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INFINITESIMAL MODELS FOR CALCULUS

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Introduction

The purpose of the present paper is to show how to construct, in explicit form, models that structure calculus in the manner it was, apparently, conceived by Leibniz, Euler and others. From a strict theoretical point of view this might be of little interest (if one is not a constructivist), since we have the so called non-standard model, due to A. Robinson [8]. Nevertheless, it is interesting, even from the mathematical point of view, that it is possible to validate infinitesimal calculus in a constructible way, and by methods far more elementary than those of Robinson or others (see [3], [4] and [11]).

To be sure, the idea of constructing infinitesimal models for calculus, in a way similar to the one we present here, is not a new idea. Previous efforts have been made as in [2], [9], [10] and [12], but they have not prospered. The main reason for it has been either the impossibility of actual manipulation or the lack of something to play the part that the extension or transfer principle plays in the non-standard theory. In our presentation this role is, so to speak, played by the concept of faithfulness of extensions, to be defined later. This concept is not necessary in the construction of the model, but it is the key for validation, i.e., to prove that the model is equivalent to the standard (ϵ, δ) model of Calculus.

The Set R*

Briefly, the construction of an infinitesimal model for calculus requires a numerical structure R^* (extension of the reals) and then extensions of real domains and real functions to R^*, in order to define calculus concepts through manipulation of these extensions (we refer to these concepts as $*$-concepts).

There are different options for R^*. To be specific, we mention one due to Levi-Civita (see [6] or [7]), which consists of elements r^* of the form

$$r^* = \sum_{i=1}^{\infty} a_i \omega^{\alpha_i}$$

where ω is a symbol, a_i, α_i are reals, $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$, unbounded. With natural definitions R^* is a non-archimedian ordered field extension of the reals, R, and R^* contains infinitesimals and infinite numbers. The details will be omitted, since they can be found in the above mentioned literature. To be sure, elements $r^* \in R^*$ will be called positive (negative) infinite numbers if $\alpha_1 > 0$ and $a_1 > 0$ ($a_1 < 0$). They will be called finite if $\alpha_1 = 0$ (in particular reals are those elements for which $\alpha_1 = 0$ and $a_i = 0$, $i = 2, 3, \cdots$). Finally, they are called positive (negative) infinitesimals if $\alpha_1 < 0$ and $a_1 > 0$ ($a_1 < 0$).

For any fixed real a_1 the set of numbers of the form

$$a_1 + \sum_{i=2}^{\infty} a_i \omega^{\alpha_i}, \; \alpha_2 < 0$$

constitute the atom (or monad) around the real a_1, that is, they are the non reals infinitely close to a_1, in particular they belong to the left (right) part of the atom if $a_2 < 0$ ($a_2 > 0$).

The way to extend sets is quite natural and simple. Given a set $A \subseteq R$, its extension $A^* \subseteq R^*$ will consist of the following elements. First, all of A is included in A^*. Second, if the real p is an accumulation point of A by the left and/or by the right, then we include in A^* the left and/or right portion of the atom of p (exclusion made of p, unless $p \in A$). Finally, if A is positively (negatively) unbounded, we include in A the positive (negative) infinite numbers.

Faithful Extensions

We now come to the problem of extending real functions $f: A \to R$ into functions $f^*: A^* \to R^*$. For this we introduce the concept of faithfulness.

Definition. An extension f^* of a real function f is said to be faithful if the following conditions are fulfilled:

1. $f^*(A^*) \subseteq [f(A)]^*$
2. Whenever there exists a sequence x_n ($x_n \neq p$) in A converging by the right and/or the left to a real p, such that the sequence $f(x_n)$ converges by the right and/or the left to a real q, then part from the right and/or left of the atom of p goes under f^* to the corresponding part of the atom of q, and conversely. Obvious modifications have to be made for the cases in which either p or q or both are $\pm\infty$.

The idea of the above definition is that the behavior of the extension f^* at the atom of a real point is, in a manner, comparable to the local behavior of f at the same real point, in such a way that one can define $*$-concepts using f^* and these concepts will coincide with the standard ϵ, δ, counterparts. To illustrate the situation, let us define one of the fundamental concepts of calculus, that of limit of a function, in the frame of our infinitesimal model (from now on, extension of function will mean faithful extension).

Definition. The $*$-limit of a real function $f(x)$, when x tends to a real p, is the real q ($*\text{-}\lim_{x \to p} f(x) = q$) if, for every $P^* \neq p$ in the atom of p (and in the domain of some extension f^*), we have that $f^*(p^*)$ belongs to the atom of q.

