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ON YUZVINSKY'S THEOREM CONCERNING ADMISSIBLE TRIPLES OVER AN ARBITRARY FIELD

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1. Notation and Intention

For any field F of characteristic not 2 (as always will be supposed here), let (F^n, q) be a quadratic space, where F^n is the usual n-dimensional vector space over F, whose elements are column vectors $x = (x_1, \cdots, x_n)^t$ (where t is the transpose operation) and $q: F^n \to F$ is the standard quadratic map given by $q(x) = x_1^2 + \cdots + x_n^2$. Now, if $B: F^n \times F^n \to F$ is the symmetric bilinear pairing determined by q, then $B(x, y) = x^t y$ is the inner product of x and y, and it can be regarded as the product of the row vector x^t by the column vector y. Trivially, (F^n, q) constructed in this form, is a regular (nonsingular) quadratic space (see [3; Chap I]).

Let (F^r, q_1), (F^s, q_2) and (F^n, q) be quadratic spaces as above, where q_1, q_2 and q are the standard quadratic maps, respectively, for the values r, s, and n. A bilinear map $\theta: F^r \times F^s \to F^n$ is a *normed map* if

$$q(\theta(x, y)) = q_1(x) q_2(y), \tag{1.1}$$

for all $x \in F^r$ and $y \in F^s$. Any map θ of this type is called a *normed pairing* of size $[r, s, n]$, and we say that a triple $[r, s, n]$ is *admissible* over F if there exists such a normed map θ.

The problem to determine if a given triple $[r, s, n]$ is admissible over an arbitrary field F, seems to be a difficult one, even for low values of r. The cases $r \leq 4$ have been decided. For $r = 3$ this is done using some results established by the author in [2]. Later, D. B. Shapiro presented in [5] an elegant new approach and an extension of the author's results. The case $r = 4$ was recently solved by S. Yuzvinsky in [6], using some clever geometric arguments. The information needed to settle this case is contained in the following.

THEOREM (1.2) (Yuzvinsky). *No triple* $[4, 4h + 1, 4h + 3]$ $(h = 1, 2, \cdots)$ *is admissible over any field* F.

In this note we will reproduce almost verbatim the proof given by Yuzvinsky of his theorem. The only novelty presented here is the form how it is established the regularity, dimension and invariance of some subspaces required in the proof. This is accomplished using canonical forms for pairs of matrices through orthogonal equivalence, as it has already been done by the author in [1] and [2]. This approach seems to make more transparent certain parts of the proof and perhaps it can be used to study other cases.

For a complete account of results and bibliography on the subject, the reader is referred to the superb expository paper [4] by Shapiro.

2. Dimension and invariance of certain subspaces

In terms of matrices, the existence of a normed map, as the one in (1.1), is equivalent to the existence of a set $N_1, \cdots, N_r$ of r rectangular $n \times s$ matrices over F, such that

$$N_i^t N_i = I_s \quad \text{if} \quad 1 \leqslant i \leqslant r, \tag{2.1}$$

$$N_i^t N_j + N_j^t N_i = 0, \quad \text{if} \quad i \neq j, \quad 1 \leqslant i, j \leqslant r. \tag{2.2}$$

The relations (2.1) and (2.2) are called the Hurwitz equations. They are as in [2; (2.5)] with a slight difference: the $n \times s$ matrices N_i here are the transpose of the $s \times n$ matrices M_i there (i.e., $N_i = M_i^t$).

As before, regard $V = F^s$ and $W = F^n$ as quadratic spaces, and consider each N_i as a linear transformation N_i: $V \to W$.

Remark: Any quotation here to a statement in [1] or [2] about matrices, should be understood as a reference to the equivalent result obtained under the transpose operation. This departure from [1] and [2] adopted here, is necessary in order to have a matrix N_i as above, operating to the *right* on column vectors.

Let V_i = image (N_i), then V_i is a regular subspace of W and dim $V_i = s$. In fact, V is isometric with V_i through the linear transformation $\bar{N}_i$: $V \to V_i$ obtained from N_i by restricting its range to its image. Also, it follows that the composition $\bar{N}_j (N_i^t \mid V_i)$: $V_i \to V_j$ is an isometry.

To establish certain properties of the subspaces $V_i \cap V_j$ and $V_i + V_j$ of W, some restrictions on s and n need to be made. Set $s = 2k + 1$ and $n = 2k + 3$. For $i \neq j$, let N_i and N_j be *any two* $n \times s$ matrices over F satisfying the Hurwitz equations. We may suppose that F is algebraically closed since this is no restriction when we are verifying nonexistence of a set of matrices over F (see [1; p 35]). If P and Q are orthogonal matrices over F, or orders n and s, respectively, then it readily follows that the matrices $E_1 = PN_iQ$ and $E_2 = PN_jQ$, also satisfy the Hurwitz equations. As will be shown, a convenient choice of P and Q will bring E_1 and E_2 to two very simple canonical forms.

