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SOME NOTES ABOUT THE RANDOM MOTION OF A PARTICLE

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1. Models and theorems

In 1968 Frank Spitzer [9] constructed the following model. In R we have a Poisson system with parameter 1, which is equivalent to two independent Poisson processes on the left and on the right of the origin. In order to have a realization $\{x_k\}_{k=-\infty}^{+\infty}$ of the model, the points x_k, $k = 0, \pm 1, \cdots$, with $x_0 = 0$, will represent the initial positions of particles, all having equal mass. For every realization $\cdots < x_{-1} < x_0 < x_1 < \cdots$ of the initial point process the model evolves according to a deterministic interaction of the form

$$\frac{dx_k}{dt} = \frac{x_{k+1}(t) + x_{k-1}(t)}{2} - x_k(t)$$

This model represents bilateral repulsions. Following Spitzer it is very easy to solve this system of differential equations explicitly. Writing $x(t) = \{x_k(t)\}_{-\infty}^{+\infty}$, the system becomes

$$\frac{dx}{dt} = (F - I)x,$$

where F is the operator $(Fx)_k = \frac{1}{2}(x_{k+1} - x_{k-1})$ and I is the identity operator. Therefore the solution can be expressed as

$$x(t) = e^{t\Omega}x, \quad x = x(0), \quad \Omega = F - I.$$

Thus Ω is the infinitesimal operator for the randomized random walk $S(t)$ (cf. Feller [4]), where

$$P[S(t) = k] = e^{-t}I_k(t), \quad k \in Z, \quad t \geq 0,$$

I_k being the Bessel function of order k, i.e.

$$I_k(t) = \sum_{j=0}^{\infty} \frac{1}{j!\Gamma(j + k + 1)} \left(\frac{t}{2}\right)^{2j+k}, \quad k \geq 0, \quad I_{-k} = I_k.$$

We have

$$(1) \qquad x_n(t) = \sum_{k=-\infty}^{\infty} P[S(t) = k]x_{n+k}(0), \qquad t \geq 0,$$

and in particular

$$x_0(t) = \sum_{k=-\infty}^{\infty} e^{-t}I_k(t)x_k(0).$$

In fact, the solution (1) is independent of whether we start from a Poisson system or from another point process (we only need to have the natural order).

To be able to prove a limit theorem for $x_0(t)$ we need some assumption about the initial positions x_k. We put

$$x_0 = 0,\ x_k = \zeta_1 + \cdots + \zeta_k,\ x_{-k} = \zeta_{-1} + \cdots + \zeta_{-k},\ k > 0,\ \zeta_k > 0,\ \zeta_{-k} < 0.$$

THEOREM (1). *If* $|\zeta_k| = |x_k - x_{k-1}|$, $k = 1, 2, \cdots$ *are independent random variables with mean* 1, *variance* σ^2 *and*

$$\sup_k E(\zeta_k{}^6) < \infty, \tag{2}$$

then

$$Y_A(t) \equiv A^{-1/4}x_0(At) \Rightarrow Y(t) \quad \text{as} \quad A \to \infty,$$

where $\Rightarrow$ *means weak convergence on* $C([0, 1], R)$ (cf. Billingsley [1]), *and* $Y(t)$ *is a Gaussian process with mean* 0 *and covariance*

$$E(Y(s)Y(t)) = \frac{\sigma^2}{\sqrt{2\pi}}(\sqrt{t} + \sqrt{s} - \sqrt{t+s}).$$

If we are interested only in convergence of the finite-dimensional distributions (proved by Gisselquist [5] in a little bit less general situation), instead of (2) it suffices to assume that the $\zeta_k{}^2$ are uniformly integrable, which is satisfied if $\sup_k E|\zeta_k|^{2+\epsilon} < \infty$ for some $\epsilon > 0$, and obviously if the ζ_k have the same distribution.

Spitzer and Gisselquist considered also the model where the evolution of the system is caused by unilateral repulsions:

$$\frac{dx_k}{dt} = x_{k+1} - x_k - 1, \quad E|x_k - x_{k-1}| = 1$$

Gisselquist proved that if the $|\zeta_k|$ have the same distribution and are independent, with $E\zeta_k{}^6 < \infty$; $\zeta_k > 0$, $k > 0$; $\zeta_k < 0$, $k < 0$, then

$$x_0(t) = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!}[x_k(0) - k],$$

and the finite-dimensional distributions of $A^{-1/2}x_0(At)$ converge to those of Brownian motion. A little observation allows us to show the following result.

