f \mid (M - S_0)$ is open.

(3) $W - q(S(f))$ is a smooth surface without boundary (not a compact surface), immersed via $\bar{f}$ into $\mathbf{R}^2$. Its smooth structure is pulled back by the local diffeomorphism $\bar{f} \mid (W - q(S(f)))$. Since the restriction of f $f \mid :M - f^{-1}(f(S(f))) \to f(M) - f(S(f))$ is a proper submersion, it follows that $q \mid q^{-1}(W - q(S(f))) \to W - q(S(f))$ is a smooth surjective submersion with fibre S^1—thus a trivial circle bundle—since M, hence $q^{-1}(W - q(S(f)))$ is orientable.

Proposition (2). *Let $x \in S_0$. There is a nbhd of x which is diffeomorphic to $I \times D_r$, where I is an open interval and D_r is a closed disc with radius r in $\mathbf{R}^2$. The components of the f-fibres in the nbhd are just the circles $t \times S_\epsilon$ where $S_\epsilon \subseteq D_r$ is the circle of radius ϵ, $t \in I$. The q-image of the nbhd $I \times D_r$ is just $I \times [0, r]$, where $q(t \times S_\epsilon) = (t, \epsilon) = f(t \times S_\epsilon)$.*

Proof. To see this, merely look at the normal form (L_0) at a point $x \in S_0$. Choosing a transverse arc $(0 \times J)$, $J = [-r^2, r^2]$ locally in the local coordinate expression for f we have

$$f^{-1}(0 \times J) = \{(0, x, y) \mid x^2 + y^2 \leq r^2\} = 0 \times D_r.$$

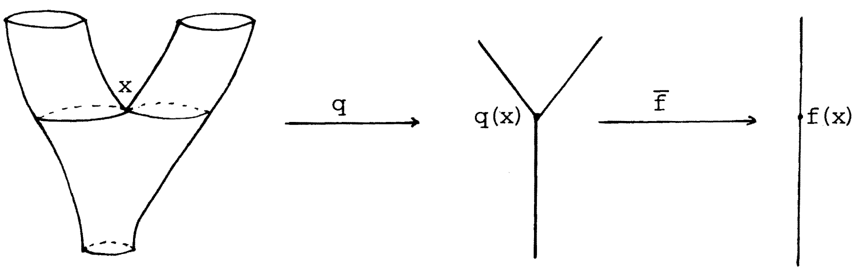
Obviously $f^{-1}(0, \epsilon)$ is empty if $\epsilon < 0$ and is $0 \times S_\epsilon$ if $\epsilon \geq 0$. □

Note. We have begun and will continue to take some liberties in the notation by suppressing some diffeomorphisms. We hope no confusion will result.

Proposition (3). *Let x be a simple point in S_1. There exists a nbhd $I \times T(x)$ of x such that $T(x)$ is a disc with two holes $\$_3$ (a sphere with three discs removed). The q-image of $T(x)$ is a Y with $q(x)$ the interior vertex. $I \times T(x)$ is mapped by q onto $I \times Y$ and by $\bar{f}$ onto $I \times J$, J a closed interval. In fact $f \mid I \times T(x)$ is equivalent to a product map.*

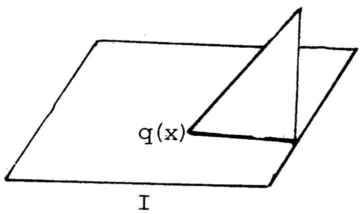
Proof. By the local form (L_2) it follows that $f{:}I \times T(x) \to I \times J$, $(t, z) \to (t, h_t(z))$ where $h_t{:}T(x) \to J$ is a Morse function which is constant on each boundary component and which has a single saddle singularity at x. Since $T(x)$ is orientable, it is $\$_3$ (see for example [H_i] §3, p. 201).

The statements about q and $\bar{f}$ are obvious from the following picture:



Since $h_{\bullet}$ is a smooth homotopy through stable maps each of which has a single saddle singular point whose image is $0 \in J$, it is easy to construct a diffeomorphism $\phi{:}I \times T(x) \to I \times T(x)$ such that $f \circ \phi(t, z) = (t, h_0(z))$ (see [G^2] Theorem 3.3, p. 104 or [L] Theorem 12.3, p. 73). □

Proposition (4). *Let $x \in C$. There exists a nbhd $I \times T(x)$ of x such that $T(x)$ is a cylinder $\$_2$ (a sphere with two discs removed). The map $f{:}I \times T(x) \to I \times J$, $(t, z) \to (t, h_t(z))$ is such that for $t < 0$ h_t has no singularities and for $t > 0$ h_t has one saddle and one extremum. As t approaches zero from the right,* *the saddle and the extremum coalesce. Thus the q-image of $I \times T(x)$ is a rectangle with a fin attached.*

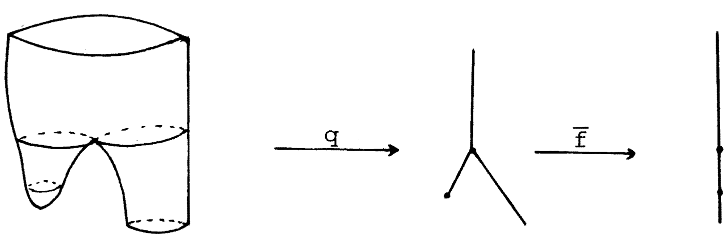


The heavy lines represent the q-images of arcs of $S(f)$ on either side of the cusp x. The top of the fin is in $q(S_0)$ and the base of the fin is in $q(S_1)$.

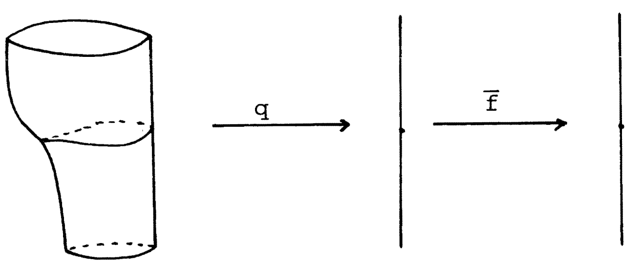
Proof: Here again, all that is needed is to examine the local form (L_2) at a cusp. In coordinates we have $(u, x, y) \to (u, y^2 + ux - x^3/3)$. If we label the coordinates in $\mathbf{R}^2$, (V, X) and take as $I \times J$ the coordinate rectangle $|V| < \epsilon$, $|X| \leq \delta$ with $9\delta^2 > 4\epsilon^3$, the image of the singular set of f is just $\{(V, X) \mid 4V^3 = 9X^2\}$.

Here the X-axis interval is a transverse arc to f. The singular arcs in M above $4V^3 = 9X^2$ are S_1-points for $X > 0$ and S_0-points for $X < 0$. We show $f \mid f^{-1}(V \times J)$ for $V > 0$, $V = 0$ and $V < 0$.

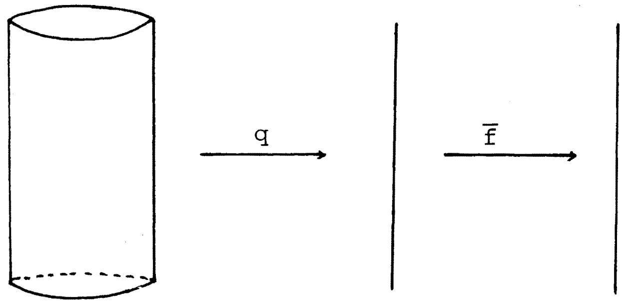
If $V > 0$:

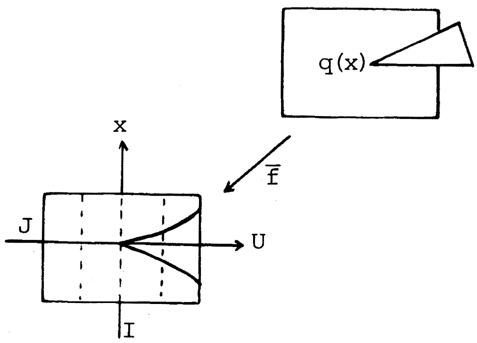


If V = 0 :



If V < 0 :





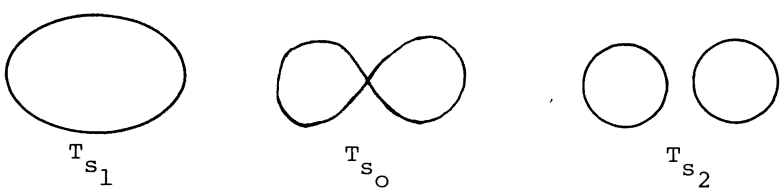
Thus the nbhd of W above $I \times J$, $q(I \times T(x))$, is the finned rectangle and the map $\bar{f}$ of this nbhd of $q(x)$ is the indicated projection. The three preceding figures show $f = \bar{f} \circ q$ in the pre-images of the three dotted verticle lines.

