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AXIAL MAPS AND CROSS-SECTIONS

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§1. Introduction

Let P^n denote the real projective n-space. We have inclusions $P^n \subset P^{n+1} \subset \cdots \subset P^{n+k}$. A mapping $u:P^n \to P^{n+k}$, $k \geq 0$ is said to be *nontrivial* if u^* is non-zero in mod 2 cohomology. If $k \geq 1$, this is equivalent to saying that u is homotopic to the natural inclusion. By *an axial map of type* (m, n, k), we mean a map $P^m \times P^n \to P^k$ which is nontrivial on both axes.

By lifting an axial map of type (m, n, k) to the universal coverings we obtain a continuous map $S^m \times S^n \to S^k$ such that $f(-x, y) = f(x, -y) = -f(x, y)$, and then by radial extension we get a continuous map $R^{m+1} \times R^{n+1} \to R^{k+1}$ such that

i) $f(kx, y) = f(x, ky) = kf(x, y)$
ii) $f(x, y) = 0 \Leftrightarrow x = 0 \quad \text{or} \quad y = 0$

Such a map is called a nonsingular skew-skew map of type $(m + 1, n + 1, k + 1)$. This process can be reversed to get:

PROPOSITION (1.1). *There is an axial map of type (m, n, k) if and only if there is a nonsingular skew-skew map of type $(m + 1, n + 1, k + 1)$.*

On the other hand, a non singular skew-linear map $R^{m+1} \times R^{n+1} \to R^{k+1}$, produces an application $S^m \to \{n + 1 \text{ frames in } R^{k+1}\}$ such that $f(x) = -f(-x)$.

This can be interpreted as a set of $n + 1$ linearly independent cross-sections for the trivial $k + 1$ dimensional bundle over S^m. Furthermore, these sections being equivariant under the Z_2 action by sign changing, produce $n + 1$ linearly independent cross-sections for $k + 1$ times the Whitney sum of the Hopf bundle over P^m.

Let ξ denote the Hopf line bundle over P^m, then since the above process can be reversed, we get:

PROPOSITION (1.2). *There are $n + 1$ linearly independent cross-sections for $(k + 1)\xi$ over P^m if and only if there is a nonsingular skew-linear map $R^{m+1} \times R^{n+1} \to R^{k+1}$.*

Now, we have the following corollary of propositions 1.1 and 1.2.

COROLLARY (1.3). *If $(k + 1)\xi$ over P^m has $n + 1$ linearly dependent cross-sections then there is an axial map of type (m, n, k).*

A theorem of Haefliger and Hirsh [4, Lemma 1.1] and James [7, Thm 4.1] establishes the converse, when $m < 2(k - n)$. Thus:

THEOREM (1.4). *If $m < 2(k-n)$, then $(k+1)\xi$ has $n+1$ linearly independent cross-sections over P^m if and only if there is an axial map of type (m, n, k).*

In other words this can be stated as follows:

If $m < 2(k - n)$ and there is a non-singular skew-skew map of type $(m + 1, n + 1, k + 1)$, then there is also one which is skew-linear.

Now, in what follows we will suppose that $m \leq n \leq k$. Notice that then $m < 2(k - n)$ implies $n < 2(k - m)$ and then there are skew-linear and linear-skew maps and in this case arise the question: when are there bilinear maps?

In [1] Adem, Gitler and James, established the following theorem:

THEOREM (1.5). *The following conditions are equivalent:*

i) *The bundle $(k + 1)\xi$ over P^n admits $n + 1$ linearly independent cross-sections.*

ii) *There is an axial map of type (n, n, k).*

iii) *There is an immersion of P^n in R^k.*

The proof of this theorem is based on the fact that if there exists axial map of type (n, n, k) then $n < 2(k - n)$ except for exceptional finite cases, $(n \leq 15)$. [1, Lemma 2.1].

Here we prove the following:

THEOREM (1.6). *If there is an axial map of type (m, n, k), then $n < 2(k - m)$ except for finite exceptional cases in which the related nonsingular maps $R^{m+1} \times R^{n+1} \to R^{k+1}$ are bilinear.*

COROLLARY (1.7). *The following conditions are equivalent:*

i) *There is an axial map of type (m, n, k).*

ii) *The bundle $(k + 1)\xi$ over P^n has $m + 1$ linearly independent cross-sections.*

And then it is also true that if there is a nonsingular skew-skew map $R^{m+1} \times R^{n+1} \to R^{k+1}$, then there is one which is linear-skew.

This result answers affirmatively a question raised by Daniel B. Shapiro in [10], relating skew-skew maps with linear-skew maps.