THEOREM (2). *Under the same assumption above,* $A^{-1/2}x_0(At)$ *converges weakly to Brownian motion on* $C([0, 1], R)$.

Now we pass to models where the dynamics is due to elastic collisions. In R we described some models starting either from the equilibrium state (Poisson system) or away from equilibrium (cf. Szatzschneider [10], Major, Szász [6]). In the multidimensional case there appears a serious problem, namely, how to construct the collisions. A discrete model in R^2 has been constructed by Dao-Quang-Tuyen and Szász [2]; they prove weak convergence to a two-dimensional Brownian motion. Another model in R^n represents a large particle in a system of smaller colliding particles (cf. Dür, Goldstein and Lebowitz [3]). They prove weak convergence of the motion of the particle to the Ornstein-Uhlenbeck process. Now we will construct the "jump-continuous" model. This class of models originated from a conversation with Professor Dobrushin. So far we have not obtained any results concerning "rich" models, and the only theorem we have covers a simple situation.

In $R^2 = \{(x, y)\}$, on each of the lines $y = k$ (integer) we construct independent Poisson systems with parameter 1. We put a distinguished particle black say, at the origin. We endow the particles with independent random horizontal velocities: $+1$, -1 with probability $\frac{1}{2}$ each. The collisions of particles against the black particle produce the exchange of their velocities and also a jump of the black particle to each one of the neighbouring lines with probability $\frac{1}{2}$ each. The remaining particles continue on their original lines. Let $\vec{y}_0(\cdot)$ denote the two-dimensional motion of the black particle. Then we have

THEOREM (3).

$$\vec{Y}_A(t) = A^{-1/2}\vec{y}_0(At) \Rightarrow \vec{B}(t), \quad \text{as} \quad A \to \infty,$$

where $\vec{B}(\cdot)$ is the two-dimensional Brownian motion. Here $\Rightarrow$ means weak convergence in the space of functions $(x(t), y(t))$ from $[0, 1]$ to R^2, such that, $x \in C([0, 1], R)$ and $y \in D([0, 1], R)$ (cf. Billingsley [1]).

Another model on R which poses similar problems is the following. We start from the equilibrium state (Poisson system). The particles collide with probability α, $0 < \alpha < 1$ and penetrate each other with probability $1 - \alpha$. In the simple case, i.e. the velocities v_k of the particles are independent and $v_k = \pm 1$ with probability $\frac{1}{2}$, it is almost obvious that this holds.

THEOREM (4).

$$A^{-1/2}y_0(At) \Rightarrow \sqrt{\alpha}\, B(t) \text{ as } A \to \alpha,$$

where $y_0(\cdot)$ is the motion on R of the 0-th particle, where $\Rightarrow$ is, as before, weak convergence on $C([0, 1], R)$, and $B(\cdot)$ is Brownian motion.

In all models constructed above in R we have assumed that $E[v_k] = 0$, where v_k is the velocity of the k-th particle.

Now we suppose that the velocities v_k are independent and identically distributed random variables and $E[v_k] > 0$. It is easy to see that in both models: (i) initial equilibrium state (Poisson system), (ii) initial positions on the integers, the following result holds.

THEOREM (5). *$Y_A(t) = A^{-1/2}y_0(At)$ tends to $+\infty$ with probability 1 as $A \to \infty$, where $y_0(\cdot)$ is, as before, the motion of 0-th particle on R. (If $v_k = \pm\gamma$ in case (ii), this fact follows from the theory of random walks).*

2. Proofs of the theorems

Proof of theorem (1).

(**a**) Convergence of the finite-dimensional distributions.
If we put $\eta_j = \zeta_j + \zeta_{-j}$, $j = 1, 2, \cdots$ we obtain

$$Y_A(t) = A^{-1/4} \sum_{j=0}^{\infty} \eta_{j+1} P(S(At) > j).$$

First we will calculate $\lim_{A\to\infty} E(Y_A(t) Y_A(s))$.

