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SCHAUDER BASIC MEASURES IN BANACH AND HILBERT SPACES

By José Luis Abreu (*) and Habib Salehi

Summary

Schauder basic measures are defined as a natural generalization of orthogonally scattered measures. It is shown that Schauder basic measures with values in a Hilbert space are precisely those measures that can be transformed, via a bounded linear operator with bounded inverse, into orthogonally scattered measures. As a consequence it is shown that sequences $\{x_n: n \in Z\}$ in Hilbert space, which have a uniformly bounded shift operator group, are precisely those which are Fourier series of Schauder basic measures on the circle. In order to prove these theorems, two basic results are established. The first one is that the space of integrable functions with respect to a Banach space valued measure becomes a Banach space when endowed with certain natural norm. The second one is an inequality that gives upper and lower bounds for the sum of the squared norms of a finite set of vectors in Hilbert space, in terms of some of their linear combinations.

§1. Introduction

Unconditional Schauder bases (see [8] §1.c) in Banach spaces play an analogous role as orthogonal bases do in Hilbert spaces. On the other hand, in Hilbert space, an orthogonally scattered measure (see [9]) is something like a "continuously distributed" orthogonal basis. It seems natural to introduce an analogous concept of "continuously distributed" unconditional Schauder basis. We define such a concept in §3 and name it Schauder basic measure.

Our motivation to study such measures (and their Fourier transforms) comes from the work of Tjöstheim and Thomas [14], Niemi [10], Abreu and Fetter [1] on second order nonstationary processes with shift operator groups. It became apparent from the mentioned articles that the existence of a shift operator group is closely related to the process being the image, under an injective linear map, of a stationary process. This in turn is equivalent to the process being the Fourier transform (or Fourier series in the discrete case) of a vector valued measure which is the image, under an injective linear map, of an orthogonally scattered measure. Thus the need to characterize such measures arises.

Let Σ be a sigma algebra of subsets of a set Ω and ξ a countably additive orthogonally scattered measure on Σ with values in a Hilbert space H. Let X be a Banach space and T: $H \to X$ a bounded linear map with a bounded inverse. Define $\mu = T \cdot \xi$. Then μ is an X-valued measure on Σ with the following properties:

(*) Visiting Professor at Michigan State University, 1983–1984.

(a) If $\{E_n\}$ is a sequence of disjoint elements of Σ then $\{\mu(E_n)\}$ is an unconditional Schauder basic sequence in X.

(b) There exists a positive constant K such that for every pair f, g of μ-integrable functions with $|f| \leqslant |g|$,

$$\| \int_\Omega f \, d\mu \| \leqslant K \| \int_\Omega g \, d\mu \|.$$

A Banach space valued measure satisfying (a) is called *basically scattered.* Such measures were introduced by Kalton, Turett and Uhl in [7]. A Banach space valued measure satisfying (b) will be called a *Schauder basic measure* in this paper (see Definition 3.1 and Propositions 3.3 and 3.5).

It turns out that some basically scattered measures, those having bounded basis constant according to the definition given in §2 of [7], are precisely the Schauder basic measures introduced in this paper (see Proposition 3.6).

In §2 we study some properties of the space of integrable functions (in the sense of Bartle, Dunford and Schwartz [2] with respect to a vector measure. In particular we show that this space is complete (see Theorem 2.7). The completeness was proved first by E. Thomas [13] for Radon measures. It can also be deduced from the extensive work of Brooks and Dinculeanu (see [3], in particular Theorem 4.6) on operator valued measures. Our approach is an adaptation of Thomas's approach to measures which are set functions. We include it for the benefit of the reader unfamiliar with Radon measures and to make our exposition self contained.

In §3 we define Schauder basic measures and use the completeness of the space of integrable functions with respect to one such measure to establish the existence of a constant K such that $\| \int f \, d\mu \| \leqslant K \| \int g \, d\mu \|$ whenever $|f| \leqslant |g|$. This constant K is analogous to the unconditional constant of an unconditional Schauder basic sequence (see [8] 1.c). The equivalence between basically scattered measures with bounded basis constant as defined in [7] and Schauder basic measures is also established.

In §4 we prove the elementary inequality:

$$\inf_{|c_j|=1} \| \Sigma \, c_j x_j \|^2 \leqslant \Sigma \, \| x_j \|^2 \leqslant \sup_{|c_j|=1} \| \Sigma \, c_j x_j \|^2$$

for an arbitrary finite set of vectors $x_1, \ldots, x_n$ in a Hilbert space (Theorem 4.2), which is crucial for our work. This inequality together with the existence of the constant K established in §3 and an important result due to Niemi [11], allows us to prove that a Schauder basic measure with values in Hilbert space is similar (via a bounded linear operator with a bounded inverse) to an orthogonally scattered measure (Theorem 4.2). Tjöstheim and Thomas in [14] and Niemi in [10] studied a class of second order nonstationary stochastic processes called uniformly bounded linearly stationary (U.B.L.S.). U.B.L.S. processes are those which have a uniformly bounded shift operator group. Using a theorem of Sz. Nagy [12] on uniformly bounded linear operators in Hilbert space it can be proved, as was done by Tjöstheim and Thomas, that a U.B.L.S. process is the image under a bounded linear operator with a bounded inverse of a stationary process. Using this characterization of U.B.L.S. processes and Theorem 4.2 one obtains that U.B.L.S. processes in Hilbert space are precisely the Fourier transforms (or Fourier series in the discrete parameter case) of Schauder basic measures (Theorem 4.3). An example is given to show that this result is false for U.B.L.S. processes in Banach space.

