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NEC SUBGROUPS IN KLEIN SURFACES

BY J. J. ETAYO-GORDEJUELA

1. Introduction

In this paper we study the number of fixed points of the elements of an odd order cyclic group of automorphisms of a Klein surface that is not a Riemann surface.

This question was solved in the particular case of Riemann surfaces by Moore [4].

If the Klein surface is $X = D/K$, where K is an NEC group, each group of automorphisms has the form $G = \Gamma/K$, Γ being another NEC group. We calculate the possible signatures of the subgroups Γ' of Γ such that Γ'/K are the subgroups of G.

The first part of the paper, devoted to the fixed points, follows the line of [4] in the case of Riemann surfaces. It is necessary, however, to keep in mind that orientation-reversing elements appear in our case. So the involved groups are NEC groups instead of fuchsian groups, and we need to consider, besides the topological genus, also the orientability and the number of period-cycles, determining thus the algebraic genus of the group.

A Klein surface, X, is a surface with or without boundary, with an open covering $\mathscr{U} = \{U_i\}_{i \in I}$, that fulfills the following two conditions:

i) For each $U_i \in \mathscr{U}$, there exists a homeomorphism ϕ_i from U_i onto a subset of $\mathbb{C}$.

ii) If $U_i, U_j \in \mathscr{U}$, $U_i \cap U_j \neq \varnothing$, then $\phi_i \phi_j^{-1}$ is an analytic or anti-analytic application defined in $\phi_j(U_i \cap U_j)$.

An automorphism of the surface is a homeomorphism $f\colon X \to X$, such that $\phi_i f \phi_j^{-1}$ is analytic or anti-analytic.

Orientable Klein surfaces without boundary are Riemann surfaces.

A non-orientable Klein surface X with topological genus g and k boundary components has algebraic genus $p = g + k - 1$; if X is orientable with boundary its algebraic genus is $p = 2g + k - 1$.

Klein surfaces and their automorphisms may be studied by means of non-Euclidean crystallographic (NEC) groups. An NEC group is a discrete subgroup of isometries of the non-Euclidean plane with compact quotient space. NEC groups include orientation-reversing isometries, reflections and glide-reflections.

NEC groups are classified according to their signatures. The signature of an NEC group is of the form

$$(*) \qquad (g, \pm, [m_1, \cdots, m_r], \{(n_{11}, \cdots, n_{1s_1}), \cdots, (n_{k1}, \cdots, n_{ks_k})\}).$$

The number g is the genus, the m_i are the proper periods, and the brackets $(n_{i1}, \ldots, n_{is_i})$ are the period-cycles. If the period-cycle is empty we note it as (—).

The group Γ with signature $(*)$ has a presentation given by generators

i) x_i, $i = 1, \cdots, r$
ii) e_i, $i = 1, \cdots, k$
iii) c_{ij}, $i = 1, \cdots, k$, $j = 0, \cdots, s_i$
iv) (with sign '+') a_j, b_j, $j = 1, \cdots, g$
(with sign '−') d_j, $j = 1, \cdots, g$

and relations
i) $x_i^{m_i} = 1$, $i = 1, \cdots, r$
ii) $e_i^{-1} c_{i0} e_i c_{is_i} = 1$ $i = 1, \cdots, k$
iii) $c_{i,j-1}{}^2 = c_{ij}{}^2 = (c_{i,j-1} c_{ij})^{n_{ij}} = 1$, $i = 1, \cdots, k; j = 1, \cdots, s_i$
iv) (with sign '+') $x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1$
(with sign '−') $x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1$.

Each NEC group has an associated area, $|\Gamma|$, given by

$$|\Gamma| = 2\pi\left(\alpha g + k - 2 + \Sigma_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\Sigma_{i=1}^{k}\Sigma_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

α being 2 with sign '+' and 1 with sign '−'.

The NEC group Γ has a canonical fuchsian subgroup Γ^+ formed by the elements that preserve orientation.