$S(At)$ can be represented as a sum of independent random variables, $[At]$ of them with the distribution of $S(1)$ and one distributed like $S(At - [At])$ ($[\cdot]$ is the integer part). For $t \in [0, 1]$ we have

$$E\,|\,S(t)\,|^3 = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!} E[\,|\,S_k\,|^3]$$

$$\leq \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!} (kE\,|\,S_k\,|^2) = t^2 + t \leqslant 2t = E(S(t))^2,$$

where S_k is the simple symmetric random walk. Therefore we can use the Berry-Essén theorem (cf. Feller [4]) to obtain

$$|\,P(S(At) > j) - \textstyle\int_{j/\sqrt{At}}^{\infty} (1/\sqrt{2\pi})\exp(-x^2/2)\,dx\,| < c(\sqrt{At})^{-1}.$$

Therefore

$$\lim_{A\to\infty} \sum_{j=0}^{\infty} A^{-1/2}\,|\,P(S(At) > j) - \textstyle\int_{j/\sqrt{At}}^{\infty} (1/\sqrt{2\pi})\exp(-x^2/2)\,dx\,|$$

$$< \sum_{j=0}^{[B\sqrt{At}]+1} A^{-1/2}C(2\sqrt{At})^{-1}$$

$$+ \sum_{j=[B\sqrt{At}]+2}^{\infty} \sqrt{t}\,(\sqrt{At})^{-1} \left\{P\left[\frac{S(At)}{\sqrt{At}} > \frac{j}{\sqrt{At}}\right] + \textstyle\int_{j/\sqrt{At}}^{\infty} (1/\sqrt{2\pi})\exp(-x^2/2)\,dx\right\}$$

Choosing B so that $\int_B^{\infty} (x^2)^{-1}\,dx < \epsilon$, and using Tchebyshev's inequality we obtain

$$\lim_{A\to\infty} \sum_{j=0}^{\infty} A^{-1/2}\,|\,P(S(At) > j) - \textstyle\int_{j/\sqrt{At}}^{\infty} (1/\sqrt{2\pi})\exp(-x^2/2)\,dx\,| = 0.$$

Hence

$$\lim_{A\to\infty} E(Y_A(t) Y_A(s))$$

$$= 2\sigma^2 \lim_{A\to\infty} \sum_{j=0}^{\infty} A^{-1/2} \textstyle\int_{j/\sqrt{At}}^{\infty} (1/\sqrt{2\pi})\exp(-x^2/2)\,dx \int_{j/\sqrt{As}}^{\infty} (1/\sqrt{2\pi}) \cdot\exp(-y^2/2)\,dy$$

$$= 2\sigma^2 \textstyle\int_0^{\infty} \int_{v/\sqrt{t}}^{\infty} (1/\sqrt{2\pi})\exp(-x^2/2)\,dx \int_{v/\sqrt{s}}^{\infty} (1/\sqrt{2\pi})\exp(-y^2/2)\,dy\,dv$$

since this is a Riemann sum converging to the corresponding integral.

After integration by parts three times we obtain

$$\lim_{A\to\infty} E(Y_A(t)Y_A(s)) = \frac{\sigma^2}{\sqrt{2\pi}}(\sqrt{t} + \sqrt{s} - \sqrt{t+s}).$$

Now we use one of the classical formulations of the multidimensional central limit theorem, namely: Let for every n,

$$\vec{X}^{(n,k)} = (X_1^{(n,k)}, \cdots, X_m^{(n,k)})$$

be independent random vectors for $k = 1, 2, \cdots, i_n$. If

(i) $\exists \vec{b} = (\beta_1, \cdots, \beta_m)\ \forall 1 < j < m\ \sum_{k=1}^{i_n} E(X_j^{(n,k)}) - \beta_j \to 0$ as $n \to \infty$,

(ii) $\exists [\sigma_{jp}]\ \forall 1 \le j, p \le m\ \sum_{k=1}^{i_n}$

$$\cdot E\{(X_j^{(n,k)} - E(X_j^{(n,k)}))\,(X_p^{(n,k)} - E(X_p^{(n,k)}))\} \to \sigma_{jp}$$

as $n \to \infty$.