§2. The space of integrable functions with respect to a Banach space valued measure

Throughout this section X is a Banach space with norm $\| \ \|$, X^* denotes its dual space, Ω is a set, Σ a sigma algebra of subsets of Ω and $M(\Omega, \Sigma; X)$ the linear space of countably additive X-valued measures on (Ω, Σ). For each $\mu \in M(\Omega, \Sigma; X)$ and $E \in \Sigma$, $\|\mu\|(E)$ denotes the semivariation and $|\mu|(E)$ the total variation of μ on E. $\|\mu\|(\cdot)$ is bounded and subadditive while $|\mu|(\cdot)$ is additive but may be unbounded. For the definition and properties of the semivariation and other aspects of the theory of integration of complex valued functions with respect to vector measures we refer to Dunford and Schwartz [6] §IV. 10. The semivariation $\|\mu\|(\Omega)$ on the whole space Ω is a norm in $M(\Omega, \Sigma; X)$.

PROPOSITION (2.1). *$M(\Omega, \Sigma; X)$ with the norm $\mu \to \|\mu\|(\Omega)$ is a Banach space.*

Proof. Let $\{\mu_n\}$ be a Cauchy sequence in $M(\Omega, \Sigma; X)$, i.e. $\|\mu_m - \mu_n\|(\Omega) \to 0$ and $m, n \to \infty$. Since for every $E \in \Sigma$, $\|\mu_m(E) - \mu_n(E)\| \leq \|\mu_m - \mu_n\|(\Omega)$, it follows that $\mu(E) = \lim \mu_n(E)$ as $n \to \infty$ exists. Thus by a generalization of Nikodym's theorem (see [6], theorem IV.10.6), μ is a countably additive measure. It only remains to show that $\|\mu_n - \mu\|(\Omega) \to 0$ as $n \to \infty$. Let $\epsilon > 0$. Choose N such that $\|\mu_n - \mu_m\|(\Omega) < \epsilon/4$ for every $m, n \geq N$. Let ϕ be a simple complex valued function such that $|\phi| \leq 1$ and

$$\|\mu_N - \mu\|(\Omega) < \epsilon/4 + \left\|\int_\Omega \phi \, d(\mu_N - \mu)\right\|.$$

Finally choose $m \geq N$ such that $\|\int_\Omega \phi \, d(\mu - \mu_m)\| < \epsilon/4$. This can be done because there are only finitely many sets involved due to the fact that ϕ is simple. Then for $n \geq N$

$$\begin{aligned} \|\mu_n - \mu\|(\Omega) &\leq \|\mu_n - \mu_N\|(\Omega) + \|\mu_N - \mu\|(\Omega) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \left\|\int_\Omega \phi \, d(\mu_N - \mu)\right\| \\ &< \frac{\epsilon}{2} + \left\|\int_\Omega \phi \, d(\mu_N - \mu_m)\right\| \\ &\quad + \left\|\int_\Omega \phi \, d(\mu_m - \mu)\right\| \\ &< \frac{3}{4}\epsilon + \|\mu_N - \mu_m\|(\Omega) < \epsilon. \end{aligned}$$

Q.E.D.

Throughout the remainder of this section μ will be a fixed countably additive measure on (Ω, Σ) with values in X. A set $E \in \Sigma$ is μ-null if $\| \mu \| (E) = 0$. If two measurable functions f and g agree everywhere except on a μ-null set, then we say that $f = g$ μ-almost everywhere (abbreviated μ a.e.). A complex valued measurable function f on (Ω, Σ) is said to be μ-integrable if there exists a sequence of simple functions $\{\phi_n\}$ that converges pointwise to f, except perhaps on a μ-null set, and for each $E \in \Sigma$, $\{ \int_E \phi_n \, d\mu\}$ is a Cauchy sequence in X. Of course $\int_E f \, d\mu$ is defined to be the limit of $\int_E \phi_n \, d\mu$ as $n \to \infty$. In [6] §IV.10 it is proved that $\int_E f \, d\mu$ is well defined and is linear as a function of f, and countably additive as a function of E. Also it is proved that every bounded measurable function is μ-integrable and if $| f | \leq M$ μ a.e. then $\| \int_E f \, d\mu \| \leq M \| \mu \| (E)$.

Our purpose in this section is to make a Banach space out of the space of equivalence classes of μ-integrable functions (two functions belonging to the same class if they agree μ a.e.). In order to do this we require the following elementary results.

PROPOSITION (2.2). *For every* $E \in \Sigma$, $\| \mu \| (E) = \sup_{|f| \leq 1} \| \int_E f \, d\mu \|$ *where the* sup *is taken over all measurable functions bounded by* 1.

Proof. Define for each $E \in \Sigma$

$c(E) = \sup\{ \| \int_E f \, d\mu \| : f$ is measurable and $| f | \leq 1\}$. From the definition of semivariation it is clear that $\| \mu \| (E) \leq c(E)$. Since for $| f | \leq 1$, $\| \int_E f \, d\mu \| \leq \| \mu \| (E)$ (c.f. paragraph following the proof of Proposition 2.1) it follows that $c(E) \leq \| \mu \| (E)$. This completes the proof.

PROPOSITION (2.3). *Let f be μ-integrable. Then* $\| f \, d\mu \| (\Omega) = 0$ *if and only if* $f = 0$ μ a.e. ($f \, d\mu$ *denotes the measure* $E \to \int_E f \, d\mu$).