§2. Proof of Theorem 1.6.

Remember that $0 \leq m \leq n \leq k$, and write $m = 2^s + a$, $n = 2^t + b$, $k = 2^r + c$, where $0 \leq a < 2^s$, $0 < b < 2^t$ and $0 \leq c < 2^r$.

If $n \geq 2(k - n)$ then $2n \geq 2k - n \geq k$ or $2^{s+1} + 2a \geq 2^r + c$, so $s + 1 \geq r \geq t \geq s$, and we have to consider three cases:

Case I. $t = r = s$

Case II. $t = r = s + 1$

Case III. $t = s$ and $r = s + 1$

Before we start to analyze these cases, let us recall the classical Hopf condition and a theorem of L. Astey [2].

THEOREM (2.1) (H. Hopf). *If there exists an axial map of type* (m, n, k) *then the binomial coefficients fulfill the following conditions*:

$$\binom{k+1}{i} = 0 (\text{mod } 2), \quad \text{for} \quad k - n + 1 \leq i \leq m.$$

Proof: Let $x_n \in H^1(P^n; Z_2)$ be the generator. If $f: P^m \times P^n \to P^k$ is an axial map, we have $f^*(x_k) = x_m + x_n$ and hence $(x_m + x_n)^{k+1} = 0$ and thus the Hopf condition.

THEOREM (2.2) (L. Astey). *There is no axial map of type* $(2n, 2k - 2n + 6t, 2k - 1)$ *if*:

$$\text{i)} \quad \binom{k}{n-t} = \cdots = \binom{k}{n-2t} = 0 (\text{mod } 2^t)$$

$$\text{ii)} \quad \binom{k+t}{n-t} = 2^t (\text{mod } 2^{t+1})$$

Recent results of D. Davis remove the condition i) of this theorem. Consequently, to apply theorem 2.2. it suffices only to verify condition ii).

Case I. In this case we are dealing with axial maps of type $(2^s + a, 2^s + b, 2^s + c)$, where $0 \leq a \leq b \leq c < 2^s$ and $2^s + b \geq 2(c - a)$. We will call any of these an axial map of type I. This case follows from the next three lemmas and a final consideration of a few cases.

LEMMA (I.1). *There is no axial map of type I if* $c < 2^s - 1$.

Proof: Let $i = c + 1$, then $c + 1 - b \leq i \leq 2^s + a$ and

$$\binom{2^s + c + 1}{i} = \binom{2^2 + c + 1}{c + 1} = 1 (\text{mod } 2)$$

and we can apply (2.1) to get the result.

LEMMA (I.2). *There is no axial map of type I if* $a \geq 2$, $b \geq 4$, $c = 2^s - 1$ *and* $s \geq 2$.

Proof: Put $k = 2^s$, $n = 2^{s-1} + 1$, $t = 1$ in (2.2) to get that there is no axial map of type $(2^s + 2, 2^s + 4, 2^{s+1} - 1)$ if $s \geq 2$, and then the lemma by restriction.

LEMMA (I.3). *There is no axial map of type I if* $a < 2$, $c = 2^s - 1$ *and* $s > 4$.

Proof: Put $k = 2^s$, $n = 2^s - 2^{s-2} + 2$, $t = 2$ in (2.2) to get that there is no axial map of type $(2^{s-1} + 8, 2^s - 2^{s-1} + 4, 2^{s+1} - 1)$ if $s \geq 4$. But if $a < 2$, then $b \geq 2^s - 2 - 2a \geq 2^s - 4$ and then we get the lemma by restriction.

Now, to cover all the cases of type I in the complement of the above three lemmas, it is enough to consider all cases where $c = 2^{s+1} - 1$ and $s \leq 3$. To see this we just have to consider the case when $a \geq 2$, $b < 4$ and $c = 2^{s+1} - 1$, but then since $2b \geq 2a > 2^s - 2 - b$, we get that $9 \geq 3b \geq 2^s - 2$, and then also $s \leq 3$.

For $s = 0, 1, 2$, and $c = 2^{s+1} - 1$, all the axial maps of type I exist as a consequence of the structure of the complex, quaternion and Cayley numbers, and then the corresponding sections also exist.

For $s = 3$ we have axial maps of type (8, 14, 15) and (8, 15, 15) as a consequence of the existence of a nonsingular bilinear map $R^9 \times R^{16} \to R^{16}$, due to Hurwitz [6], but then also the corresponding cross-sections exist. That there are no more axial maps of type I for $s = 3$ follows from the fact that there is no axial map of type (9, 10, 15) since 16ξ does not have 11 sections over P^9, [9].