COROLLARY. *A component of $S(f)$ is either completely in S_0 or S_1 or it has an even number of cusps on it which divides the component into arcs of type S_0 and S_1.*

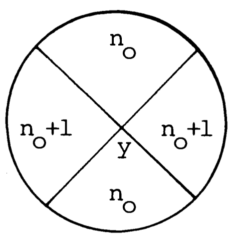
1.3. The Local Structure at Non Simple S_1-Points

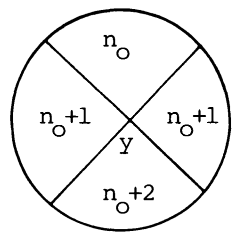
Let x be a simple indefinite point and let $h: T \to J$ be the associated Morse function (§1.2, Prop. 3). The fibres of this map are circles (one or two) except over the image s_0 of the one saddle singularity of h where it is a figure eight. If we let $h^{-1}(s) = T_s$, we see that the transition from $T_{s_1}(s_1 < s_0)$ to $T_{s_2}(s_2 > s_0)$ is effected by one surgery.

To analyze the non-simple S_1-points, we must consider all pairs of such surgeries on a circle.



Let x be a non-simple indefinite point. We know that $q^{-1}(q(x)) = \{x, x'\}$, and two small open arcs of S_1 say $c \ni x$, and $c' \ni x'$ have q-images in W and f-images in $\mathbf{R}^2$, which cross transversally at $q(x)$ in W and at $f(x) = y \in \mathbf{R}^2$. Take a small rectangular nbhd V of y in $\mathbf{R}^2$ and let U be the component of $f^{-1}(V)$ which contains x. We assume that $f(c \cup c')$ cuts V into four regions. Label each of these open regions of V with the number of components of U above it (i.e. the number of components of the fibre of $f \mid U$ above each point is constant in each open connected region of $V - f(c \cup c')$). Label the regions as indicated with n_0, the smallest. We know that the number of components in abutting regions differ by one. Thus there are only two possibilities:

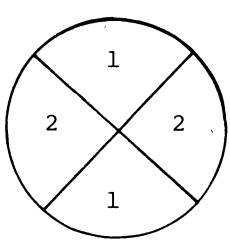




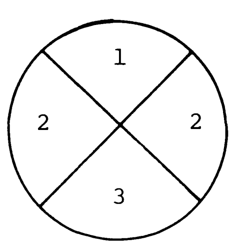
Lemma. *In the local description at a non-simple S_1 point $n_0 = 1$.*

Proof: Consider the picture of V with the regions of $V - f(c \cup c')$ labelled n_i, $i = 0, 1, 2, 3$ as above. If we follow the fibre above the arc passing from the n_0-region to the n_1-region and from the n_0-region to the n_2-region, two surgeries occur. Each surgery involves a single component of the fibre over the n_0-region, since any surgery involving two circles replaces two circles by one which would contradict the minimality of n_0. On the other hand, following the fibre above the points of a transverse arc travelling from the n_0-region through y to the n_3-region, both of the above mentioned surgeries, occur simultaneously above y. If the surgeries were performed on distinct circles, the $f \mid U$ pre-image of the transverse arc at x would not be connected—contradiction. Hence both surgeries—the one effecting the transition across the $(n_0 - n_1)$-arc as well as that across the $(n_0 - n_2)$-arc, both involve one circle in the fibre above the n_0-region. Hence $n_0 = 1$. □

Consequence. The non-simple S_1-points are described by the two diagrams:



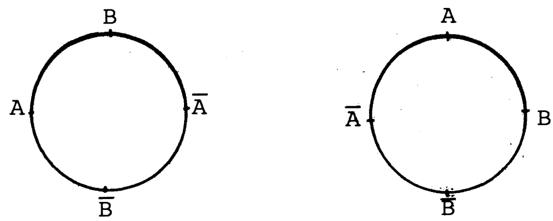
(1·2·2·1) or double cone point



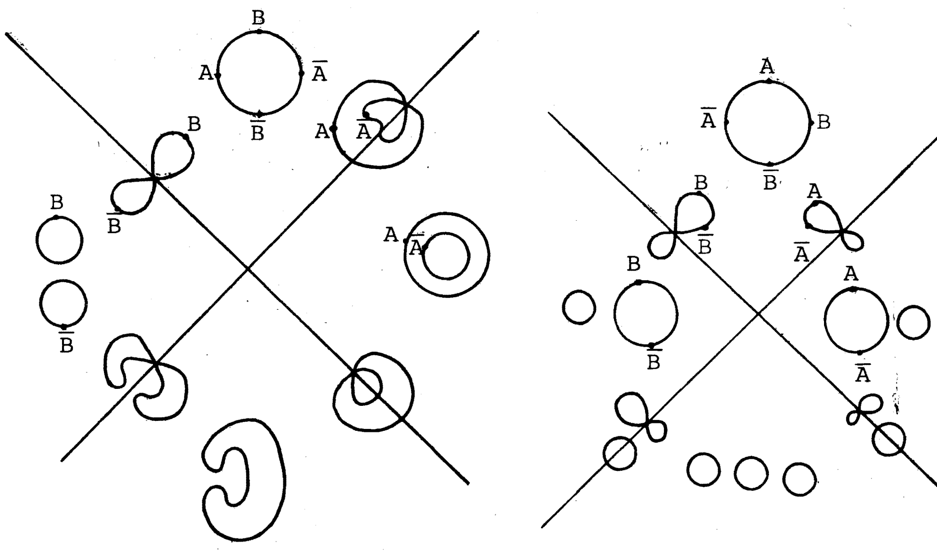
(1·2·2·3) or trident point

Again, in this case, $f: I \times T(x) \to \dot{I} \times J$, $(t, z) \to (t, h_t(z))$. $T(x)$ is a torus with two holes Υ_2, if x is a double cone point. If x is a trident point, the connected component of the inverse image of a transverse arc joining (1–3) regions, $T(x)$ is $\$_4$: a sphere with four discs removed. It is enough to consider two possible ways: two surgeries can be performed on one circle. If we orient the circle arbitrarily, a surgery replaces $\uparrow\downarrow$ by $\rightleftarrows$.