Proof. Let $E_n = \{\omega \in \Omega : 0 < | f(\omega) | < 1/n\}$. $\{E_n\}$ is a decreasing sequence of measurable sets with an empty intersection. Let λ be a finite positive measure on (Ω, Σ) such that $\| \mu \| (E) \to 0$ as $\lambda(E) \to 0$. The existence of such λ is granted by Lemma IV.10.5 in [6]. Thus $\lambda(E_n) \to 0$ and hence $\| \mu \| (E_n) \to 0$ as $n \to \infty$. Let $N = \{\omega \in \Omega : f(\omega) \neq 0\}$. Then, using Proposition 2.2,

$$\| \mu \| (N \backslash E_n) = \sup_{|h| \leq 1} \| \textstyle\int_{N \backslash E_n} h \, d\mu \| \leq \sup_{|h'| \leq 1} \| \int_{N \backslash E_n} h' n f \, d\mu \|$$

$$\leq n \| f \, d\mu \| (N \backslash E_n), \quad \text{since}$$

$$\{h 1_{N \backslash E_n} : | h | \leq 1\} \subset \{h' n f : | h' | \leq 1\}.$$

This shows that if $\| f \, d\mu \| (\Omega) = 0$ then $\| \mu \| (N) \leq \| \mu \| (N \backslash E_n) + \| \mu \| (E_n) \leq 0 + \| \mu \| (E_n) \to 0$ as $n \to \infty$ and hence $\| \mu \| (N) = 0$, i.e., $f = 0$ μ a.e. Conversely if $f = 0 \mu$ a.e. then the constant sequence $\{0\}$ converges to $f \, d\mu$ a.e. and hence $\int_E f \, d\mu = 0$ for every $E \in \Sigma$, thus showing that $\| f \, d\mu \| (\Omega) = 0$. This completes the proof.

Definition (2.4). Let $I(\mu)$ be the linear space of equivalence classes of complex valued μ-integrable functions f (two functions being in the same class if they are equal μ almost everywhere), equipped with the norm $|||\, f\, ||| = \| f\ d\mu \| (\Omega)$. It is easy to see, using Proposition 2.3 and the subadditivity of the semivariation, that $I(\mu)$ is a normed linear space. Our objective now is to show that $I(\mu)$ is actually a Banach space. We need two lemmas.

LEMMA (2.5). *Let $f, g, \in I(\mu)$ and $E \in \Sigma$. If $| f | \leq | g |$ on E then $\| f\, d\mu \| (E) \leq \| g\, d\mu \| (E)$. If $| f | = | g |$ on Ω then $|||\, f\, ||| = |||\, g\, |||$.*

Proof. Using Proposition 2.2 we see that $\| g\, d\mu \| (E) = \sup_{|h| \leq 1} \| \int_E hg\, d\mu \| \geq \sup_{|h| \leq 1} \| \int_E h(f/g)\, g\, d\mu \| = \| f\, d\mu \| (E)$.

This proves the first part. The second part is an immediate consequence of the first.

Q.E.D.

LEMMA (2.6). *The simple functions are dense in $I(\mu)$,* i.e. *given $f \in I(\mu)$ and $\epsilon > 0$ there is a simple function ϕ such that $|||\, \phi - f\, ||| < \epsilon$.*

Proof. Let $f \in I(\mu)$. Let λ be a finite positive measure on (Ω, Σ) such that $\lim \| f\ d\mu \| (E) = 0$ as $\lambda(E) \to 0$. For each positive integer n define $E_n = \{\omega \in \Omega\colon | f(\omega) | > n\}$. Then $\lambda(E_n) \to 0$ and hence $\| f\, d\mu \| (E_n) \to 0$. Let $\epsilon > 0$. Choose n such that $\| f\, d\mu \| (E_n) < \epsilon/2$ and choose ϕ simple such that $\phi = 0$ on E_n and $| \phi - f | < \epsilon/2 \| \mu \| (\Omega \backslash E_n)$ on $\Omega \backslash E_n$. Then by Lemma 2.5,

$$\| (\phi - f) d\mu \| (\Omega) \leq \| (\phi - f) d\mu \| (\Omega \backslash E_n) + \| f\, d\mu \| (E_n) < \epsilon.$$

Thus $|||\, \phi - f\, ||| < \epsilon$.

Q.E.D.

THEOREM (2.7). *$I(\mu)$ is a Banach space.*

Proof. The fact that $I(\mu)$ is a normed space was observed in 2.4. It only remains to prove that $I(\mu)$ is complete. Let $\{ f_n \}$ be a Cauchy sequence in $I(\mu)$. Using Lemma 2.6 we can construct a sequence of simple functions ϕ_n such that $|||\, \phi_n - f_n\, ||| \to 0$ as $n \to \infty$. In particular $\{\phi_n\}$ is also a Cauchy sequence in $I(\mu)$. Now we extract a subsequence $\{\phi_{n_k}\}$ such that

$$\textstyle\sum_{k=1}^{\infty} |||\, \phi_{n_{k+1}} - \phi_{n_k}\, ||| < \infty.$$

Define

$$N = \{\omega \in \Omega\colon \textstyle\sum_{k=1}^{\infty} | \phi_{n_{k+1}}(\omega) - \phi_{n_k}(\omega) | = \infty\}.$$

We will show that $\| \mu \| (N) = 0$. Suppose $\| \mu \| (N) = \epsilon > 0$. Let M be any positive real number such that

$$\textstyle\sum_{k=1}^{\infty} \| \phi_{n_{k+1}} - \phi_{n_k} \| < M.$$

Define for every positive integer m,

$$N_m = \left\{\omega \in \Omega\colon \textstyle\sum_{k=1}^{m} | \phi_{n_{k+1}}(\omega) - \phi_{n_k}(\omega) | > \dfrac{2M}{\epsilon}\right\}, \quad \text{and} \quad N' = \textstyle\bigcup_{m=1}^{\infty} N_m.$$