The relation between Klein surfaces and NEC groups comes from the following two results:

THEOREM A [5]. *Let X be a Klein surface of topological genus g, k boundary components, and algebraic genus ≥ 2. Then X may be represented as D/K, where $D = \{z \in \mathbb{C}: \mathrm{Im}(z) > 0\}$ and K is an NEC group with signature*

$$\left(g, \pm, [-], \left\{(-), \overset{k}{\cdots}, (-)\right\}\right),$$

with sign '+' if X is orientable, and '−' if X is nonorientable.

THEOREM B [3]. *A finite group G is a group of automorphisms of the Klein surface D/K if and only if $G = \Gamma/K$, where Γ is an NEC group of which K is a normal subgroup.*

2. Fixed points

Let X be a Klein surface, and G a cyclic group of automorphisms of X, of odd order n. Let t be a generator of G, and $d' \mid n$. Then $t^{d'}$ generates a subgroup of G, of order $d = n/d'$, that we shall call G_d.

If $X = D/K$, K being the NEC group with signature

$$\left(g, \pm, [-], \left\{(-), \overset{k}{\cdots}, (-)\right\}\right)$$

($k > 0$ with sign '+'), then $G = \Gamma/K$, and by [1] Γ has signature

$$\left(\gamma, \pm, [m_1, \cdots, m_s], \left\{(-), \overset{k'}{\cdots}, (-)\right\}\right),$$

where the signs of K and Γ are the same, and $k = 0$ if and only if $k' = 0$.

LEMMA (1). *There exists an NEC group, Γ_d, which is a normal subgroup of Γ, such that $\Gamma_d/K = G_d$.*

Proof. The group Γ_d is formed by the lifts in Γ of the elements of G_d. □

LEMMA (2). *Let θ be the natural epimorphism from Γ onto $G = \Gamma/K$. The restriction of θ is an isomorphism between those stabilizers of points of D, generated by elliptic elements of Γ, and the stabilizers of the images of those points by the canonical projection $p\colon D \to X$.*

Proof. As the signature of K has neither proper periods nor non-empty period-cycles, the group X has no elliptic elements. So the proof of [4] holds. □

As Γ has signature $\left(\gamma, \pm, [m_1, \cdots, m_s], \left\{(-), \overset{k'}{\cdots}, (-)\right\}\right)$ we shall call r_q the number of proper periods m_i such that $m_i = q$. Now θ is an epimorphism from Γ onto $G = Z/n$, whose kernel has no orientable element of finite order, and so Γ has no proper periods bigger than n. Thus we may reorder the elliptic generators of Γ as follows:

$$x_{21}, \cdots, x_{2r_2}, x_{31}, \cdots, x_{3r_3}, \cdots, x_{n1}, \cdots, x_{nr_n},$$

subjected to the conditions $x_{ij}{}^i = 1$.

Let us consider again a generator t of G. If $d \mid n$, we shall denote $t^{n/d}$ by t_d. Let $y \in X$ be a fixed point of G, such that stab(z) has order q. If y is a fixed point of t_d, then $d \mid q$. We shall classify the fixed points of G in classes C_q characterized by the order q of the stabilizer. If $y \in C_q$, every point of the G-orbit of y is in C_q.

LEMMA (3). *There is a bijection between the G-orbits of fixed points of C_q and the conjugacy classes of the x_{qi}, $i = 1, \cdots, r_q$.*

Proof. Analogous to [4]. □

The number of the points in the G-orbit of $y \in C_q$ is $n/q = q'$. The number of conjugacy classes in G of elements of order q is r_q. Hence the number of points in C_q is $q'r_q$.

