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A NOTE ON HYPERPLANE SECTIONS OF REAL ALGEBRAIC SETS

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It is well known that the set of hyperplanes which meet a non zero-dimensional algebraic set over an algebraically closed ground field contains a non-empty Zariski open set. Although a real analogue of this result, changing the Zariski topology to the order topology, can be easily proven using [6] and [7], many hyperplane sections of a non zero-dimensional algebraic set V over a real closed ground field R may well be empty, and the "size" of the set of all such hyperplanes reflects the extent of $V \subset R^n$.

We study in this note the size of the set of hyperplanes which meet V in a "good way". More precisely, let us denote by V_c the locus of central points of V, i.e., the closure, in the order topology of R^n, of the set of regular points of V. In [6] it was proven the following:

THEOREM 1. (Dubois-Recio). *There exists a finite number of linear isomorphisms of R^n such that, given a "generic" hyperplane, at least one of its transforms meets V_c.*

We state here a sharper result:

THEOREM 2. *There exists a linear isomorphism σ of R^n such that for every "generic" hyperplane H of R^n, either H meets V_c or its transform by σ meets V_c.*

Set $x = (x_1, \cdots, x_n)$, $y = (y_0, \cdots, y_n)$ indeterminates. Let $p \subset R[x]$ be the ideal of V and $q = p \cdot R[x, y] + gR[x, y]$, where $g = y_0 + y_1x_1 + \cdots + y_nx_n$ is the equation of a generic hyperplane. We call $G \subset R^n \times R^{n+1}$ the set of zeroes of q and $\pi: R^n \times R^{n+1} \to R^{n+1}$ the standard projection. It is obvious that a hyperplane $H_y: y_0 + y_1x_1 + \cdots + y_nx_n = 0$ meets V if and only if $y = (y_0, \cdots, y_n) \in \pi(G)$. Moreover, it was proved in 3.2 of [5] that H_y meets V_c if and only if $y \in \pi(G)$. So, theorem 2 can be reformulated in the following way:

THEOREM 2′. *Given a non zero-dimensional algebraic set $V \subset R^n$, there exists a projective automorphism σ of $R(y)$ such that $\pi(G_c) \cup \sigma(\pi(G_c))$ is dense in R^{n+1}.*

We are interested in hyperplanes meeting V_c instead of all hyperplanes meeting V because V_c plays a fundamental role in the theory of ordered function fields, in case V is irreducible. For instance, it is well known the equivalence between:

1) $a \in V_c$.
2) There is a real place of the field $R(V)$ of rational functions of V centered at a.
3) There is an ordering in $R(V)$ which contains all $f \in R(V)$ verifying $f(a) > 0$.

Moreover, if V is irreducible, and keeping all notations above, p and q are prime ideals and it is easy to check that $q \cap R[y] = \{0\}$. Then, let $E = R(y)$, $F = qf(R[x, y]/q)$ and $j: E \to F$ the canonical inclusion. We denote by X_E the space orderings of E.

3. *Definition.* A subset A of projective automorphisms of E is said admissible if for every $\alpha \in X_E$ there is $\sigma \in A$ such that $\sigma(\alpha)$ extends to F.

Using theorem 3.2 in [5] and the correspondence introduced in [5] and [8] between semialgebraic subsets of V and clopen (closed and open) subsets in the space of orderings of $R(V)$, the theorem 1 quoted above is equivalent to:

THEOREM 1′. *There exist finite admissible subsets* (*in fact with cardinality* $\leq n + 1$).

In the same paper [5] it is defined the number $D(V)$ = minimum cardinality of admissible subsets. In this language our theorem 2′ states $D(V) \leq 2$.

4. REMARK. *This is in fact, the best bound, because* $D(V) \neq 1$ *if* V *is bounded.*

Proof: Let r be a positive element of R such that

$$V \subset \{x \in R^n: x_1^2 + \cdots + x_n^2 < r^2\}$$

Then

$$U = \{y = (y_0, \cdots, y_n) \in R^{n+1}: y_0^2 - r(y_1^2 + \cdots + y_n^2) > 0\}$$

is a non empty open subset of R^{n+1} and $U \cap \pi(G) = \emptyset$ so $\pi(G_c)$ is not dense in R^{n+1} or equivalently, $D(V) \neq 1$.