Then $\| \mu \| (N' \backslash N_m) \to 0$ as $m \to \infty$. (This is a consequence of Lemma IV.10.5 in [6]). Since

$$\epsilon = \| \mu \| (N) \leqslant \| \mu \| (N') \leqslant \| \mu \| (N_m) + \| \mu \| (N' \backslash N_m)$$

there exists m_0 such that $\| \mu \| (N_{m_0}) > \frac{\epsilon}{2}$. Hence

$$\begin{aligned} \textstyle\sum_{k=1}^{\infty} ||| \phi_{n_{k+1}} - \phi_{n_k} ||| &\geqslant \textstyle\sum_{k=1}^{m_0} \| (\phi_{n_{k+1}} - \phi_{n_k}) d\mu \| (N_{m_0}) \\ &\geqslant \| \textstyle\sum_{k=1}^{m_0} | \phi_{n_{k+1}} - \phi_{n_k} | \, d\mu \| (N_{m_0}) \\ &\geqslant \left\| \frac{2M}{\epsilon} d\mu \right\| (N_{m_0}) = \frac{2M}{\epsilon} \| \mu \| (N_{m_0}) > M \end{aligned}$$

which contradicts our choice of M. We conclude that $\| \mu \| (N) = 0$. Now define

$$f(\omega) = \begin{cases} \phi_{n_1}(\omega) + \sum_{k=1}^{\infty} (\phi_{n_{k+1}}(\omega) - \phi_{n_k}(\omega)) & \text{if} \quad \omega \in \Omega \backslash N \\ 0 \quad \text{if} \quad \omega \in N. \end{cases}$$

Then $\phi_{n_k} \to f$ pointwise μ a.e. as $k \to \infty$. Also, for every $E \in \Sigma$, $\{ \int_E \phi_{n_k} d\mu \}$ is a Cauchy sequence in X since for every pair of positive integers k, j

$$\begin{aligned} \| \textstyle\int_E \phi_{n_{k+j}} d\mu - \int_E \phi_{n_k} d\mu \| &\leqslant \| (\phi_{n_{k+j}} - \phi_{n_k}) d\mu \| (\Omega) \\ &\leqslant \textstyle\sum_{l=k}^{\infty} ||| \phi_{n_{l+1}} - \phi_{n_l} ||| \end{aligned}$$

which goes to zero as $k \to \infty$. This proves $f \in I(\mu)$. From the completeness of $M(\Omega, \Sigma; X)$ (Proposition 2.1) it follows that there exists a $\nu \in M(\Omega, \Sigma; X)$ such that $\phi_{n_k} d\mu \to d\nu$ in the norm of $M(\Omega, \Sigma; X)$. But then for every $E \in \Sigma$

$$\textstyle\int_E \phi_{n_k} d\mu \to \nu(E) \quad \text{as} \quad k \to \infty$$

and hence $d\nu = f \, d\mu$. This shows that $\| (\phi_{n_k} - f) d\mu \| (\Omega) \to 0$ as $k \to \infty$, i.e. $||| \phi_{n_k} - f ||| \to 0$ as $k \to \infty$. This implies, by the way we chose ϕ_{n_k}, that $||| f_n - f ||| \to 0$, and completes the proof of Theorem 2.7.

COROLLARY (2.8). *$\{ f \, d\mu : f \in I(\mu) \}$ is a closed linear subspace of $M(\Omega, \Sigma; X)$.*

§3. Schauder basic measures in Banach space

Throughout this section X is a complex Banach space with norm $\| \ \|$, X^* denotes its dual space, Ω is a set, Σ is a sigma algebra of subsets of Ω and μ is an X-valued countably additive measure on (Ω, Σ). $I(\mu)$ and $M(\Omega, \Sigma; X)$ are as defined in §2.

Definition (3.1). μ is called a *Schauder basic measure* if the map $T: I(\mu) \to X$ given by $Tf = \int_\Omega f \, d\mu$ is one to one and its range is closed.

An unconditional Schauder basic sequence $\{x_n\}$ (see [8] §1.c) can be viewed as a Schauder basic measure on a discrete space, provided Σx_n converges. Also a bounded countably additive orthogonally scattered measure (with values in a Hilbert space) is a Schauder basic measure.

PROPOSITION (3.2). *Let μ be a Schauder basic measure on (Ω, Σ) with values in X and let Q be a bounded linear operator on X with a bounded inverse. Then $Q\mu$ is a Schauder basic measure too. Also μ and $Q\mu$ have the same integrable functions.*

Proof. Let $\eta(E) = Q\mu(E)$ for every $E \in \Sigma$. It is clear that η is an X-valued countably additive measure on (Ω, Σ). It is clear too that μ and η have the same null sets, and it is not difficult to see that $f \in I(\eta)$ if and only if $f \in I(\mu)$. Suppose $f \in I(\eta)$ and $\int_\Omega f\, d\eta = 0$. Then $Q^{-1} \int_\Omega f\, d\eta = \int_\Omega f\, d\mu = 0$ and therefore $f = 0$ μ-almost everywhere and hence also η-almost everywhere. This shows that the map $f \longrightarrow \int_\Omega f\, d\eta$ from $I(\eta)$ into X is one to one. Now let $x \in X_\eta$, the closed linear space generated by $\{\eta(E): E \in \Sigma\}$. Since $X_\eta = QX_\mu$ and μ is a Schauder basic measure, there exists $f \in I(\mu)$ such that $Q^{-1}x = \int_\Omega f\, d\mu$. Hence $f \in I(\eta)$ and $x = Q \int_\Omega f\, d\mu = \int_\Omega f\, d\eta$, thus proving that the range of $f \longrightarrow \int_\Omega f\, d\eta$ is closed. This completes the proof.

PROPOSITION (3.3). *Let μ be a Schauder basic measure. Then there exists a constant $K > 0$ such that whenever $f, g \in I(\mu)$ and $|f| \leqslant |g|$ on Ω then $\| \int_\Omega f\, d\mu \| \leqslant K \| \int_\Omega g\, d\mu \|$.*

Proof. Let X_μ denote the closed linear subspace generated by $\{\mu(E): E \in \Sigma\}$. From the hypothesis that μ is a Schauder basic measure it follows that the map T of Definition 3.1 has X_μ as its range. Since $\| Tf \| \leqslant ||| f |||$ for every $f \in I(\mu)$, and since $I(\mu)$ is a Banach space (see Theorem 2.7), T has a bounded inverse $T^{-1}: X_\mu \longrightarrow I(\mu)$. Let K be the norm of T^{-1}. Let $c = f/g$ where $g \neq 0$ and $c = 0$ if $g = 0$. Then $|c| \leqslant 1$ and $\| \int_\Omega f\, d\mu \| = \| \int_\Omega cg\, d\mu \| \leqslant \| g\, d\mu \| (\Omega) = ||| g ||| = ||| T^{-1}Tg ||| \leqslant K \| Tg \| = K \| \int_\Omega g\, d\mu \|$.