Take now the case of a function $f(x)$ that is standard continuous in its domain (i.e., $\lim_{x \to p} f(x) = f(p)$). Then clearly, if we define, for example $f^*(x + i) = f(x) + i$ where i represents an infinitesimal), this extension is faithful and, of course, $*\text{-}\lim_{x \to p} f(x) = f(p)$, so that $f(x)$ is $*$-continuous. To be sure, one can not always take just any faithful extension of the function involved. For example, in the case of the derivative, it is not the limit of $f(x)$ but that of $F(h) = \dfrac{f(x + h) - f(x)}{h}$, that is to be considered, so that the extension f^* of f should be such that $F^*(i) = \dfrac{f^*(x + i) - f(x)}{i}$ is faithful with respect to $F(h)$ around $h = 0$.

As a further illustration consider Dirichlet's function ($d(x) = 1$ is x is rational, $d(x) = 0$ otherwise). Then a faithful extension of d^* is given by:

$$d^*(x + i) = 1 \qquad \text{if } i \text{ is "rational"}$$

$$d^*(x + i) = 0 \qquad \text{otherwise,}$$

where i being "rational" means that the first non zero coefficient of the representation of i is rational. If now one defines (in the obvious way) the concepts of $*\text{-}\overline{\lim}$ and $*\text{-}\underline{\lim}$, they will be 1 and 0, respectively, and coincide with the standard counterparts.

"Canonical" Extensions

If $f(x)$ is continuous, certainly the extension $f^*(x + i) = f(x) + i$ is faithful, but also is rather arbitrarily related to $f(x)$. We mention here a more canonical way to extend functions. Consider a function $f: A \to R$. In order to extend f to A^*, we need to say what values f^* will take at the atom of any point p that is an accumulation point of A. Now, assume that at a certain neighborhood V of p (relative to A), the values of $f(p + h)$, $h \neq 0$, $p + h \in V$, are given by a certain explicit "formula" $g(p, h)$. We will then define f^* for all elements $p + i$ (in the relevant part of the atom of p) by computing $g(p, i)$ whenever that makes sense directly, or can be adequate to make sense and render f^* faithful. So, in the case of algebraic functions, it is clear that $g(p, i)$ will make sense, since R^* is a field. In the case of analytic functions, that is when

$$g(p, h) = \textstyle\sum_{n=0}^{\infty} \alpha_n(p)h^n,$$

then we can adequate $g(p, i)$ by taking it to mean:

$$g(p, i) = \textstyle\sum_{n=0}^{m} \alpha_n(p)i^n,$$

where m is chosen so that the extension is adequate for the problem at hand. For example, in the computation of limits it may happen, because of cancellations, that one has to take a relatively big m in order to get the right answer. To illustrate, consider the simple example of computing $*\text{-}\lim_{h\to 0}(\cos h - 1)/h^2$, in this case,

$$g(p, h) = g(o, h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots,$$

taking $m = 1$ would mean taking $g(0, i) = 1$ and rendering the wrong result that the limit is zero. The square in the demoninator warns us to make $m \geqslant 2$. In fact, if we make $m = 2$ we get the right result that the limit is $-\frac{1}{2}$, taking m greater does not change things.

In the preceding section we saw another example of adequation of $g(p, h)$, for the case of Dirichlet's function.

As a matter of fact, we can prove the following.

THEOREM. *Let f: $A \to R$ be any given function and p be an accumulation point of A. Then there exists a "canonical" faithful extension f^* to the atom of p.*

Proof. Let V be a neighborhood of p (relative to A) and let L be the set of all limit points of sequences $f(x_n)$, where $x_n \in V$, $x_n \to p$. Separate the non-reals of the atom of p in as many disjoint subsets as elements are in L, now we define f^* on every subset in such a way that its image goes to the atom of the corresponding limit point in L. This extension is certainly faithful.

The Equivalence Theorem

In this section we will simply state the result that, on the basis of the faithfulness concept, validates this infinitesimal model of calculus in relation to the standard ϵ, δ model.

The definition of $*$-concepts of calculus can be worded in the same way as is done in non-standard analysis, see [8] or [11], so we will not give these definitions here.

THEOREM. *For any real function $f\colon A \to R$ that has an adequate faithful extension $f^*\colon A^* \to R^*$, the $*$-concepts of calculus are equivalent to their corresponding standard ϵ, δ concepts.*

The adjective "adequate" is to remind that in some cases we need more than just faithfulness of f^*, as illustrated before.

The proof of this theorem follows from a straight forward manipulation of correlated definitions and the faithfulness condition, and will therefore be omitted.

Final Comments

On the basis of models like the one we have described here, one can structure a more plausible introduction to calculus, than those structured on the basis of the so called, non-standard models, see [1] and [4]. It also seems simpler to program calculus into a computer.

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