In the geometrical language that means that σ cannot be omitted in theorem 2′. We shall see in 5 that bounded sets are not the only ones with $D(V) \neq 1$.

Proof of theorem 2′. If dim $V > 1$, the central locus of V contains the central locus of a curve. So, it is enough to study the case dim $V = 1$.

We shall work, first, in case R is a Cantor field, see [2], [3] and [4]. To pass to an arbitrary real closed field we shall use a trick which appears in [1].

Let us choose a regular point $O \in V$, a neighborhood U of $O \in R$ and analytic functions $f_1, \cdots, f_n: U \to R$ such that $f_i(0) = 0$ and the image of

$$f: U \to R^n: t \to (f_1(t), \cdots, f_n(t))$$

is contained in V_c.

Set

$$f_j(t) = \textstyle\sum_{i=1}^{\infty} b_{ij} t^i, \quad t \in U$$

and let us consider, for each

$$y = (y_0, \cdots, y_n) \in R^{n+1}, \quad F_y(t) = y_0 + \textstyle\sum_{i=1}^{\infty} (\sum_{j=1}^{n} b_{ij} y_j) t^i$$

From 2.11 in [1] and the usual rules of calculus it is immediate that $F_y: (-r/n, r/n) \to R$ is a well defined analytic function, r being the minimum of the radii of convergence of $f_1, \cdots, f_n$.

Choose $t_0 \in (0, r/n)$ such that $p = (f_1(t_0), \cdots, f_n(t_0)) \neq 0$, and let M be the set of $y \in R^{n+1}$ with $F_y(0) \cdot F_y(t_0) < 0$. Clearly $M \subset \pi(G_c)$ because $F_y(0) \cdot F_y(t_0) < 0$ implies the existence of $t \in (0, t_0)$ verifying

$$y_0 + \sum_{j=1}^{n} y_j f_j(t) = 0$$

or, equivalently, $(f(t), y) \in \pi(G_c)$.

So, it is enough to find σ such that

$$M \cup \sigma(M) = R^{n+1}$$

But M is the set of $y \in R^{n+1}$ such that H_y meets the segment joining the origin 0 with p and, after a linear change of coordinates, we can assume that $p = (0, 0, \cdots, 0, 1)$.

Then

$$M = \{y \in R^{n+1}\colon (y_0 + y_n)y_0 < 0\} \cup \{y \in R^{n+1}\colon y_0 = y_n = 0\},$$

and choosing a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with coefficients in R such that

$$b(b + d) < 0$$

$$a(a + c) < 2ab + ad + bc < 0,$$

the automorphism σ of $R(y)$ which fixes y_j, $0 \neq j \neq n$ and sends (y_0, y_n) to

$$A\begin{pmatrix} y_0 \\ y_n \end{pmatrix}$$

verifies $M \cup \sigma(M) = R^{n+1}$.

Otherwise, there would exist $y \in R^{n+1}$ with $(y_0, y_n) \neq (0, 0)$, such that

$$(y_0 + y_n)y_n \geq 0$$

$$(ay_0 + by_n)((a + c)y_0 + (b + d)y_n) \geq 0$$

By the choice of A we deduce that $y_0^2 + y_n^2 \leq 0$. Absurd.

We shall establish now our result for an arbitrary real closed ground field R. Obviously, we can assume V is irreducible.