Q.E.D.

The constant K obtained in the previous proposition plays a similar role as the unconditional constant of the unconditional Schauder basic sequences (see [8], §1.c). For example the following corollary, which is clearly equivalent to Proposition 1.c.7 in [8], can be deduced from Proposition 3.3.

COROLLARY (3.4). *Let $\{x_n\}$ be an unconditional Schauder basic sequence in X. Then there exists a constant $K > 0$ such that whenever $|a_n| \leqslant |b_n|$ for every n and*

$$\sum_{n=1}^{\infty} b_n x_n \text{ converges then } \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leqslant K \left\| \sum_{n=1}^{\infty} b_n x_n \right\|.$$

Proof. Let Ω be the set of positive integers and Σ the sigma algebra of all subsets of Ω. Define $\mu(E) = \Sigma_{n \in E} \dfrac{x_n}{\| x_n \| n^2}$ for every $E \in \Sigma$. The rest of the proof is a straightforward application of Proposition 3.3.

The converse of Proposition 3.3 is also true and may be used as an alternative definition of basic Schauder measures. Here we state it as a proposition.

PROPOSITION (3.5). *Suppose μ is an X-valued measure and K a positive constant such that whenever $f, g \in I(\mu)$ and $|f| \leqslant |g|$ then $\| \int_\Omega f\, d\mu \| \leqslant K \| \int_\Omega g\, d\mu \|$. Then μ is a basic Schauder measure.*

The proof is straight forward and is omitted.

The measure μ is called *basically scattered* (see [7]) if for any sequence $\{E_n\}$ of disjoint sets in Σ, $\{\mu(E_n)\}$ is a Schauder basic sequence ($\{\mu(E_n)\}$ is then necessarily an unconditional basic sequence).

The following definition is given in [7] for real Banach spaces. Here we extend it to complex Banach spaces.

Definition. A *basically scattered* measure μ is said to have *bounded basic constant* on a set $E \in \Sigma$ if there is a constant K such that for any sequence $\{E_n\} \subset \Sigma$ of disjoint subsets of E, for any sequence $\{a_n\}$ of complex numbers and for any pair m, p of positive integers,

$$\| \textstyle\sum_{n=1}^{m} a_n \mu(E_n) \| \leqslant K \| \textstyle\sum_{n=1}^{m+p} a_n \mu(E_n) \| .$$

We say μ has *bounded basis constant* if it has bounded basis constant on Ω.

PROPOSITION (3.6). *μ is a Schauder basic measure if and only if it is basically scattered and has bounded basis constant.*

Proof. Suppose μ is a Schauder basic measure and let K be the constant from Proposition 3.3. Let $\{E_n\}$ be a sequence of disjoint elements of Σ. Clearly the elements of $\{\mu(E_n)\}$ are linearly independent. Suppose $x \in$ c.l.s. $\{\mu(E_n)\}$. Then $x = \lim_{k\to\infty} \int f_k\, d\mu$ where $\{f_k\}$ is a Cauchy sequence of simple functions of the form $f_k = \Sigma\, a_n^k I_{E_n}$. Thus $f_k \to f$ in $I(\mu)$ and $f = \Sigma\, a_n I_{E_n}$. Therefore

$$x = \int f\, d\mu = \Sigma\, a_n \mu(E_n).$$

This representation of X is unique and the series is unconditionally convergent. Thus $\{\mu(E_n)\}$ is an unconditional Schauder basic sequence. This proves μ is basically scattered. Also:

$$\| \textstyle\sum_{n=1}^{m} a_n \mu(E_n) \| = \| \int_\Omega \sum_{m=1}^{m} a_n I_{E_m}\, d\mu \| \leqslant K \| \int_\Omega \sum_{n=1}^{m+p} a_n I_{E_n}\, d\mu \|$$

$$= K \| \textstyle\sum_{n=1}^{m+p} a_n \mu(E_n) \|$$

for every sequence $\{a_n\}$ of real (or complex) numbers. This proves μ has bounded basis constant on Ω.

Conversely, suppose μ is basically scattered and has bounded basis constant on Ω, more specifically, let $M > 0$ be such that for every sequence $\{a_n\}$ of complex numbers, any positive integers m and p, and any disjoint sequence $\{E_n\}$ in Σ,

$$\| \textstyle\sum_{n=1}^{m} a_n \mu(E_n) \| \leqslant M \| \textstyle\sum_{n=1}^{m+p} a_n \mu(E_n) \| .$$

Now, suppose $\{a_n\}$ and $\{b_n\}$ are sequences of complex numbers such that $|a_n| \leqslant |b_n|$. Let m be any positive integer $a_n = \alpha_n b_n$ with α_n real and $|\alpha_n| \leqslant 1$.