We indicate the paired points by $(A, \bar{A})$ and $(B, \bar{B})$; they can appear on the circle over n_0-region as follows:

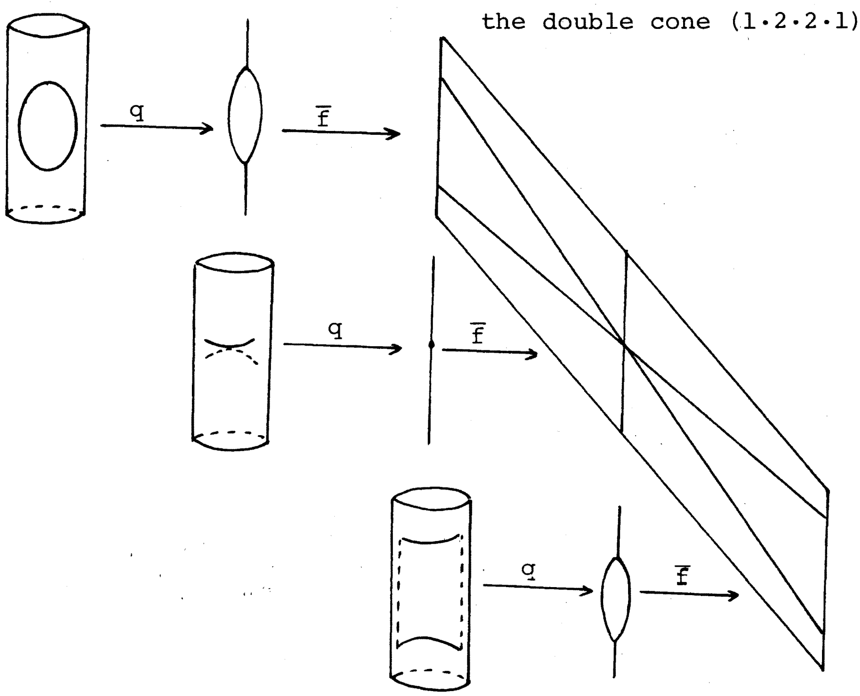


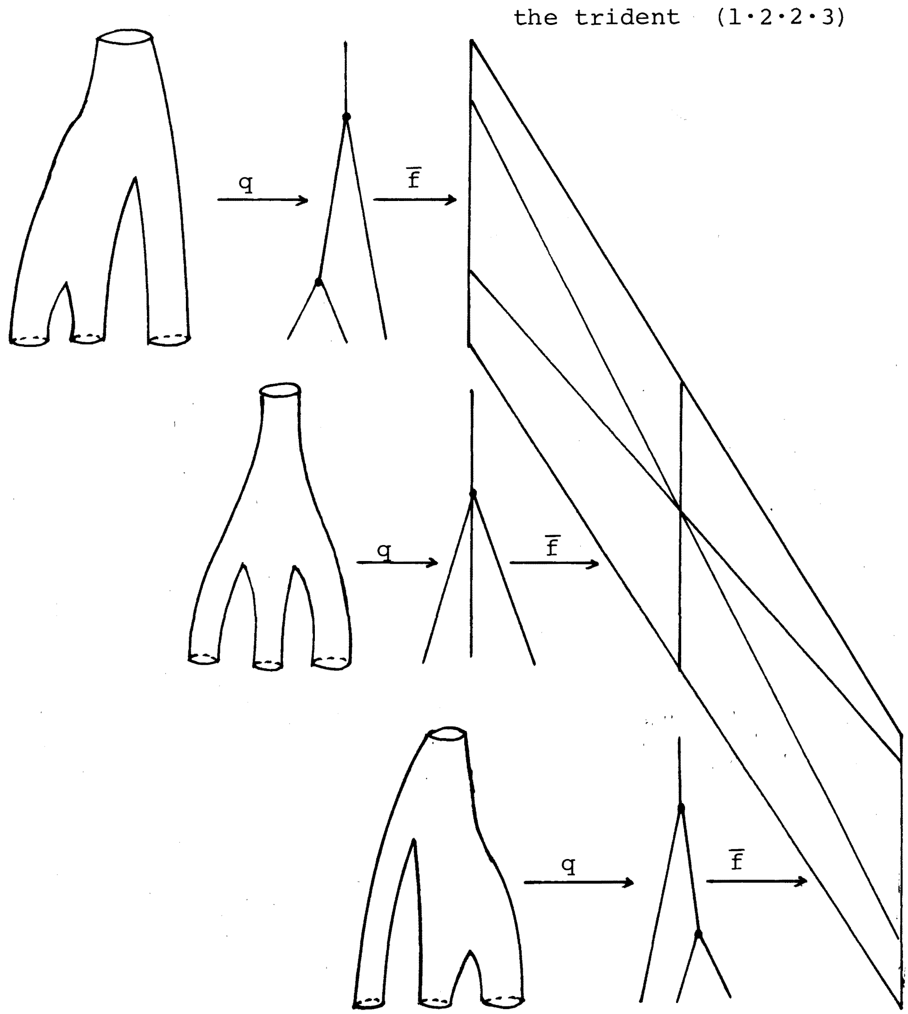
We follow these circles around the diagram $\otimes$.



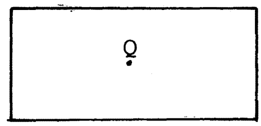
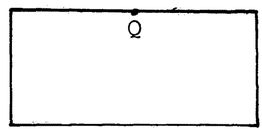
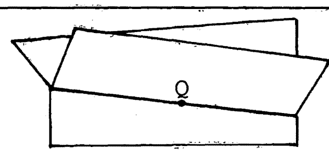
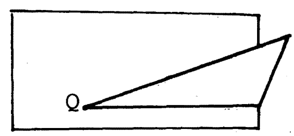
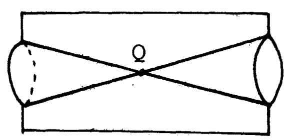
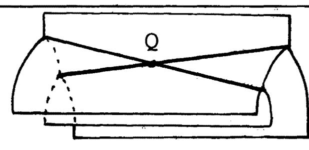
The manifolds $T(x)$ are uniquely specified by the number of boundary components and by the fact that there is a Morse function on the manifold, constant on each boundary component, with two singular points—both saddles. □

Remark. Although at each non-simple S_1-point and cusp point, there is a nbhd of the form $I \times T$, the map f to $I \times J \subset \mathbf{R}^2$ is not equivalent to a product. This fact is reflected by the fact that the q-image of the nbhd is not a product in W. Just as we did for the other singular points we will illustrate the map $f \mid f^{-1}(t \times J)$ where $t < 0$, $t = 0$, $t > 0$ for the double cone and the trident singularities.





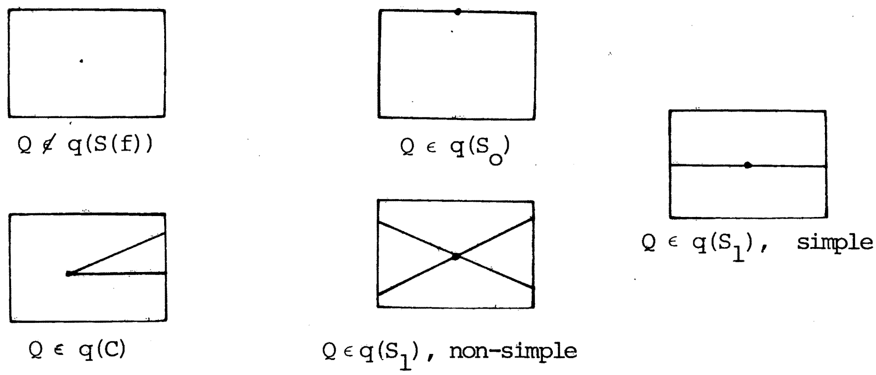
We now give a complete catalog of local descriptions of W_f which can arise for f a stable map of a (compact) orientable manifold into $\mathbf{R}^2$.

If $q(x) = Q \in$	$q(I \times T(x))$ is:	$T(x)$ is:
$W - q(S(f))$		$\$_2$
$q(S_0)$		$\$_1$
$q(S_1)$, simple		$\$_3$
$q(C)$		$\$_2$
$q(S_1)$ double cone (1·2·2·1)		T_2
$q(S_1)$ trident (1·2·2·3)		$\$_4$

The heavy lines represent the q-images of $S(f)$ near Q and as usual $\$_k$ (and Υ_k) mean the 2-sphere (and the 2-torus) with k discs removed. Notice that for all $Q \in W$, the pre-image of a nbhd $I \times T$ about Q is homeomorphic to the join of Q with the boundary of the q-image. Thus:

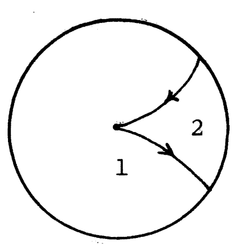
Definition. for $Q \in W$, a *conical nbhd* of Q is the q-image of any nbhd $I \times T$ of a point of $q^{-1}(Q)$.