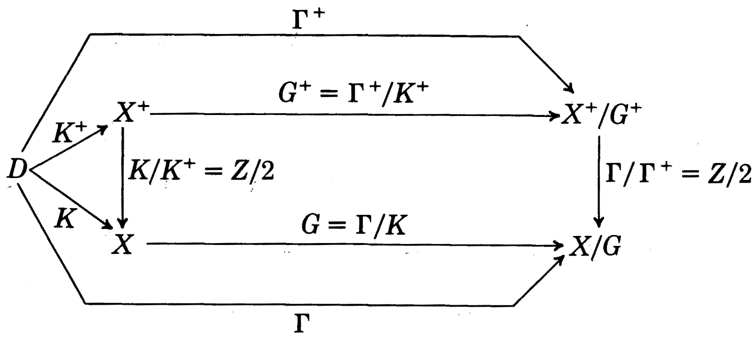
THEOREM (1). *Let r_q be the number of periods $m_i = q$; if $s' \mid n$, $s' \neq n$, the number of fixed points of $t^{s'}$ is*

$$N(t^{s'}) = \Sigma_{\delta\delta'=s'}\ \delta' r_{\delta s}, \qquad ss' = n.$$

Proof. Analogous to [4]. □

COROLLARY (1). *If* $t^{s'} \in \Gamma/K$ *has N fixed points, then* $t^{s'} \in \Gamma^+/K^+$ *has* $2N$ *fixed points.*

Proof. Observe that Γ/K acts on X, and Γ^+/K^+ acts on its two-fold cover $D/K^+ = X^+$. Thus we consider the diagram



From lemma 2, each fixed point in X appears twice in X^+, and the result holds. □

We shall obtain now another expression for the number of fixed points of $t^{s'}$.

Let μ be the Möbius function, defined as $\mu(1) = 1$, $\mu(d) = 0$ if d has a square factor, and $\mu(d) = (-1)^k$ if d has k different prime factors; and let ϕ be the Euler function, where $\phi(n)$ is the number of integers not greater than n and relatively prime with it.

PROPOSITION (1). *The number of fixed points of* $t^{s'} \in \Gamma/K$ *is given by*

$$N(t^{s'}) = \frac{1}{\phi(s)} \Sigma_{dd'=s}\, d\mu(d')(1 - p_d),$$

where p_d *is the algebraic genus of* Γ_d.

Proof. As n is odd, so is d. By lemma 1, there exists a group Γ_d, such that $|\Gamma_d{:}K| = d$, and whose signature is

$$\left(\gamma_d, \pm, [\mu_1, \cdots, \mu_r], \left\{(-), \overset{k_d}{\cdots}, (-)\right\}\right),$$

where the signs of Γ_d and K are the same, and $k_d = 0$, if and only if $k = 0$. Let p_d be the algebraic genus of Γ_d. The signature of $\Gamma_d{}^+$ is $(p_d, +, [\mu_1, \mu_1, \cdots, \mu_r, \mu_r], \{-\})$ [6]. By [4] the number of fixed points of $t^{s'} \in \Gamma^+/K^+$ is

$$\frac{1}{\phi(s)} \Sigma_{dd'=s}\, d\mu(d')(2 - 2p_d).$$

Applying corollary 1, the number of fixed points of $t^{s'} \in \Gamma/K$ is then

$$N(t^{s'}) = \frac{1}{\phi(s)} \Sigma_{dd'=s}\, d\mu(d')(1 - p_d). \quad \square$$

3. The signature of Γ_d

THEOREM (2). *Let d be a divisor of n. Then the algebraic genus of* Γ_d *is*

$$p_d = 1 + \frac{p-1}{d} - \frac{1}{d} \Sigma_{bb'=n}\, b' r_b(gcd(b, d) - 1).$$

Proof. Γ_d having signature

$$\left(\gamma_d, \pm, [\mu_1, \cdots, \mu_r], \left\{(-), \overset{k_d}{\cdots}, (-)\right\}\right)$$

and algebraic genus p_d, the signature of Γ_d^+ is $(p_d, +, [\mu_1, \mu_1, \cdots, \mu_r, \mu_r], \{-\})$. So from [4],

$$p_d = 1 + \frac{p-1}{d} - \frac{1}{2d} \Sigma_{bb'=n}\, b' r_b^+(gcd(b, d) - 1),$$

where r_b^+ is the number of proper periods of Γ_d^+ that equal b; thus $r_b^+ = 2r_b$, and the result is obtained. $\square$