Let C be the convex hull of the set $\{\sum_{n=1}^{m} \delta_n b_n \mu(E_n) : \delta_n = \pm 1\}$Then

$$\sup\{\| x \| : x \in C\} = \sup\{\| \textstyle\sum_{n=1}^{m} \delta_n b_n \mu(E_n) \| : \delta_n = \pm 1,\ n = 1, \ldots, m\}.$$

Now, for any choice of $\delta_1, \ldots, \delta_m$,

$$\begin{aligned} \| \textstyle\sum_{n=1}^{m} \delta_n b_n \mu(E_n) \| &\leqslant \| \textstyle\sum_{\delta_n=1} b_n \mu(E_n) \| + \| \textstyle\sum_{\delta_n=-1} b_n \mu(E_n) \| \\ &\leqslant 2M \ \| \textstyle\sum_{n=1}^{m} b_n \mu(E_n) \| \end{aligned}$$

and since $\sum_{n=1}^{m} a_n \mu(E_m) \in C$, we have

$$\| \textstyle\sum_{n=1}^{m} a_n \mu(E_n) \| \leqslant 2M \ \| \textstyle\sum_{n=1}^{m} b_n \mu(E_n) \| .$$

Now suppose only $| a_n | \leqslant | b_n |$. Write $a_n = (\alpha_n + i\beta_n) b_n$ with α_n, β_n real. Then $| \alpha_n |$, $| \beta_n | \leqslant 1$ and

$$\begin{aligned} \| \textstyle\sum_{n=1}^{m} a_n \mu(E_n) \| &\leqslant \| \textstyle\sum_{n=1}^{m} \alpha_n b_n \mu(E_n) \| + \| \textstyle\sum_{n=1}^{m} \beta_n b_n \mu(E_n) \| \\ &\leqslant 4M \ \| \textstyle\sum_{n=1}^{m} b_n \mu(E_n) \| \end{aligned}$$

This proves that for any pair of simple functions f, g such that $| f | \leqslant | g |$, we have

$$\| \textstyle\int_\Omega f \, d\mu \| \leqslant 4M \ \| \textstyle\int_\Omega g \, d\mu \|$$

and this in turn extends to all pairs of μ-integrable functions f, g such that $| f | \leqslant | g |$, thus proving μ is a Schauder basic measure.

Q.E.D.

§4. Schauder basic measures in Hilbert space

Throughout this section H is a complex Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$.

THEOREM (4.1). *Let $x_1, \ldots, x_n$ be vectors in a complex Hilbert space. Then*

$$\inf \ \| \textstyle\sum_{k=1}^{n} c_k x_k \|^2 \leqslant \textstyle\sum_{k=1}^{n} \| x_k \|^2 \leqslant \sup \textstyle\sum_{k=1}^{n} c_k x_k \|^2$$

where the inf *and the* sup *are taken over all collections of numbers $c_1, \ldots, c_n$ such that $c_k = \pm 1$ for $k = 1, \ldots, n$.*

Proof. Let $c_1, \ldots, c_n$ be independent, identically distributed random variables with values ± 1. Then

$$\begin{aligned} E \ \| \textstyle\sum_{k=1}^{n} c_k x_k \|^2 &= E \ \textstyle\sum_{k=1}^{n} c_k c_j \langle x_k, x_j \rangle \\ &= \textstyle\sum_{k=1}^{n} \sum_{j=1}^{n} E(c_k c_j) \langle x_k, x_j \rangle = \sum_{k=1}^{n} \| x_k \|^2. \end{aligned}$$

The inequalities follow from this equality. Q.E.D.

(The authors wish to thank Prof. S. D. Chatterji for providing this proof.)

Throughout the remainder of this section Ω is a set, Σ is a sigma algebra of subsets of Ω and μ is an H-valued countably additive measure on (Ω, Σ).

THEOREM (4.2). *The following statements are equivalent:*

(a) μ *is a Schauder basic measure.*

(b) *There exists a bounded linear operator* Q *on* H *with a bounded inverse such that* $Q\mu$ *is orthogonally scattered.*

Proof. That (b) implies (a) follows from Proposition 3.2. Suppose (a) holds. Let k be as in Proposition 3.3. Let $\{E_1, E_2, \ldots, E_n\} \subset \Sigma$ be a partition of Ω and let $a_1, a_2, \ldots, a_n$ be complex numbers. From Theorem 4.1. we have

$$\inf_{|c_j|=1} \| \textstyle\sum_{j=1}^n c_j a_j \mu(E_j) \|^2 \leq \sum_{j=1}^n |a_j|^2 \ \|(E_j)\|^2$$
$$\leq \sup_{|c_j|=1} \textstyle\sum_{j=1}^n c_j a_j \mu(E_j) \|^2.$$

The property of the constant K in Proposition 3.3 shows that

$$\frac{1}{K} \sup_{|c_j|=1} \| \textstyle\sum_{j=1}^n c_j a_j \mu(E_j) \| \leq \| \sum_{j=1}^n a_j \mu(E_j) \|$$
$$\leq K \inf_{|c_j|=1} \| \textstyle\sum_{j=1}^n c_j a_j \mu(E_j) \|.$$

Combining these inequalities we obtain

$$\frac{1}{K^2} \textstyle\sum_{j=1}^n |a_j|^2 \| \mu(E_j) \|^2 \leq \| \sum_{j=1}^n a_j \mu(E_j) \|^2 \leq K^2 \sum_{j=1}^n |a_j|^2 \| \mu(E_j) \|^2.$$

According to Hannu Niemi ([11] Theorem 3) that this last inequality holds is equivalent to the existence of a bounded orthogonally scattered measure μ_o on (Ω, Σ) with values in some Hilbert space H_0, and a bounded linear transformation $A: H_{\mu_o} \to H_\mu$ with a bounded inverse (where H_{μ_o} and H_μ are the closed linear subspaces of H_o and H generated by $\{\mu_o(E): E \in \Sigma\}$ and $\{\mu(E): E \in \Sigma\}$, respectively), such that $\mu = A\mu_o$. Let U be a unitary transformation from H_{μ_o} onto H_μ and define $\xi = U_{\mu_o}$. Now define $Q = UA^{-1}$ on H_μ and Q equal to the identity on the orthogonal complement $H \Theta H_\mu$. Then ξ is orthogonally scattered, Q is a bounded linear operator on H with a bounded inverse and $\xi = Q\mu$.
Q.E.D.