We see that the $\bar{f}$-image of a conical nbhd of $Q \in W$ is always a rectangle in which $f(S(f))$ looks like:

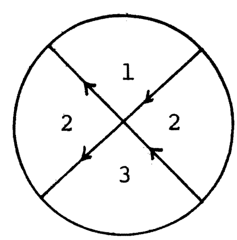


1.4. An Orientation of $S(f)$ – {(1·2·2·1)-Points}

We assign to each connected component, Ω, of $\mathbf{R}^2 - f(S(f))$ an integer $n_f(\Omega)$ = the number of components of the fibre above each point of Ω. Obviously, $\bar{f}^{-1}(\Omega) \xrightarrow{\bar{f}} \Omega$ is a covering space and $n_f(\Omega)$ is the number of times $\bar{f}^{-1}(\Omega)$ covers Ω. Every arc of $f(S(f))$ − {f-images of double points and cusps} is contained in the boundary of two components of $\mathbf{R}^2 - f(S(f))$ whose n_f-values are different. Orient that arc so that the region with the larger n_f-value is on the left. These orientations induce orientations on the arcs of $S(f) - C \cup$ {double points}. Notice that all of S_0 can be oriented in a way compatible with this partial orientation; it is oriented so that locally the image of M lies to the left of the immersed curve, $f(S_0)$; or so that $q(S_0)$ is the boundary of W. Also notice that the arcs of simple S_1-points can be oriented consistently. Where a cusp separates a S_0-arc from the S_1-arc the orientation are also compatible; the orientations also extend through the (1·2·2·3)-points as well.

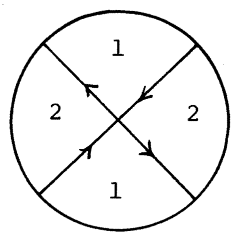


cusp



(1·2·2·3)-point

However the orientations *do not* extend through the double cone $(1\cdot 2\cdot 2\cdot 1)$-points.



double cone (1·2·2·1)-point

Resuming. Each componet of $S(f) - \{(1\cdot 2\cdot 2\cdot 1)\text{-points}\}$ can be oriented so that the value of n_f is always greater to the left of the image than to the right. The orientation does not extend through any $(1\cdot 2\cdot 2\cdot 1)$-points.

Consequently, on any component of $S(f)$, there is an even number of $(1\cdot 2\cdot 2\cdot 1)$-points.

Note: This orientation on $S(f) - \{(1\cdot 2\cdot 2\cdot 1)\text{-points}\}$ induces an orientation on its q-image.

1.5. Decomposing M and W_f

In this section we decompose M and $W = W_f$ into simple pieces. As a matter of fact, M is the union of:

(1) Trivial circle bundles over 2-dimensional regions R of W, $B(R)$.

(2) Trivial disc bundles over the components of S_0, $B(c)$, where c is the q-image of the component.

(3) $\$_3$-bundles over the components of S_1, $B(c)$ (again c is the q-image of the component). If the component is a circle, the bundle is either trivial or the one in which two of the holes of $\$_3$ are exchanged.

(4) Neighborhoods $B(v)$ containing the q-fibre above the image v of a non-simple S_1-points or a cusp. $B(v) \cong I \times T(v)$ where $T(v)$ is $\$_2$ if $v \in q(C)$; it is Υ_2, if v is the image of a double cone point and it is $\$_4$, if v is the image of a trident point.

We now give some notation. Let $\Sigma = q(S(f))$ and let the set of components of $W - \Sigma$ be denoted by $\mathscr{R}$. For each $R \in \mathscr{R}$ (a surface without boundary), the same arguments as used in §1.2 show that $\bar{R}$ is a surface with boundary (possibly with corners). Moreover $\bar{f} \mid R$ is an immersion. $B(R)$ is a trivial circle bundle over R. In Σ, let V be the set of q-images of cusp points and non-simple points of S_1—called vertices of W. Let $\mathscr{S}$ be the set of components of $\Sigma - V$. An arc $c \in \mathscr{S}$ is immersed by $\bar{f}$ (with double points at worst). An element $v \in V$ is the end point of at most four distinct arcs in $\mathscr{S}$. We write $\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_1$, where $\mathscr{S}_i$ is the set of q-images of components of S_i—{non-simple points}.

About each $v \in V$, choose a conical nbhd $N(v)$ (see the end of §1.3). We know that $B(v) = q^{-1}(N(v))$ is diffeomorphic to an interval I times the transverse manifold $T(v)$ at v. Let $c \in \mathscr{S}$ and let $N_{\bar f}$ be the normal bundle of the immersion $\bar f \mid c{:}c \to \mathbf{R}^2$ and let $p{:}N_{\bar f} \to c$ be the bundle projection. We may assume that we have an immersion $e{:}N_{\bar f} \to \mathbf{R}^2$ which is an embedding on the fibres and such that $\bar f \mid c = e \circ$ (zero section).

THEOREM. *For each component $c \in \mathscr{S}$, there is nbhd $B(c)$ of $q^{-1}(c)$ with smooth boundary and a map $\pi{:}B(c) \to c$ making $B(c)$ a bundle over c. The fiber of this bundle is $T(c)$, introduced in §1, at any point of $q^{-1}(c)$. Further, there is a fibre preserving map $\tilde f{:}B(c) \to N_{\bar f} \mid c$ over c, such that $e \circ \tilde f = f \mid B(c)$.*

Remark: Since $q{:}S(f) \cap q^{-1}(c) \to c$ is a homeomorphism, we may regard $B(c)$ as a bundle over $q^{-1}(c) \cap S(f)$ as well as over c.

Proof. Choose a covering of c by a collection $\mathscr{I}$ of open intervals such that for each $I \in \mathscr{I}$, $e \mid p^{-1}(I)$ is an embedding. Let $\phi_I{:}I \times \mathbf{R} \to p^{-1}(I)$ be a trivialization. It is no restriction to assume that for each I, if $J = [-1, 1]$, then $e \circ \phi_I = \psi_I{:}I \times J \to \mathbf{R}^2$, satisfies:

(1) $\text{proj} \circ \psi_I^{-1} \circ f{:}f^{-1}(\psi_I(I \times J)) \to I$ is a trivial bundle, and

(2) $\bigcup_{I \in I} \phi_I(I \times J) = \tilde N_c \subseteq N_{\bar f} \mid c$ is a smooth submanifold with smooth boundary $\partial \tilde N_c$ transverse to the fibres of p. $\tilde N_c$ is a sub-bundle of $N_{\bar f} \mid c$ with fibre J.

As we did in §1, construct a nbhd $I \times T(c)$ for every point of $q^{-1}(I)$:

$$\begin{array}{ccc} I \times T(c) & \xrightarrow{\Phi_I} & M \\ {\scriptstyle 1 \times h_I}\downarrow & & \downarrow{\scriptstyle f} \\ I \times J & \xrightarrow{\psi_I} & \mathbf{R}^2 \end{array}$$

where Φ_I is a diffeomorphism onto $B(c) \mid I$, the component of $f^{-1}(\psi_I(I \times J))$ containing $q^{-1}(I)$. We define $\tilde f_I$ and π_I by the commutativity of:

$$\begin{array}{ccccc} I \times T(c) & & \xrightarrow{\Phi_I} & & B(c) \mid I \\ {\scriptstyle 1 \times h_I}\downarrow & \searrow^{\text{proj}} & I & \swarrow^{\pi_I} \quad \swarrow^{\tilde f_I} & \downarrow{\scriptstyle f} \\ & & \uparrow p & & \\ I \times J & \xrightarrow{\phi_I} & \tilde N_{c \mid I} & \xrightarrow{e} & R^2 \end{array}$$

i.e., $\pi_I = \text{proj} \circ \Phi_I^{-1}$, $\tilde f_I = (e \mid \tilde N_{c \mid I})^{-1} \circ f$.