THEOREM (3). *The signature of* Γ_d *is*

$$\left(\gamma_d, \pm, \left[\left(\frac{m_i}{q_i}\right)^{n/dq_i} : m_i \neq q_i \qquad i = 1, \cdots, s\right], \left\{(-), \overset{k_d}{\cdots}, (-)\right\}\right),$$

where the sign is the same as that of K; $k_d = 0$ *if and only if* $k = 0$; $(\cdots)^a$ *means that the proper period is repeated a times; the numbers* q_i, *that are the exponent of* x_i *modulo* Γ_d, *are subjected to the condition*

$$n\, \Sigma_{i=1}^{s} \frac{1}{q_i} = n(p - 1 + s) - p + 1 - \Sigma_{bb'=n}\, b' r_b(gcd(b, d) - 1),$$

and the algebraic genus is given by theorem 2.

Proof. $|\Gamma:\Gamma_d| = d'$, $|\Gamma_d: K| = d$, and d, d' are odd. So Γ_d has the same sign that Γ and K have. Let us calculate now the proper periods. By [1] if $m_1, \cdots, m_s$ are the proper periods of Γ, and q_i is the exponent of x_i modulo Γ_d, the proper periods of Γ_d are

$$\left[\left(\frac{m_i}{q_i}\right)^{n/dq_i} : m_i \neq q_i, \qquad i = 1, \cdots, s\right].$$

Further, as $|\Gamma: \Gamma_d| = d'$, their associated areas $|\Gamma_d|$ and $|\Gamma|$ satisfy $d' =$

$|\Gamma_d|/|\Gamma|$ [6], and so

$$\frac{n}{d} = \frac{p_d - 1 + \Sigma_{i=1}^{s} \frac{n}{dq_i}\left(1 - \frac{1}{m_i/q_i}\right)}{p - 1 + \Sigma_{i=1}^{s}\left(1 - \frac{1}{m_i}\right)}.$$

Then

$$n = \frac{p - 1 + \Sigma_{bb'=n}\, b'r_b(gcd(b, d) - 1) + \Sigma_{i=1}^{s} \frac{n}{q_i} - \Sigma_{i=1}^{s} \frac{n}{m_i}}{p - 1 + \Sigma_{i=1}^{s}\left(1 - \frac{1}{m_i}\right)}.$$

We get

$$\Sigma_{i=1}^{s} \frac{n}{q_i} = n\left(p - 1 + \Sigma_{i=1}^{s}\left(1 - \frac{1}{m_i}\right)\right)$$

$$- p + 1 - \Sigma_{bb'=n}\, b'r_b(gcd(b, d) - 1) + \Sigma_{i=1}^{s} \frac{n}{m_i}$$

$$= n(p - 1 + s) - p + 1 - \Sigma_{bb'=n}\, b'r_b(gcd(b, d) - 1). \quad \square$$

COROLLARY (2). *If d is prime, the signature of Γ_d is*

$$\left(\gamma_d, \pm, [(d)^{n/m_i}: m_i \neq q_i, \qquad i = 1, \cdots, s], \left\{(-), \overset{k_d}{\cdots}, (-)\right\}\right),$$

where the sign is the same of K, $k_d = 0$, if and only if $k = 0$, and the algebraic genus is given by theorem 2.

Proof. As K is a subgroup of Γ_d of index d, we apply [2, th. 4] and [6, th. 2] and so

$$\frac{m_i}{q_i} \mid d, \qquad i = 1, \cdots, s.$$

If d is prime, and $m_i \neq q_i$, then

$$\frac{m_i}{q_i} = d, \quad \text{and so} \quad \frac{n}{dq_i} = \frac{n}{m_i}. \quad \square$$

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UNIVERSIDAD COMPLUTENSE
MADRID, ESPAÑA

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