As a corollary we now prove the following results on the spectral representation of a continuous uniformly bounded linearly stationary (U.B.L.S.) stochastic processes. $\{x_t\} \subset H$ is said to be a U.B.L.S. process if there exists a constant M such that

$$\| \textstyle\sum_{j=1}^k a_j x_{t+t_j} \| \leq M \ \| \sum_{j=1}^k a_j x_{t_j} \|$$

for all t, t_j in $\mathbb{R}$ and all complex numbers a_j, $j = 1, \ldots, k$. It turns out that continuous U.B.L.S. processes admit a spectral representation

$$x_t = \textstyle\int_{\mathbb{R}} e^{2\pi i \omega t} \ \mu(d\omega)$$

where μ is a countably additive H valued measure on $\mathbb{R}$ (see [9]).

THEOREM (4.3). *Let* $\{x_t: t \in \mathbb{R}\} \subset H$. *Then the following statements are equivalent:*

(a) *$\{x_t\}$ is a continuous uniformly bounded linearly stationary stochastic process.*

(b) $x_t = \int_{\mathbb{R}} e^{2\pi i\omega t}\,\mu(d\omega)$ *for some Schauder basic measure μ on the Borel sets of $\mathbb{R}$.*

(c) *There exists a continuous stationary process $\{y_t\} \subset H$ and a bounded linear operator B with bounded inverse such that $x_t = By_t$.*

Proof. Let $\{x_t : t \in \mathbb{R}\}$ be a continuous U.B.L.S. process. Let μ be a countably additive H valued measure on $\mathbb{R}$ such that

$$x_t = \int_{\mathbb{R}} e^{2\pi i\omega t}\,\mu(d\omega).$$

From Theorem 4 in [10] we know that there exists a countably additive orthogonally scattered measure ξ on the Borel sets of $\mathbb{R}$ and a bounded linear operator B with bounded inverse such that $\mu = B\xi$. Thus by Theorem 4.2 μ is a Schauder basic measure. This proves that (a) implies (b). Conversely, if $x_t = \int_{\mathbb{R}} e^{2\pi i\omega t}\,\mu(d\omega)$ with μ being a Schauder basic measure then by Theorem 4.2 there exists a countably additive orthogonally scattered H valued measure ξ and a bounded linear operator B with bounded inverse such that $\mu = B\xi$. Let $y_t = \int_{\mathbb{R}} e^{2\pi i\omega t}\,\xi(d\omega)$. Then $\{y_t\}$ is stationary and $x_t = By_t$. Hence x_t is a continuous U.B.L.S. process. This shows (b) implies (a). The fact that (b) and (c) are equivalent follows immediately from Theorem 4.2.

A minor modification of the above proof shows that a sequence $\{x_n : n \in Z\} \subset H$ is U.B.L.S. if and only if it is the "Fourier series" of a Schauder basic measure and this holds if and only if $\{x_n\}$ is "equivalent" to a stationary sequence.

§5. Notes and comments

The main result of §4, namely Theorem 4.2, can be specialized to yield the following characterization of unconditional Schauder basic sequences in Hilbert space. Let $\{x_n : n = 1, 2, \ldots\}$ be a sequence in a Hilbert space H. Then $\{x_n\}$ is an unconditional Schauder basic sequence if and only if there exists a bounded linear operator Q on H with a bounded inverse such that $Qx_n \perp Qx_k$ for $n \neq k$, $n, k = 1, 2, \ldots$.

The proof follows applying Theorem 4.2 to the measure

$$\mu(E) = \sum_{n \in E} \frac{x_n}{\| x_n \| n^2}$$

defined on all subsets E of the positive integers. A more direct proof may be given using Theorem 4.1 to show that if $\{x_n\}$ is a Schauder basic sequence in H and $\{e_n\}$ is an orthonormal basis of c.l.s. $\{x_n\}$ then $Q\,\Sigma\, a_n x_n = \Sigma\, a_n \| x_n \| e_n$ defines a bounded linear operator with a bounded inverse on c.l.s. $\{x_n\}$.

This characterization of unconditional Schauder basic sequences in a Hilbert space is actually equivalent to the well known result that all unconditional

Schauder bases in a Hilbert space, which are bounded (above and below), are equivalent. [See 15, Notes and Comments to Chapter 1, Section 9 in p. 208].

Combining Theorem 3 of Niemi [11] and our Theorem 4.3 one obtains that the following statements (a) through (d) are equivalent for a Hilbert space valued countably additive measure μ on (Ω, Σ):

(a) μ is a Schauder basic measure.

(b) There exists a bounded linear operator Q with a bounded inverse such that $Q\mu$ is orthogonally scattered.

(c) There exists a finite positive measure m on (Ω, Σ) and a positive constant M such that for all simple complex valued functions ϕ on Ω

$$\frac{1}{M} \int_\Omega |\phi|^2 \, dm \leqslant \| \int_\Omega \phi \, d\mu \|^2 \leqslant M \int_\Omega |\phi|^2 \, dm.$$

(d) There exists a positive constant K such that for any finite collection $\{E_1, \ldots, E_n\} \subset \Sigma$ of disjoint sets, one has

$$\frac{1}{K^2} \sum_{j=1}^n |a_j|^2 \| \mu(E_j) \|^2 \leqslant \| \sum_{j=1}^n a_j \mu(E_j) \|^2 \leqslant K^2 \sum_{j=1}^n |a_j|^2 \| \mu(E_j) \|^2$$

$$\text{for all} \quad a_j \in C, j = 1, 2, \ldots, n.$$

Niemi proved the equivalence of (b), (c), and (d). Our contribution consists on the observation that (b) $\Rightarrow$ (a) and the proof that (a) $\Rightarrow$ (d). Actually Niemi's results also contemplate the finitely additive case while ours is restricted to the countably additive case since we rely on the Bartle, Dunford and Schwartz theory of integration of scalar functions with respect to a countably additive Banach space valued measure.