Setting $B(c) = \bigcup_{I \in \mathscr{I}} B(c) \mid I$, it is obvious by construction that the maps π_I

cohere to define a global projection $\pi: B(c) \to c$, making $B(c)$ a bundle over c, with fibre $T(c)$. Since $q^{-1}(I) = \Phi_I((1 \times h_I)^{-1}(I \times 0)) \subseteq B(c) \mid I$, it is clear that $B(c)$ is a nbhd of $q^{-1}(c)$. Also since $B(c)|_{I \cap I'} \subseteq (f^{-1} \circ e)(\tilde{N}_c|_{I \cap I'})$ and $\tilde{f}_I = (e \mid \tilde{N}_c \mid I)^{-1} \circ f \mid B(c) \mid I$, $\tilde{f}_I \mid B(c) \mid I \cap I' = \tilde{f}_{I'} \mid B(c) \mid I \cap I'$.

Thus the $\tilde{f}_I$ cohere to give a fibre preserving map of $B(c)$ onto $\tilde{N}_c \subseteq N_{\tilde{f}}$ such that $e \circ \tilde{f} = f \mid B(c)$. □

Then, we have M as union of three types of subsets:

(1) $B(R)$: trivial circle bundles over R, for $R \in \mathscr{R}$.
(2) $B(c)$: $T(c)$-bundles over c, for $c \in \mathscr{C}$.
(3) $B(v)$: diffeomorphic images of $I \times T(v)$, for $v \in V$, where $T(v)$ is the transverse manifold to a vertex v.

The possible types for $T(c)$ and for $T(v)$ are given in §1.2 and §1.3 respectively and this completes our statement about the decomposition of M.

Given any bundle $\pi: B(c) \to c$ as in the preceding Theorem, let $N(c) = q(B(c))$ and define $\Pi: N(c) \to c$ by $\pi = \Pi \circ q$. By construction, $B(c) = q^{-1}(N(c))$ and $\Pi: N(c) \to c$ is a bundle with fibre $q(T(c))$.

Remark. $N(c)$ is diffeomorphic to $c \times [0, 1]$ if $c \in \mathscr{C}_0$ where c is identified with $c \times \{0\}$. If $c \in \mathscr{C}_1$, then $N(c)$ is either $c \times Y$, or the non trivial bundle over a circle c in which two arms of the Y are exchanged. In both cases c is identified as $c \times$ (branching point). To see that these are the only possibilities for Y-bundles over a circle, c, we argue as follows. Since $N(c) - c$ is immersed in $\mathbf{R}^2$, it consists of a number of cylinders; the circle is oriented so that two arms are on the left and the stem is always on the right of the image. Thus, one cylinder is immersed on the right of the image of c. The rest of $N(c) - c$, double covers the left of the image of c. If the rest of $N(c) - c$ is one cylinder, then the bundle is the one which the two arms of the Y are exchanged. If the rest of $N(c) - c$ is two cylinders the bundle is trivial. Then, if we take q-images of the pieces of M we obtain W as the union of:

(1′) Regions $\bar{R}$, $R \in \mathscr{R}$, oriented surfaces with boundary (possibly with corners).
(2′) $c \times [0, 1] \cong N(c)$ for $c \in \mathscr{C}_0$.
(3′) $N(c) \cong$ either $c \times Y$ or the non trivial Y bundle over c (described in the preceding remark) for $c \in \mathscr{C}_1$.
(4′) $N(v)$, a conical nbhd of $v \in V = \{$images of cusps and non-simple S_1-points$\}$.

It is no restriction to assume that we can replace the conical nbhds $N(v)$ by smaller nbhds, $N'(v)$ so that $N(\Sigma) = (\cup_{v \in V} N'(v)) \cup (\cup_{c \in \mathscr{C}} N(c))$ has a smooth boundary. Since $\partial N(\Sigma)$ is in the q-image of the regular points of f, $q^{-1}(N(\Sigma)) = B(\Sigma) = (\cup B'(v)) \cup (\cup B(c))$ will be a nbhd of $q^{-1}(\Sigma)$ with smooth boundary. That is:

$$\partial B(\Sigma) \text{ is a smooth circle bundle over } \partial N(\Sigma).$$

For later use it will also be convenient to replace each $R \in \mathscr{R}$ with slightly smaller surface which has a smooth boundary contained in $N(\Sigma)$. We will continue to refer to the smoothed, shrunken version of $\bar{R}$ as R. Thus $B(R) = q^{-1}(R)$ is a circle bundle with boundary a circle bundle over ∂R. ∂R is homeomorphic with (and parallel to) $\partial N(\Sigma) \cap R$ and $\partial B(R)$ is equivalent to $\partial B(\Sigma) \cap B(R)$.

2.1. Lifting W_f to R^4

Here we show that the map $\bar{f}: W_f \to \mathbf{R}^2$ for $f: M \to \mathbf{R}^2$, where f is a stable map, can be lifted to a map $g: W = W_f \to \mathbf{R}^4$ with pleasant properties. This is only a subsidiary result to the next one, in §2.2, which is the main result of §2. We use the notation and the results of §1.5.

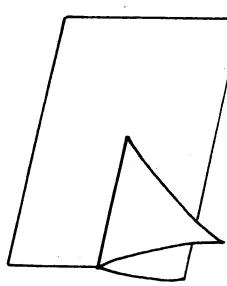
Definition. A continuous map $g: W \to \mathbf{R}^2 \times \mathbf{R}^2$, $x \to (\bar{f}(x), h(x))$ is called a lift of $\bar{f}$ into $\mathbf{R}^4$ if:

(1) $g \mid (W - \Sigma)$ *is an immersion with normal crossings.*
(2) $g \mid \Sigma$ *is 1:1 and is an embedding on* $\Sigma - V$.
(3) $g \mid N(\Sigma)$ *is 1:1,* $g \mid (N(\Sigma) - \Sigma)$ *is an embedding.*

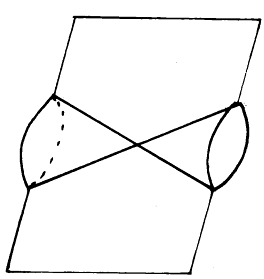
THEOREM. *For any stable map $f: M \to \mathbf{R}^2$, there is always a lift g of $\bar{f}$ into $\mathbf{R}^4$.*

Proof: We consider $\mathbf{R}^2 \subset \mathbf{R}^4$ as $\mathbf{R}^2 \times 0$. For each $v \in V$, define $g \mid (N(v) \cap \Sigma) = \bar{f} \mid (N(v) \cap \Sigma)$. By the definition of the conical nbhd, we know that $\bar{f} \mid (N(v) \cap \Sigma)$ is 1:1. It is obvious that g can be extended to an embedding of all of Σ above $\bar{f}$. All that is required is the separation of a finite number of double points of $\bar{f} \mid (\Sigma - V)$, which is easily accomplished by lifting one of the crossing curves in the direction of one of the extra dimensions. (Recall that $N(v)$ are chosen so that the double points of $\bar{f} \mid (\Sigma - V)$ are disjoint from $N(v)$). Thus we have satisfied (2). We now extend this q to all of $N(\Sigma)$.

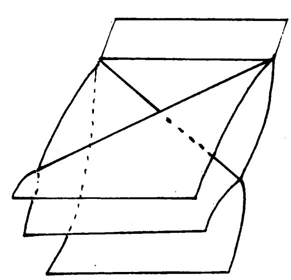
We begin by lifting the whole nbhd $N(v)$ about each $v \in V$ into $\mathbf{R}^3 = \mathbf{R}^3 \times 0 \subset \mathbf{R}^4$. The simplest way to explain this is to draw figures. In these figures the image of $N(v) \cap \Sigma$ is drawn with heavy lines and lies in the horizontal plane. The angle at the branches of the Y's in the image of a nbhd of the boundary of $N(v)$ (including everywhere in $N(v) - N'(v)$) is always a standard fixed angle, δ, and the stems are always horizontal.



cusp



double cone



trident

To extend this definition to all of $N(\Sigma)$ we work in each $N(c)$ separately. Recall that the projection of $N(c)$ to c is denoted by Π. For all the bundles $N(c)$ for which the fibres are mapped 1:1 by $\bar{f}$, we define $g:N(c) \to \mathbf{R}^3$, $x \to (\bar{f}(x), h(\Pi(x)))$.