(iii) $\forall \epsilon > 0\ \forall 1 \le j \le m\ \sum_{k=1}^{i_n}$

$$\cdot E\{(X_j^{(n,k)} - EX_j^{(n,k)})^2 \chi(|X_j^{(n,k)} - EX_j^{(n,k)}| > \epsilon)\} \to 0$$

as $n \to \infty$.

then

$$\sum_{k=1}^{i_n} (\vec{X}^{(n,k)} - \vec{b}) \overset{(D)}{\to} N(\vec{0}, [\sigma_{jp}])$$

where $\overset{(D)}{\to}$ means convergence in distribution.

Let

$$X_j^{(n,k)} = A_n^{-1/4}\, \eta_k P[S(A_n t_j) > k].$$

We choose i_n such that for every $j = 1, \cdots, m$

$$\sum_{k>i_n} A_n^{-1/4} P[S(A_n t_j) > k] < 1/A_n.$$

We have $Y_{A_n}(t_j) - \sum_{k=1}^{i_n} X_j^{(n,k)} \to 0$ with probability 1, and also

$$E[(\textstyle\sum_{k=1}^{i_n} X_j^{(n,k)})(\sum_{k=1}^{i_n} X_p^{(n,k)})]$$

$$\to (\sigma^2/\sqrt{2\pi})(\sqrt{t_j} + \sqrt{t_p} - \sqrt{t_j + t_p}) \quad \text{as} \quad n \to \infty.$$

Then we only need to check (iii). It suffices to see that for every $\epsilon > 0$.

$$\sum_{k=0}^{\infty} A^{-1/2} P^2(S(At) > k) \int_{\epsilon A^{1/4}}^{\infty} x^2 P_{\eta_k}(dx) \to 0,$$

but this results from the uniform integrability of $\eta_k{}^2$.

(b) Weak convergence.

According to Billingsley [1] (theorem (12.3)), it suffices to show that there exist $\gamma > 0$, $\alpha > 1$, such that for every A and $t > s$

$$E\{|Y_A(t) - Y_A(s)|^\gamma\} < C(t-s)^\alpha,$$

where C is a constant.

First we will prove

LEMMA.

$$\sum_{j=0}^{\infty} (P[S(At) > j] - P[S(As) > j]) \le E|S(A(t-s))| \le [A(t-s)]^{1/2}, \qquad t > s.$$

Proof of the lemma.

$$\begin{aligned} &\sum_{j=0}^{\infty} \{P[S(At) > j] - P[S(As) > j]\} \\ &\le \sum_{j=0}^{\infty} P[\sup_{s<\tau\le t} S(A\tau) > j,\ S(As) \le j] \\ &= \sum_{j=0}^{\infty} \sum_{k=-\infty}^{j} P[\sup_{s<\tau\le t} S(A\tau) > j,\ S(As) = k] \\ &= \sum_{j=0}^{\infty} \sum_{k=-\infty}^{j} P(\sup_{s<\tau\le t} S(A\tau) > j \mid S(As) = k) P(S(As) = k) \\ &= \sum_{j=0}^{\infty} \sum_{k=-\infty}^{j} P(\sup_{0<\tau\le t-s} S(A\tau) > j - k) P(S(As) = k) \end{aligned}$$

Because the process $S(t)$ has the strong Markov property and symmetrically distributed increments, it satisfies a reflection principle, i.e.

$$P[\sup_{0<\tau\le t} S(\tau) > j] \le 2P[S(t) > j].$$

Next we have

$$\begin{aligned} &\sum_{j=0}^{\infty} (P[S(At) > j] - P[S(As) > j]) \\ &\le 2 \sum_{j=0}^{\infty} \sum_{k=-\infty}^{j} P[S(A(t-s)) > j - k] P[S(As) = k] \\ &< 2 \sum_{n=0}^{\infty} P[S(A(t-s)) > n] \\ &= 2\tfrac{1}{2} E|S(A(t-s))| \le (E[S(A(t-s))]^2)^{1/2} = [A(t-s)]^{1/2}, \end{aligned}$$

where we have used the Schwarz inequality. Now we will apply the following theorem of Marcinkiewicz-Zygmund [7]:

Let ζ_i be independent random variables such that $E\zeta_i = 0$, $E|\zeta_i|^p < \infty$, $p > 1$, then

$$B_p E(\sum_{k=1}^{n} \zeta_k^2)^{p/2} \le E|\sum_{k=1}^{n} \zeta_k|^p \le C_p E(\sum_{k=1}^{n} \zeta_k^2)^{p/2},$$

where B_p and C_p are positive constants which depend only on p.