Case II. We need just one lemma for this case.

LEMMA (II.1). *There is no axial map of type* $(2^s + a, 2^{s+1} + b, 2^{s+1} + c)$ *if* $b + 2a \geq 2c$.

Proof: We have $b < 2^{2+1}$, so $2^{s+1} + 2a > b + 2a \geq 2c$ and then $c + 1 \leq 2^s + a$. Since $b \leq c$, $b + 2a \geq 2c \geq b + c$, so $c \leq 2a \leq 2^{s+1} - 2$ or $c + 1 < 2^{s+1}$. Now, let $i = c + 1$ and apply (2.1) to get the lemma.

Case III. In this case, we are dealing with axial maps of type $(2^s + a, 2^s + b, 2^{s+1} + c)$, where $0 \leq a \leq b < 2^s$ and $b + 2a \geq 2^s + 2c$. We will call any of these an axial map of type III. Case III follows as I from some lemmas and few final cases.

Before we start with the lemmas let us consider one simple consequence of the numerical hypothesis for this case; $b + 2a \geq 2^s + 2c$ and $b < 2^s$ imply $c < a$ and we have $0 \leq c < a \leq b < 2^s$.

LEMMA (III.1) *There is no axial map of type III if* $a \geq 2$ *and* $b \geq c + 6$.

Proof: First we see from (2.2) that if $\ell \leq 2^{s-1} - 3$, then there is no axial map of type $(2^s + 2\ell + 6, 2^s + 2, 2^{s+1} + 2\ell + 1)$.

Then, since $b \geqslant c + 6$, we have $c \leq b - 6 \leq 2^s - 5$ and the lemma follows by restriction of the above by taking $c = 2\ell + 1$ or $c = 2\ell$ depending on whether it is odd or even since in any case $c \leq 2^s - 5$ implies $\ell \leq 2^{s-1} - 3$.

LEMMA (III.2). *There is no axial map of type III if* $a \leq 1$ *and* $s \geq 5$.

Proof: Again, first apply (2.2) to see that there is no axial map of type $(2^s - 2^{s-2} + 4, 2^s + 2^{s-1} + 8, 2^{s+1} + 2^{s-2} - 1)$ for $s \geq 5$.

Now, $a \leq 1$ implies $a = 1$ and $c = 0$. Also, $b \geq 2^s + 2c - 2a \geq 2^s - 2$. Then we get the lemma by restriction of the above.

LEMMA (III.3). *There is no axial map of type III if* $b \leq c + 5$ *and* $s \geq 6$.

Proof: Again, by (2.2) there is no axial map of type $(2^s + 2^{s-1} + 4, 2^s + 2^{s-1} + 8, 2^{s+1} + 2^s - 1)$ if $s \geq 4$.

Now, we know that $a \geq 2^s + 2(c - b)$ and since we suppose $c - b > 5$ we get $b \geq a \geq 2^s - 10$ and we have the lemma by restriction of the above for $s \geq 6$.

Now, analyzing the axial maps of type III with $s \leq 5$ and not covered by the above lemmas we find the following:

For $s = 0$ there are no cases of axial maps of type III. For $s = 1$ or 2 all axial maps of this type exist as a consequence of the structure of the quaternion and Cayley numbers and then also the corresponding sections exist.

For $s = 3$ there are three axial maps of type III: (11, 11, 16), (12, 12, 18) and (15, 15, 22). The first one as a consequence of a nonsingular bilinear map $R^{12} \times R^{12} \to R^{17}$, [8], and the other two because P^{12} immerses in R^{18} and P^{15} immerses in R^{22}, [3], and then also the corresponding cross-sections exist.

The nonexistence of more axial maps of type III with $s = 3$ follows, for example, from the following facts [9]:

a) The maximal number of cross-sections for 19ξ over P^{13} is 10.
b) The maximal number of cross-sections for 18ξ over P^{12} is 9.
c) The maximal number of cross-sections for 22ξ over P^{13} is 12.

For $s = 4$ there are no axial maps of type III. This follows, for example, from the following facts in [1] and [9]:

a) There is no axial map of type $(n, n, n + k)$ if $n > 16$ and $n > 2k$.
b) The maximal number of cross-sections for 35ξ over P^{17} is 21.
c) The maximal number of cross-sections for 37ξ over P^{17} is 23.
d) The maximal number of cross-sections for 45ξ over P^{23} is 25.

That there are no axial maps of type III and $s = 5$ follows from:

a) There is no axial map of type (52, 56, 95).
b) There is no axial map of type (52, 42, 87).

and these two cases follow from (2.2).

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