Let k be the field obtained adjoining to Q all the coefficients of a set of generators of p. Then the real closure K of k with respect to $k \cap R^2$ is a Cantor field (see [3] and [4]). So, if p_K and q_K are, respectively, the ideal generated by p and q in $K[x]$ and $K[x, y]$, there exists a projective automorphism σ_K of $K(y)$ such that $\pi(G_c{}^K) \cup (\pi(G_c{}^K)) = K^{n+1}$; clearly G^K denotes the set of zeroes of q_K in $K^n \times K^{n+1}$. Using 3.2 in [5] this implies that, for a given ordering α in $K(y) = E_1$, either α or $\sigma_K(\alpha)$ extends to $F_1 = qf(K[x, y]/q_K)$. Let σ be the trivial extension of σ_K to $E = R(y)$.

To prove the density of $\pi(G_c) \cup \sigma(\pi(G_c))$ it suffices, again by 3.2 in [5], to show that either β or $\sigma(\beta)$ extends to $F = qf(R[x, y]/q)$ for every order β in E.

For this, we use Serre's criterion [9]. Let us suppose that $\alpha = \beta \cap E_1$ extends to F_1. Then, if $p_1, \cdots, p_m \in \beta$ and $h_1, \cdots, h_n \in F$ verify $p_1 h_1{}^2 + \cdots + p_n h_n{}^2 = 0$, we construct the field L adjoining to K the coefficients of each p_i and h_i. Since α extends to F_1 and L is a finitely generated extension of K with $K = L$ or tr.d. $L/K \geq 1$, α extends to an ordering α^* in $L(y)$ which extends to $qf(L[x, y]/q)$. So each $h_i = 0$ and β extends to F.

In case α does not extend to F_1, we know that $\sigma_K(\alpha)$ extends to F_1 and repeating the same argument above we conclude that $\sigma(\beta)$ extends to F.

5. Final remarks

(**a**) As was indicated in 4, we can find unbounded set V with $D(V) = 2$. More precisely, $D(V) = 2$ for every irreducible conic. In fact it is enough to compute $D(V)$ for $V_1 = \{x_2 = x_1{}^2\}$ and V_2: $\{x_1 \cdot x_2 = 1\}$. But, in the first case $\pi(G_c) = \{y \in R^3: y_1 - 4y_0 y_2 \geq 0\}$, and in the second one $\pi(G_c) = \{0\} \cup \{y \in R^3: y_2 = 0, y_0, y_1 \neq 0\} \cup [\{y \in R^3: y_2 \neq 0, y_0{}^2 - 4y_1 y_2 \geq 0\} \cap (\{y_1 \neq 0\} \cup \{y_1 = 0, y_0 \neq 0\})]$.

In both cases $\pi(G_c)$ is not dense in R^3.

(**b**) $D(V) = 1$ if V contains a line. It is enough to consider V: $\{(x_1, x_2) \in R^2: x_2 = 0\}$. Then $\pi(G_c) = \{y \in R^3: y_1 \neq 0\}$ is a dense subset of R^3.

(**c**) $D(V)$ is not invariant under ambient algebraic isomorphism. Let us consider F: $R^2 \to R^2$ defined by $F(x, y) = (2x + \frac{1}{2}y^2, 2x + y + \frac{1}{2}y^2)$ whose inverse is G: $R^2 \to R^2$: $(u, v) \to (\frac{1}{4}(2u - (v - u)^2, v - u)$.

From (**b**) $D(V) = 1$ if V: $\{y = x\}$. However $W = F(V)$ is the irreducible conic $(y - x)(4 + y - x) - 2x = 0$ and, from (**a**), $D(W) = 2$.

(**d**) The number $D(V)$ gives a measure of the extent of the locus of central points of V. If we want to know the extent of V we must know $\Delta = \pi(G) - \pi(G_c)$. This is a "thin" set in case V is a curve, as was pointed out in 3.4 of [5]: Δ is contained in a proper algebraic subset of R^{n+1}. But the same does not hold in higher dimensions: for example, if

$$V: \{(1 - x_3{}^2)x_1{}^2 = x_1{}^4 + x_2{}^4\}$$

is the Coste-Coste Roy Balloon, Δ contains

$$\{y \in R^4: y_3 \neq 0, y_0{}^2 - 3(y_1{}^2 + y_2{}^2 + y_3{}^2) > 0\}.$$

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