This takes care of all of $N(c)$ with $c \in \mathscr{S}_0$ the fibre being a closed interval with one end point on c. For $N(c)$ with $c \in \mathscr{S}_1$, the fibres are Y's and we lift all of the stems horizontally exactly as in the case of $c \in \mathscr{S}_0$. For the arm components of $N(c) - c$, the lift has the form $g(x) = (\bar{f}(x), h(\Pi(x)) + Z(x))$, where $Z(x) \in \mathbf{R}^2$ and if $\Pi(x_1) = \Pi(x_2)$ and $\bar{f}(x_1) = \bar{f}(x_2)$, but $x_1 \neq x_2$, then $Z(x_1) = -Z(x_2)$, and $|Z(x_1)| = |\bar{f}(x_1) - \bar{f}(\Pi(x_1))| \tan \delta/2$. Thus the arms are lifted into $\mathbf{R}^4$ and make an angle δ with the horizontal.

If c has v_0 as its initial end point we can clearly define Z so that the lift of $N(c)$ in $N(v_0)$ agrees with the lift of $N(v_0)$. If c is a closed curve, we begin the lift on a piece of $N(c)$ say $\Pi^{-1}(I)$ for some small interval $I \subseteq c$; we lift $\Pi^{-1}(I)$ into $\mathbf{R}^3$ just spreading the arms of the Y fibres about the horizontal in the angle δ. We extend the definitions of Z in both cases travelling along c in its positive direction so that the lifting of all of $N(c)$ is consistent with that of $N(v_1)$ if c terminates at v_1 or with the lifting of $\Pi^{-1}(I)$ as c returns to I. This extension is either trivial, that is, can be accomplished in $\mathbf{R}^3$ by lifting the arms symmetrically about the horizontal or must be done in $\mathbf{R}^4$ if in the process of travelling along c, the arm exchange.

In that case the pairs of values $Z(x_1) = -Z(x_2)$ (for $\bar{f}(x_1) = \bar{f}(x_2)$ and $\Pi(x_1) = \Pi(x_2)$ and $x_1 \neq x_2$) rotate about a half circle in $\mathbf{R}^2$ as we travel along c. This completes the lifting of $N(\Sigma)$ so as to satisfy (2) and (3). We must now lift each region $R \in \mathscr{R}$ so that the lift agrees with that already defined in $R \cap N(\Sigma)$. It is no restriction to assume that we have a tubular nbhd B of the boundary of R in R on which g is defined and such that $R \cap N(\Sigma) \subset B^0$. (Here B^0 is the interior of B in R). Define $g_0:R \to \mathbf{R}^4$ by $g_0 \mid (R \cap N(\Sigma)) = g \mid (R \cap N(\Sigma))$ and $g_0 \mid (R - B^0) = \bar{f} \mid (R - B^0)$. On $B - N(\Sigma)^0$, which we may assume to be a finite union of annuli, we just make a smooth transition from $\bar{f}$ on one boundary component to g on the other boundary component of each annulus.

Let $\cup_{R \in \mathscr{R}} R = A$ and let $N(\Sigma) \cap A = E$, a nbhd of the boundary of A, a finite set of closed annuli. We have defined an immersion $g_0:A \to \mathbf{R}^4$ such that $g_0 \mid E = g \mid E$, where g_0 is above $\bar{f}$. We define a regular homotopy $g_t:A \to \mathbf{R}^4$ such that $g_t \mid E = g_0 \mid E$ for all t and g_1 is an immersion with normal crossings and such that $\Pi_1 \circ g_t:A \to \mathbf{R}^2$ is an immersion for all t, where $\Pi_1:\mathbf{R}^4 \to \mathbf{R}^2$ is just the projection on the first two coordinates. This g_t is easily obtained as follows. Let k_t be a regular homotopy of A into $\mathbf{R}^4$ such that $k_0 = g_0$ and which stays close enough to g_0 so that $\Pi_1 \circ k_t$ is an immersion in $\mathbf{R}^2$ for all t, $k_t \mid E$ is an embedding for all t and k_1 is an immersion with normal crossings. It is easy to construct a diffeotopy ϕ_t of $\mathbf{R}^4$ so that $\phi_0 = 1_{\mathbf{R}^4}$ and so that $k_t \mid E = \phi_t \circ k_0 \mid E$ (see [P]). Define $g_t = \phi_t^{-1} \circ k_t$.

The immersion of g_1 surely extends g to all of W; the only problem is that $g_1 \mid A$ may not lie over $\bar{f}$ except on E. We now construct a diffeotopy $\psi_t: A \to A$ such that $\psi_t \mid E = 1 \mid E$, and $\Pi_1 \circ g_t \circ \psi_t = \bar{f} \mid A$. For this we apply the following lemma to $h_t = \Pi_1 \circ g_t: A \to \mathbf{R}^2$.

LEMMA. *Let A be a compact surface with boundary and let E be a nbhd of the boundary of A. Let $h_t: A \to \mathbf{R}^2$ be a regular homotopy such that $h_t \mid E = h_0 \mid E$. Then there is a diffeotopy $\psi_t: A \to A$ such that $\psi_0 = 1_A$, $\psi_t \mid E = \psi_0 \mid E$ and $h_t \circ \psi_t = h_0$ for all t.*

Proof: Let $H: A \times I \to \mathbf{R}^2 \times I$, $(x, t) \to (h_t(x), t)$. Since $h_t \mid E = h_0 \mid E$ we may apply Theorem 3.3 of Chap. 5 of [G^2] by virtue of which it suffices to find a vector field ξ in $A \times I$ such that:

(1) $T\pi'(\xi) = 0$ where $\pi': A \times I \to I$ is the projection and

(2) $TH\left(\frac{d}{dt}\right) = -TH(\xi) + \left(\frac{d}{ds} \circ H\right)$, where $\frac{d}{dt}$ $\left(\text{respectively } \frac{d}{ds}\right)$ is the vector field which at (z_0, t_0) is tangent to $(t \to (z_0, t_0 + t))$ for $(z_0, t_0) \in A \times I$ (resp. $\mathbf{R}^2 \times I$).

For each $(x, t) \in A \times I$, $(Th_t)_x: TA_x \to T\mathbf{R}^2{}_{h_t(x)}$ is an isomorphism, so we define ξ by:

$$(Th_t)_x(\xi(x, t)) = -TH\left(\frac{d}{dt}\right)_{(x,t)} + \left(\frac{d}{ds}\right)_{H(x,t)}$$

where we have identified TA_x with the kernel of $T\pi'$ in $T(A \times I)_{(x,t)}$. As defined, ξ has compact support and vanishes identically on $E \times I$.

Thus we can integrate ξ as in the proof of the theorem cited above to give the required diffeotopy ψ_t of A for all $t \in I$. □

Note: The Lemma is obviously false without the constancy of h_t on the boundary of A. For example just slide a disc around in the plane.

2.2. Lifting M to $\mathbf{R}^4$

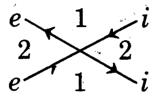
Having lifted $\bar{f}: W \to \mathbf{R}^2$ to $g: W \to \mathbf{R}^4$, we now give a sufficient condition for the existence of a lifting of $f: M \to \mathbf{R}^2$ to an immersion F of M into $\mathbf{R}^4$. It is a really interesting result, in the same direction of [H_a], even if unnecessarily strong conditions are assumed, objectifying the simplicity of its proof. The problem will be considered with more generality in a future work.