We put $n = \infty$ (the series converge) and $p = 6$. Then

$$\begin{aligned} E[Y_A(t) - Y_A(s)]^6 &\le A^{-3/2} C_6 E\{\sum_{j=0}^{\infty} \eta_j^2 [P(S(At) > j) - P(S(As) > j)]\}^3 \\ &\le A^{-3/2} C[\sum_{j=0}^{\infty} \{P(S(At) > j) - P(S(As) > j)\}]^3 \le C(t-s)^{3/2}, \end{aligned}$$

as was to be shown.

Proof of theorem (2).

We have

$$x_0(t) = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!} [x_k(0) - k].$$

Consequently, using the inequality of Marcinkiewicz-Zygmund for $p = 4$,

$$E\left[\frac{x_0(At) - x_0(As)}{\sqrt{A}}\right]^4 = A^{-2}E\{\sum_{k=1}^{\infty} [\zeta_1 + \cdots + \zeta_k - k]\}$$
$$\cdot \left[\frac{[\exp(-At)](At)^k - [\exp(-As)](As)^k}{k!}\right]^4$$
$$< CA^{-2}E[\sum_{k=0}^{\infty} (\zeta_{k+1} - 1)^2\{P[S(At) > k] - P[S(As) > k]\}^2]^2$$

where $S(\cdot)$ is the Poisson process with parameter 1. Therefore

$$E\left[\frac{x_0(At) - x_0(As)}{\sqrt{A}}\right]^4 < \text{const } A^{-2}(\sum_{k=0}^{\infty} (P[S(At) > k] - P[S(As) > k]))^2$$
$$= \text{const } A^{-2}(\sum_{k=0}^{\infty} P[S(At) > k,\ S(As) \le k])^2$$
$$= \text{const } A^{-2} (\sum_{k=0}^{\infty} \sum_{j=0}^{k} (P[S(A(t - s)) > k - j]P[S(As) = j]))^2$$
$$= \text{const } A^{-2} (\sum_{n=0}^{\infty} P[S(A(t - s)) > n])^2 = \text{const}(t - s)^2.$$

Proof of theorem (3).

To begin with, it is convenient to mention some additional facts about Spitzer's model [8] and to give another proof of the weak convergence of the real normalized motion of the 0-th particle to Brownian motion in the "simple case", i.e., $v_k = \pm 1$ with probability $\frac{1}{2}$. Spitzer's model in the "simple case" is the following. Initially the particles form a Poisson system with parameter 1 in R and have independent velocities $v_k = \pm 1$ with probability $\frac{1}{2}$ independent by of the positions. If two particles meet they simply exchange velocities. The real motion of the 0-th particle is denoted $y_0(t)$.

Note 1.

The times of the consecutive collisions, $\tau_1, \tau_2, \cdots$ of the 0-th particle are independent exponentially distributed random variables with mean 1.

Clearly the τ_k are finite with probability 1 and we may assume the absence of multiple collisions. We conclude this note immediately by using the nature of Poisson system. In fact, at the random time τ_k the particles do not form a Poisson system. For example set $v_0 = 1$. In $(-\tau_1, \tau_2)$ there does not exist any particle with $v_k = -1$, but such a particle could never influence the motion of the 0-th particle in the future. (If we consider the real motion of the 0-th particle, the remaining particles can be considered as penetrating each other). Therefore

$$y_0(At) = \eta_1 + \cdots + \eta_{\delta_{At}} + \epsilon_{At}(\omega),$$

where $|\eta_i| = \tau_i$ and δ_{At} is the number of collisions in the interval $[0, At]$.

We have three facts:

(a) $\dfrac{\delta_{At}}{At} \to 1$ with probability 1 as $A \to \infty$. (δ_{At} has Poisson distribution with mean At).

(b) $\dfrac{\epsilon_{At}(\omega)}{\sqrt{A}} \to 0$ almost everywhere as $A \to \infty$.