Consider the matrices A and B, respectively, of orders $s \times s$ and $s \times 2$, defined by

$$A = \operatorname{diag}[0, C, \underset{k\text{times}}{\cdots} C] \text{ where } C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

Now, set E_1 and E_2 as the $n \times s$ matrices defined below

$$E_1 = [I_s, 0]^t \quad \text{and} \quad E_2 = [A, B]^t, \tag{2.3}$$

where 0 in the expression for E_1 represents the $s \times 2$ matrix of zeros.

We have the following

LEMMA (2.4). *Set $s = 2k + 1$ and with $i \neq j$, let N_i and N_j be two $(s + 2) \times s$ matrices over an algebraically closed field. Moreover, suppose that the matrices N_i and N_j satisfy the Hurwitz equations. Then, there exist orthogonal matrices P and Q, such that*

$$E_1 = PN_iQ \quad \text{and} \quad E_2 = PN_jQ, \tag{2.5}$$

where E_1 and E_2 are the matrices of (2.3).

Proof. The work for the proof was already done elsewhere. Here, as a reference we may say that it readily follows by combining in a single pair the orthogonal matrices used to obtain, first [1; (3.2)], and then [2; (2.8)].

Let $U_h = E_h(V)$ for $h = 1, 2$, and then regard $V_i \cap V_j$, $V_i + V_j$, $U_1 \cap U_2$ and $U_1 + U_2$ as quadratic subspaces of W. The following isometries ($\backsimeq$) hold.

$$V_i \cap V_j \backsimeq U_1 \cap U_2 \quad \text{and} \quad V_i + V_j \backsimeq U_1 + U_2. \tag{2.6}$$

More precise is the next

LEMMA (2.7). *Let $P\colon W \to W$ be the transformation induced by the $n \times n$ orthogonal matrix P of* (2.4). *Then, the linear maps*

$$\tau\colon V_i \cap V_j \to U_1 \cap U_2 \quad \text{and} \quad \theta\colon V_i + V_j \to U_1 + U_2,$$

defined by $\tau(u) = Pu$ and $\theta(v + w) = P(v + w)$, where $u \in V_i \cap V_j$ and $v \in V_i$, $w \in V_j$, are isometries.

Proof. It is omitted since it follows directly from (2.5) and the definition of the terms used in the arguments.

To write explicitly U_1 and U_2 as subspaces of W, let $x \in V$, so that $x = (x_1, \cdots, x_s)^t$. Then, it follows from (2.3) that,

$$E_1(x) = (x_1, \cdots, x_s, 0, 0)^t \quad \text{and} \tag{2.8}$$

$$E_2(x) = (0, -x_3, x_2, \cdots, -x_s, x_{s-1}, x_1, 0)^t. \tag{2.9}$$

Therefore,

$$U_1 = \{x \mid x = (x_1, \cdots, x_s, 0, 0)^t\} \quad \text{and}$$

$$U_2 = \{y \mid y = (0, y_1, \cdots, y_s, 0)^t\},$$

where x_i and y_i for $1 \leqslant i \leqslant s$, are arbitrary elements of F.

Consequently,

$$U_1 \cap U_2 = \{u \mid u = (0, u_1, \cdots, u_{s-1}, 0, 0)^t\} \quad \text{and} \tag{2.10}$$

$$\mathrm{U}_1 + \mathrm{U}_2 = \{\mathrm{v} \mid \mathrm{v} = (\mathrm{v}_1, \cdots, \mathrm{v}_{s+1}, 0)^t\}, \tag{2.11}$$

where u_i and v_j in their respective range of indices, are also arbitrary elements of F.

Hence, from (2.6) it follows that $V_i \cap V_j$ and $V_i + V_j$ are regular subspaces of W, and that

(2.12) $\dim(V_i \cap V_j) = s - 1$ and $\dim(V_i + V_j) = s + 1$.

LEMMA (2.13). (Yuzvinsky (see [4; p 247])). *Let N_i and N_j be as in* (2.4) *and set $f_{ij} = N_i N_j^t$: $W \to W$. Then, the subspace $V_i \cap V_j \subseteq W$ is invariant under each of the maps f_{ij} and f_{ji}. Furthermore, for N_i, N_j: $V \to W$ it follows that*

(2.14) $$N_i^{-1}(V_i \cap V_j) = N_j^{-1}(V_i \cap V_j).$$

Proof. Let $g_{12} = E_1 E_2^t$ and $g_{21} = E_2 E_1^t$. From (2.5) it follows that $f_{ij} = P^t g_{12} P$ and that $f_{ji} = P^t g_{21} P$. Now, if $u \in U_1 \cap U_2$, using the explicit expressions given by (2.8), (2.9) and (2.10), it follows directly that $g_{12}(u)$ and $g_{21}(u)$ are in $U_1 \cap U_2$.