Sufficient Condition C

C_1. Except for circle components of $\mathscr{S}_1$, the nbhds $N(c)$ can be embedded in $\mathbf{R}^3 \subset \mathbf{R}^4$ in the lifting of $\bar{f}$. The arms of the Y-fibres are not interchanged as we go around a $N(\Sigma)$-loop, except if the loop is $N(c)$ for c a circle component.

C_2. The arcs of $S_1 - \{(1 \cdot 2 \cdot 2 \cdot 1)\text{-points}\}$ can be labelled e or i so that:

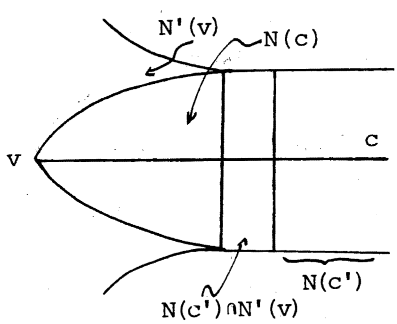
(i) The label changes as you travel through a $(1 \cdot 2 \cdot 2 \cdot 1)$-point
(ii) at the q-image of a $(1 \cdot 2 \cdot 2 \cdot 1)$-point the labelling is:



THEOREM. *If $f: M \to \mathbf{R}^2$ is a stable map which satisfies condition C (above), then there is an immersion $F: M \to \mathbf{R}^4$ which lifts f.*

Proof: The lifting of M to $\mathbf{R}^4$ is analogous to that of W in §2.1. Here, however, we follow a different sequence. First we lift $B(R)$ about the g-images of R, then $B(c)$ for $c \in \mathscr{C}_0$, then for $c \in \mathscr{C}_1$, and finally $B(v)$. We do the first two steps; these make no use of our assumptions. Since $B(R) \cap B(R') = \varnothing$ for $R \neq R'$, we work with one $R \in \mathscr{R}$ at a time. Let $\phi_R: R \times S^1 \to B(R)$ be a diffeomorphism such that $q \circ \phi_R(x, z) = x$. Here we think of $S^1 \subset R^2$, the unit circle.

Define $\Phi_{R,r}: R \times S^1 \times \mathbf{R} \to \mathbf{R}^4$, $(x, z, r) \to g(x) + (0, r, z)$ choose an r small enough, say r_0 so that $\Phi_{R,r_0} | (R \cap (\cup N(v) \cup \cup N(c)) \times S^1$ is an embedding for all $R \in \mathscr{R}$. Call $\Phi_R = \Phi_{R,r_0}$ and let $F_R = \Phi_R \circ \phi_R^{-1}: B(R) \to \mathbf{R}^4$. Instead of trying to extend this map to $B(c)$, $c \in \mathscr{C}$, we cut the sets $N(c)$ down in case c is *not* a closed curve. Recall that for each $v \in V$, $N'(v) \subset N(v)$ is such that $(\cup_v N'(v)) \cup (\cup_{c \in \mathscr{C}} N(c)) = N(\Sigma)$ has a smooth boundary, and $\partial R \subset N(\Sigma)$. If c has end point v_0 and v_1, let c' be a compact arc of c such that if $N(c') = \Pi^{-1}(c')$, $c' \cap N'(v_1)$ is a non-empty arc for $i = 0, 1$ and $N(c') \cap N'(v_i) = \Pi^{-1}(c' \cap N'(v_i))$.

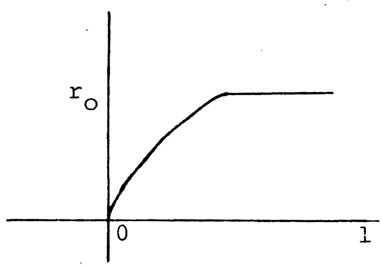


Notice that for each $v \in V$, $N(v) - N'(v) \subset \cup R$ and $N(c) \cap (\cup R) \supset \partial N(c)$ for c closed, so F is already defined on those sets.

To avoid too much notation, $c' = c$ if c is a circle component of $\mathscr{L}$ and c' will be the compact subarc just described if c begins and ends at elements of V. For $c \in \mathscr{L}_0$, there is a homeomorphism $\theta_c : c' \times [0, 1] \times N(c')$. We may assume if $N(c') \cap R \neq \emptyset$ that the θ_c pre-image of $N(c') \cap R$ is $c' \times [s, 1]$ and that the diffeomorphism $\phi_c : c' \times D \to B(c')$ and $\phi_R : R \times S^1 \to B(R)$ are compatible in $B(R) \cap B(c')$; that is:

$$\phi_R^{-1} \circ \phi_c : c' \times D - D_s^0 \to R \cap N(c') \times S^1,\ (t, z) \to \left(\theta_c(t, |z|), \frac{z}{|z|}\right).$$

Choose a smooth function $\delta : [0, 1] \to [0, r_0]$ whose graph is



Define $F_c = \Phi_c \circ \phi_c^{-1}$ where $\Phi_c : c' \times D \to \mathbf{R}^4$ and

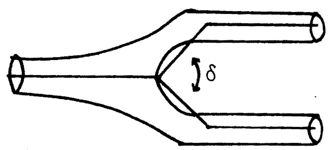
$$(t, z) \to g(\theta_c(t, |z|)) + \left(0, \delta\left(\frac{z}{|z|}\right) z\right).$$

To see that F_c and F_R agree on $B(c') \cap B(R)$ it suffices to show that Φ_c and $\Phi_R \circ \phi_R^{-1} \circ \phi_c$ agree on $c' \times (D - D_s^0)$. Evaluating

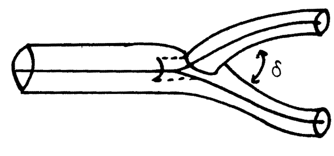
$$\Phi_R \circ \phi_R^{-1} \circ \phi_c(t, z) = \Phi_R\left(\theta_c(t, |z|), \frac{z}{|z|}\right)$$

$$= g(\theta_c(t, |z|)) + \left(0, r_0 \frac{z}{|z|}\right) = \Phi_c(t, z).$$

Thus we have defined F on all of $(\cup_{R\in\mathscr{R}} B(R)) \cup (\cup_{C\in\mathscr{C}_0} B(c'))$. Before extending F to $B(c')$ for $c' \in \mathscr{C}_1$ and to $B(v)$ we recall our notation $B(c') \xrightarrow{\pi} c'$ is a bundle with fibre $\$_3$, a disc with two holes and $N(c') \xrightarrow{\Pi} c'$ is a Y bundle. We consider two maps, λ_e, λ_i an embedding and an immersion of $\$_3$ into $\mathbf{R}^3$.



$\underline{\lambda}_e$ (the embedding)



$\underline{\lambda}_i$ (the immersion)

Both maps set over the embedding μ of Y, as indicated in the figures

$$\begin{array}{ccc} \$_3 & \xrightarrow{\lambda} & \mathbf{R}^3 \\ {\scriptstyle q}\downarrow & \nearrow_{\mu} & \\ Y & & \end{array}$$

For each point $x \in \$_3$, $\lambda(x) = \mu(q(x)) + v_.(x)$ where $v_.:\$^3 \to (0, R^2)$. We assume that $|v_.(x)| = r_0$ for all x in a collar nbhd of all the boundary components. For c' for which there are diffeomorphisms (compatible as before with ϕ_R, etc).

$$\begin{array}{ccc} c' \times \$_3 & \xrightarrow{\phi_c} & B(c') \\ \downarrow & & \downarrow q \\ c' \times Y & \xrightarrow{\theta_c} & N(c') \end{array}$$