We would like to emphasize the fact that in many cases of interest the measure m in statement (c) above, can be constructed explicitly. Suppose there exists a complex valued countably additive measure β on $(\Omega \times \Omega, \Sigma \times \Sigma)$ such that $(\mu(A), \mu(B)) = \beta(A \times B)$ for every $A, B \in \Sigma$; and suppose further that the diagonal $\Delta = \{(s, t) \in \Omega \times \Omega : s = t\}$ of $\Omega \times \Omega$ is the countable intersection of sets of the form $\cup_{j=1}^n E_j \times E_j$, where $\{E_1, \cdots, E_n\}$ is a measurable partition of Ω. Then the measure m can be taken to be the "diagonal" of β, i.e., $m(E) = \beta(E \times E \cap \Delta)$. Indeed m defined in this way is countably additive and finite, and for every $E \in \Sigma$,

$$m(E) = \lim \sum_{j=1}^n \| \mu(E \cap E_j) \|^2 = \lim \beta(E \times E \cap \cup_{k=1}^n E_j \times E_j),$$

where the limit is taken over the sequence of sets $\cup_{k=1}^n E_j \times E_j$ whose intersection is Δ. These conditions are satisfied in many cases of interest. In particular, they are satisfied when μ is the spectral measure of a strongly harmonizable sequence $\{x_n\}$, i.e., when $x_n = \int_0^1 e^{2\pi i n t} \mu(dt)$ and $\beta(A \times B) = (\mu(A), \mu(B))$ extends to a countably additive measure on the Borel sets of the unit square.

Suppose μ is a Schauder basic measure on the Borel sets of the interval [0, 1) with values in a Banach space X. Define $x_n = \int_0^1 e^{2\pi int}\,\mu(dt)$. It is easy to see that $\{x_n\}$ is a U.B.L.S. sequence in X.

The following example, kindly supplied by Alberto Alonso, shows that not every U.B.L.S. sequence in X is of this form.

Example (5.1). Let X be the Banach space $l^1(\mathbb{Z})$ of absolutely summable complex sequences on the integers. Let $\delta_n \in X$ denote the element of $l_1(\mathbb{Z})$ defined by $\delta_n(j) = 0$ if $j \neq n$ and $\delta_n(n) = 1$. It is easy to see that $\{\delta_n\}$ is a U.B.L.S. sequence in X. Actually $\| \Sigma\, a_n \delta_{n+k} \| = \| \Sigma\, a_n \delta_n \| = \Sigma\, | a_n |$. Now suppose there exists an X valued Schauder basic measure μ on the Borel sets of [0, 1) such that

$$\delta_n = \int_0^1 e^{2\pi int}\,\mu(dt)$$

for every integer n. The map $T{:}I(\mu) \to X$ given by $TF = \int_0^1 f\,d\mu$ sends every μ-integrable function into the sequence of its Fourier coefficients, but this sequence must be absolutely summable and it is known that there are continuous functions on [0, 1) for which the sequences of their Fourier coefficients are not absolutely summable. Thus such a Schauder basic measure μ cannot exist.

IIMAS-UNAM
Apartado Postal 20-726
01000 México, D. F.

Departments of Mathematics and Statistics
Michigan State University
East Lansing, Michigan 48824
U.S.A.

References

[1] J. L. Abreu and H. Fetter. *The shift operator of a non-stationary sequence in Hilbert space.* Bol. Soc. Mat. Mexicana **28**(1983) 49–57.

[2] R. G. Bartle, N. Dunford and J. T. Schwartz. *Weak compactness and vector measures.* Canad. J. Math. **7**(1955) 289–305.

[3] J. K. Brooks and N. Dinculeanu. *Lebesgue-Type spaces for vector integration, linear operator, weak completeness and weak compactness.* J. Math. Anal. Appl. **54**(1976) 348–389.

[4] J. Diestel and J. J. Uhl. Vector Measures, A.M.S. Mathematical Surveys **15**, Providence, R. I., 1977.

[5] J. L. Doob. Stochastic Processes, Wiley, New York, 1953.

[6] N. Dunford and J. T. Schwartz. Linear Operators, Interscience Publishers, Inc., New York, 1955.

[7] N. J. Kalton, B. Turett and J. J. Uhl. *Basically scattered vector measures.* Ind. Univ. Math. J. **28**,5(1979) 803–816.

[8] J. Lindenstrauss and L. Tzafriri. Classical Banach Spaces I, Springer Verlag, Berlin-Heidelberg-New York, 1977.

[9] P. Masani. *Orthogonally scattered measures.* Advances in Math. **2**(1968) 61–117.

[10] H. NIEMI. *On the linear prediction problem of certain non-stationary stochastic processes.* Math. Scand. **39**(1976) 146–160.

[11] ———. *Diagonal measure of a positive definite bimeasure.* Measure Theory, Proc. Conf., Oberwolfach 1981, Lecture Notes in Math. Springer Verlag, Berlin-Heidelberg-New York, **945**(1982) 237–246.

[12] B. SZ. NAGY. *On uniformly bounded linear operators.* Acta Sci. Math. (Szeged) **11**(1947) 152–157.

[13] E. THOMAS. *L'integration par rapport à une mesure de Radon vectorielle.* Ann. Inst. Fourier (Grenoble) **20**(1970), 55–191.

[14] D. TJÖSTHEIM AND J. B. THOMAS. *Some properties and examples of random processes that are almost wide sense stationary.* IEEE Trans. Information Theory **21**(1975) 257–262.

[15] R. M. YOUNG. An Introduction to Nonharmonic Fourier Series. Academic Press, New York, 1980.