(c) For each k

$$\eta_1 + \cdots + \eta_k = \begin{cases} \tau_1 - \tau_2 + \cdots \pm \tau_k & \text{with probability } \tfrac{1}{2} \\ -\tau_1 + \tau_2 - \cdots \pm \tau_k & \text{with probability } \tfrac{1}{2}. \end{cases}$$

Now, to obtain the other proof of Spitzer's result we apply the method of random change of time (§17 of Billingsley [1]).

For obvious reasons we can consider the horizontal coordinate of $\vec{y}_0(At)$ as if there were no jumps, and it is not worth while to introduce another notation for the horizontal motion of $\vec{y}_0(At)$ between consecutive collisions.

Now,

$$\vec{y}_0(At) = (\eta_1 + \cdots + \eta_{\delta_{At}} + \epsilon_{At}, \gamma_1 + \cdots + \gamma_{\delta_{At}})$$

where $\gamma_i = \pm 1$ with probability $\frac{1}{2}$. Moreover, $(\eta_i)_1^\infty$, $(\gamma_i)_1^\infty$ are independent systems of independent random variables, and once more $\delta_{At}/At \to 1$ and $\epsilon_{At}/\sqrt{A} \to 0$ almost everywhere.

Hence, using the random change of time we may prove Theorem 3.

Note 2.

The horizontal coordinate converges in $C([0, 1], R)$ and the vertical one in $D([0, 1], R)$, but because we work with the product space it is not necessary to explain better the type of convergence. The proof shows what problems appear in the general case.

Proof of theorem (4).

The proof is trivial if we notice the absence of repeated collisions and the fact that, if two particles penetrate each other, one of them can not influence the motion of the other in the future.

Proof of theorem (5).

(i) Spitzer's model.

Because $y_0(t) = \int_0^t v(\tau)\, d\tau$ we have

$y_0(t) = \int_0^t (v(\tau) - E(v))\, d\tau - tE[v]$,

$$(1/t) \int_0^t (v(\tau) - E(v))\, d\tau \to 0 \qquad \text{(See Spitzer [8]).}$$

Therefore, $y_0(t)$ tends to $+\infty$ with probability 1.

(ii) Initial positions on the integers (cf. Szatzschneider [10]). We will show that for any integer $k > 0$, $y_0(n) > k$, for every $n > N_\omega$ and almost all ω. We will use Spitzer's lemma, namely

$$y_0(n) > k \equiv L_k(n) - R_k(n) > k,$$

where $L_k(n)$ (resp. $R_k(n)$) is the number of lines $x + v_k t = k$ (in the plane (x, t)) hitting the line $x = k$ from the left (resp. right) before time t. Therefore

$$P(y_0(n) \le k) = P(L_k(n) - R_k(n) \le k) = P(L_0(n) - R_0(n) \le k)$$
$$= P\{(\textstyle\sum_{j=-\infty}^{0} \chi[j + v_j n > 0] - \sum_{j=0}^{\infty} \chi[j + v_j n < 0]) \le k\}. \quad (3)$$

We have

$$|\textstyle\sum_{j=0}^{\infty} P[v_j n > j] - \sum_{j=0}^{\infty} P[v_j n < -j] - nE[v]| \le 2.$$

Let δ_n be the difference of the sums in (3). Then

$$P(\delta_n \le k) \le P\{|\delta_n - E\delta_n| > \frac{n}{2} E[v]\}$$

for large n. We use now Tchebyshev's inequality for the fourth moment. We put

$$\eta_{2i-1} = \chi[i + v_i n < 0] - E(\chi[i + v_i n < 0])$$
$$\eta_{2i} = -\chi[-i + v_{-i} n > 0] + E(\chi[-i + v_{-i} n > 0]) \quad i = 1, 2, \cdots$$

The η_i are independent random variables with mean 0.

$$E(\textstyle\sum_{i=0}^{\infty} \eta_i)^4 < C(\sum_{i=0}^{\infty} E\eta_i^2)^2$$
$$\le C\{\textstyle\sum_{i=0}^{\infty} E(\chi[i + v_i \cdot n < 0]) + \sum_{i=-\infty}^{0} E(\chi[i + v_i \cdot n > 0]\}^2 \sim n^2(E[v])^2.$$

Finally we complete the proof by using the Borel-Cantelli lemma, since the series $\sum_{n=1}^{\infty} P(y(n) \le k)$ converges.

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