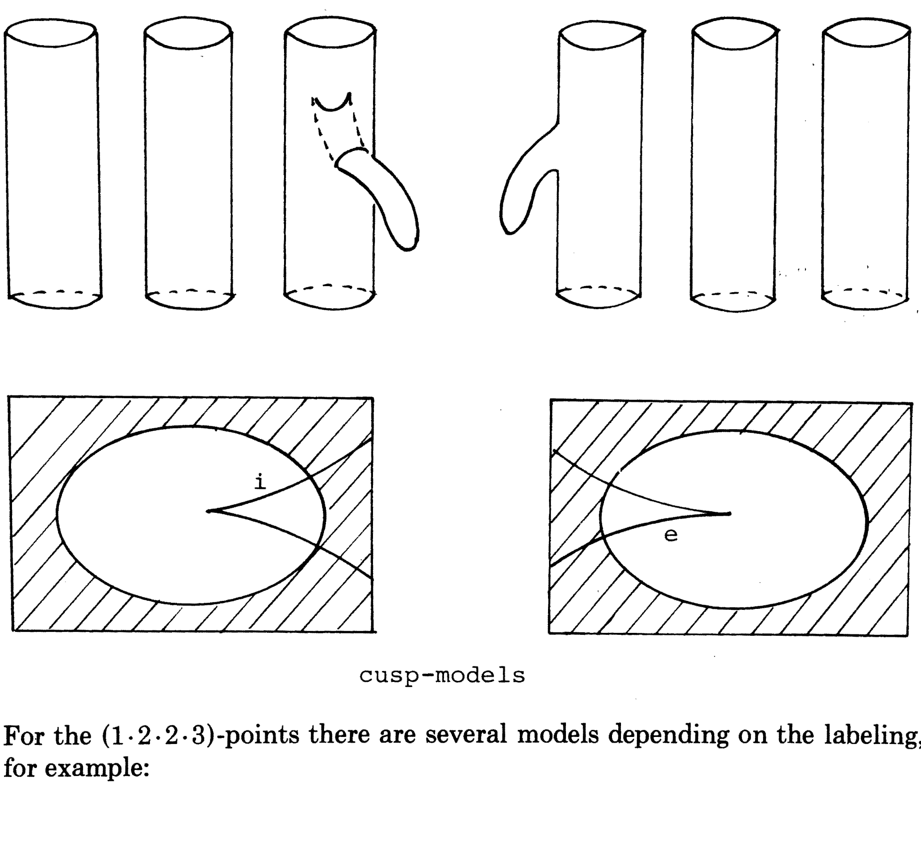
we define $\Phi_c:c' \times \$_3 \to \mathbf{R}^4$, $(t, x) \to g(\theta_c(t, q(x))) + v_.(x)$ and define $F_c = \Phi_c \circ \phi_c^{-1}$. In the expression for θ_c, the term $v_.$ is v_e if c is labelled with an e, and is v_i if c is labelled with an i. If c is a closed component such that $N(c)$ is the non-trivial Y-bundle, we pull back both $B(c)$ and $N(c)$ over an interval, I:

$$\begin{array}{ccc} I \times S_3 & \xrightarrow{\phi_c} & B(c) \\ \downarrow & & \downarrow q \\ I \times Y & \xrightarrow{\theta_c} & N(c) \end{array}$$

We suppose that $g(\theta_c(t \times Y)) \subset \bar{f}(\theta_c(t \times Y)) \times L_t$ where L_t is the line in $\mathbf{R}^2$ through the origin making an angle $t\pi$ with the first axis. Define

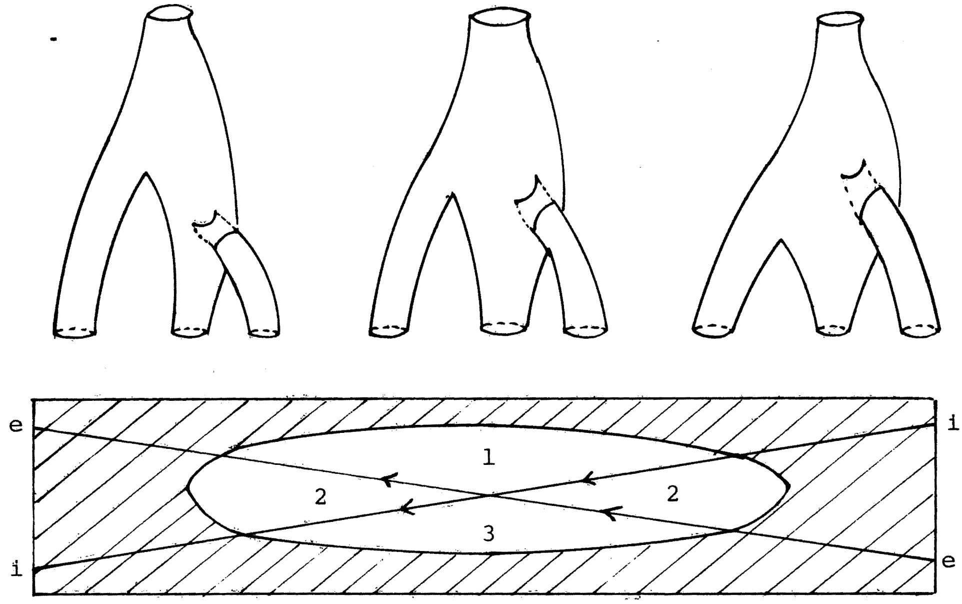
$$\Phi_c:I \times \$_3 \to \mathbf{R}^4,\ (t, x) \to g(\theta_c(t, q(x)) + \alpha(t)(v_e(x)),$$

where $\alpha(t):\mathbf{R}^2 \to \mathbf{R}^2$ rotates the plane counter-clockwise through an angle $t\pi$. It is no restriction to assume that $\Phi_c \circ \phi_c^{-1}$ is a well defined embedding $F_c:B(c) \to \mathbf{R}^4$ and that these new maps are compatible with those already defined. So we have our immersion F defined on $(\bigcup_{c\in\mathscr{S}} B(c')) \cup (\bigcup_{R\in\mathscr{R}} B(R))$. To complete our lifting to all of M we need only define it on our product nbhds $B(v)$. For this we use specific models, there being no difficulty at either the cusps or the (1·2·2·3)-points, what a nbhd of a (1·2·2·1)-point must look like is guaranteed by our condition C_2. We examine our local models in the figures, F is already defined above the shaded region in $\bar{f}(N(v))$

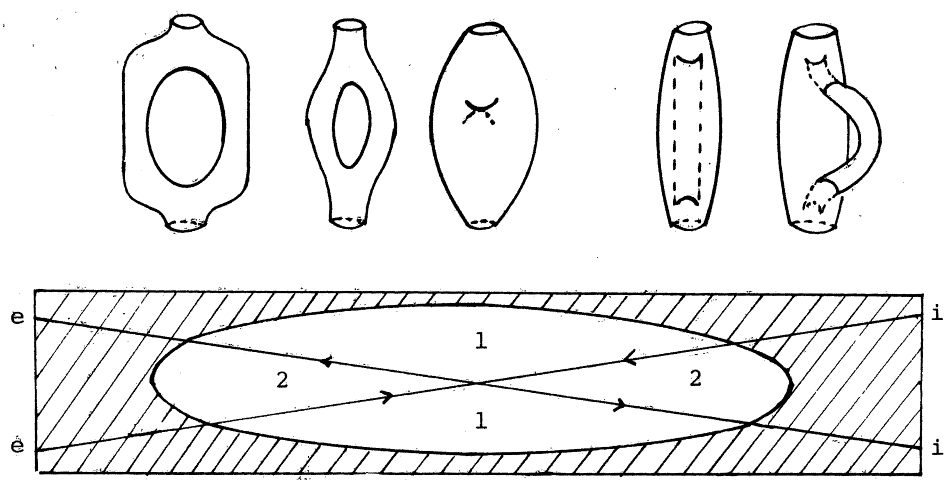


cusp-models

For the (1·2·2·3)-points there are several models depending on the labeling, for example:



Finally, for the $(1\cdot 2\cdot 2\cdot 1)$-points our model is:



The extensions we are looking for are obvious. □

COROLLARY. *Any stable map $f: M \to \mathbf{R}^2$ which has no crossings of type* $(1\cdot 2\cdot 2\cdot 1)$ *can be lifted to an immersion in* $\mathbf{R}^4$.

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