Let τ^{-1}: $U_1 \cap U_2 \to V_i \cap V_j$ be the inverse of the isometry τ defined in (2.7). Then, for $v \in U_1 \cap U_2$, $\tau^{-1}(v) = P^t v$ where P^t is the transpose of the orthogonal matrix P. Let $u \in V_i \cap V_j$ and $v = Pu$. Then, $f_{ij}(u) = P^t g_{12}(v)$, and since $g_{12}(v) \in U_1 \cap U_2$, it follows that $f_{ij}(u) \in V_i \cap V_j$. Analogously, the same holds for $f_{ji}(u)$. Actually, f_{ij} and f_{ji} are isometries of $V_i \cap V_j$.

To prove (2.14), suppose $x \in N_i^{-1}(V_i \cap V_j)$. Then, $N_i(x) = u$ with $u \in V_i \cap V_j$ and, from (2.1), it follows that $x = N_i^t(u)$. Hence, $N_j(x) = N_j N_i^t(u) = f_{ji}(u) \in V_i \cap V_j$. Therefore, $x \in N_j^{-1}(V_i \cap V_j)$. The argument can be reversed and this completes the proof of (2.13).

3. Proof of theorem (1.2)

If we use the propositions already established, the proof of (1.2) can be formulated in a few lines. This is accomplished in the last paragraph of Yuzvinsky's paper [6]. For completeness, it is also presented here, and the method already developed will allow us to exhibit explicitly the subspaces required in this part of the proof.

Assume that theorem (1.2) is false. Then, there exists an admissible triple $[4, 4h + 1, 4h + 3]$ over some field F and, if $s = 4h + 1$ and $n = 4h + 3$, this is equivalent to have four $n \times s$ matrices M_i, for $1 \leqslant i \leqslant 4$, over F, fulfilling the Hurwitz equations. We will show that this is not possible.

Clearly, if P and Q are orthogonal matrices over F, of orders n and s, respectively, then the new set of matrices PM_iQ, for $1 \leqslant i \leqslant 4$, also satisfy the Hurwitz equations and they can be used to replace the original set. Now, in accordance with (2.4), choose P and Q such that $PM_1Q = E_1$ and $PM_2Q = E_2$. Then, set $N_i = PM_iQ$ for $1 \leqslant i \leqslant 4$. Hence, the assumption that (1.2) is false, is equivalent to have the above four matrices as follows: $N_1 = E_1$, $N_2 = E_2$, N_3 and N_4.

As before, regard $V = F^s$ and $W = F^n$ as quadratic spaces, and consider each N_i as a linear transformation N_i: $V \to W$. Consider the four subspaces $V_i \subseteq W$, defined as the images of N_i and observe that $V_1 = U_1$ and $V_2 = U_2$.

If $i \neq j$, we have [see (2.6), (2.12)] that $V_i + V_j$ is a regular subspace of dimension $4h + 2$. If $V_k \subseteq V_i + V_j$ for some $k \neq i, j$, then, restricting the range of N_i, N_j and N_k, will give transformations $V \to V_i + V_j$ that would show the existence of a normed pairing of size [3, $4h + 1$, $4h + 2$]. But this contradicts [2; (3.1)]. Therefore, $V_k \not\subseteq V_i + V_j$ and, this implies that $V_i + V_j + V_k = W$. Now, let $D_1 = V_i + V_k$ and $D_2 = V_j + V_k$. Then, $W = D_1 + D_2$ and, using the well known formula

$$\dim(D_1 \cap D_2) = \dim D_1 + \dim D_2 - \dim(D_1 + D_2),$$

it follows that $\dim(D_1 \cap D_2) = 4h + 1$. Hence, from the inclusions

$$V_k \subseteq (V_i \cap V_j + V_k) \subseteq D_1 \cap D_2,$$

we conclude that $\dim(V_i \cap V_j + V_k) = 4h + 1$. Then, $V_i \cap V_j \subseteq V_k$ and consequently, $V_0 = V_i \cap V_j$ is the same subspace for all $i \neq j$. Hence, $V_0 = U_1 \cap U_2$ is the subspace of W explicitly given in (2.10). Now, from (2.14) it follows that $U = N_i^{-1}(V_0)$ is independent of i. Thus, $U = E_1^{-1}(V_0)$ and, from (2.8) and (2.10), it follows that

$$U = \{x \mid x = (0, x_2, \cdots, x_s)^t\}.$$

The fact that $N_i(U) \subseteq V_0$, for $1 \leqslant i \leqslant 4$, implies that the maps N_i do restrict to $N_i'\colon U \to V_0$, and this gives a normed pairing of size [4, $4h$, $4h$]. Now, let us consider the orthogonal complements $U^\perp$ and $V_0^\perp$, respectively, of U in V and of V_0 in W. They are,

$$U^\perp = \{x \mid x = (x_1, 0, \cdots, 0)^t\} \quad \text{and}$$

$$V_0^\perp = \{y \mid y = (y_1, 0, \cdots, 0, y_{s+1}, y_{s+2})^t\}.$$

Since the maps N_i preserve inner products, it follows that $N_i(U^\perp) \subseteq V_0^\perp$, for $1 \leqslant i \leqslant 4$. So, their restriction to $N_i''\colon U^\perp \to V_0^\perp$ sets [4, 1, 3] as an admissible triple, which is obviously false. This contradiction ends the proof of (1.2).

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